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FRÉCHET FRAMES FOR SHIFT INVARIANT WEIGHTED SPACES[1]

Suzana Simić[2]

Abstract. In the present paper we analyze Fréchet frame of the form $\{\varphi(\cdot - j) \mid j \in \mathbb{Z}^d\}$. With a known condition on φ, we show that the given sequence constitutes a frame for a test space isomorphic to the space of periodic smooth functions so that its dual is the multiple of the space of periodic distributions by $\widehat{\varphi}$.

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1. Introduction

Frame theory was introduced in [9] and up to now it has been developed very well in connection to wavelet theory, time frequency analysis and sampling theory (see [1], [2], [5], [7], [10], [13], [14], [15], ...). Shift invariant spaces are generated by the frames of the form $\{\varphi(x - na)\}_{n\in\mathbb{Z}^d}$ and in Banach spaces, especially L^p spaces, has been developed by Aldroubi, Sun and Tang [4], who studied frames of the form $\{\varphi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \leqslant i \leqslant r\}$ in L^p spaces. On the other hand, in [16] and [17] the authors introduced Fréchet frames and in this way enabled the analysis of various test function spaces and their duals spaces of distributions.

In Section 2 we recall from [16] and [17] the definitions concerning Fréchet frames. Section 3 contains preliminary results on shift invariant weighted spaces, extensions of the corresponding results in [4]. Our main result is given in Section 4. We prove that $\{\varphi(\cdot - j) \mid j \in \mathbb{R}^d\}$ is a frame for weighted shift invariant spaces through several equivalent conditions. In the end we conclude that $\{\varphi(\cdot - j) \mid j \in \mathbb{R}^d\}$ forms a Fréchet frame for a space of test functions $X_F = \mathcal{F}^{-1}(\widehat{\varphi}\cdot\mathcal{P}(-\pi, \pi))$, where $\mathcal{P}$ is the space of periodic test functions.

2. Notation and notions

We will recall basic notions following [6], [11], [16].

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[2]Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, Kragujevac 34000, Serbia, e-mail: suzanasimic@kg.ac.rs

We denote by $(X, \|\cdot\|)$ a Banach space, by $(X^*, \|\cdot\|^*)$ its dual space, $(\Theta, |||\cdot|||)$ is a Banach sequence space. If the coordinate functionals on Θ are continuous, or, equivalently, if the convergence in Θ implies the convergence of the corresponding coordinates, then Θ is called a BK-space.

We refer to [11] for the basic definitions of frames. p-frames in shift-invariant spaces of L^p were considered in [4], while p-frames in general Banach spaces were studied in [8].

Let $\{(Y_s, |\cdot|_s)\}_{s\in\mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, be a family of separable Banach spaces such that

$$\{\mathbf{0}\} \neq \bigcap_{s\in\mathbb{N}_0} Y_s \subseteq \cdots \subseteq Y_2 \subseteq Y_1 \subseteq Y_0, \tag{1}$$

$$|\cdot|_0 \leqslant |\cdot|_1 \leqslant |\cdot|_2 \leqslant \cdots, \tag{2}$$

$$Y_F := \bigcap_{s\in\mathbb{N}_0} Y_s \text{ is dense in } Y_s, \quad s\in\mathbb{N}_0. \tag{3}$$

Then Y_F is a Fréchet space with the sequence of norms $|\cdot|_s$, $s\in\mathbb{N}_0$.

We will always assume that $\{(X_s, \|\cdot\|_s)\}_{s\in\mathbb{N}_0}$ and $\{(\Theta_s, |||\cdot|||_s)\}_{s\in\mathbb{N}_0}$ are sequences of Banach spaces which satisfy (1)-(3). For fixed $s\in\mathbb{N}_0$, an operator $V:\Theta_F\to X_F$ will be called s-bounded if there exists a constant $K_s>0$ such that $\|V(\{c_i\}_{i\in\mathbb{N}})\|_s \leqslant K_s|||\{c_i\}_{i\in\mathbb{N}}|||_s$ for all $\{c_i\}_{i\in\mathbb{N}}\in\Theta_F$. If V is s-bounded for every $s\in\mathbb{N}_0$, then V will be called F-bounded.

Let $\{(\Theta_s, |||\cdot|||_s)\}_{s\in\mathbb{N}_0}$ be a sequence of BK-spaces, as well. Then a sequence $\{g_i\}_{i\in\mathbb{N}}\in(X_F^*)^{\mathbb{N}}$ is called a pre-F-frame for X_F with respect to Θ_F if for every $s\in\mathbb{N}_0$, there exist constants $0<A_s\leqslant B_s<+\infty$ such that

$$\{g_i(f)\}_{i\in\mathbb{N}}\in\Theta_F, \quad f\in X_F, \tag{4}$$

$$A_s\|f\|_s \leqslant |||\{g_i(f)\}_{i\in\mathbb{N}}|||_s \leqslant B_s\|f\|_s, \quad f\in X_F. \tag{5}$$

The constants B_s and A_s, $s\in\mathbb{N}_0$, are called resp. upper and lower bounds for $\{g_i\}_{i\in\mathbb{N}}$. If $A_s=B_s$, $s\in\mathbb{N}_0$, then the pre-F-frame is called tight. If there exists an F-bounded operator $V:\Theta_F\to X_F$ such that $V(\{g_i(f)\}_{i\in\mathbb{N}})=f$ for all $f\in X_F$, then a pre-F-frame $\{g_i\}_{i\in\mathbb{N}}$ is called an F-frame (Fréchet frame) for X_F with respect to Θ_F and V is called an F-frame operator for $\{g_i\}_{i\in\mathbb{N}}$. When (4) holds and at least the upper inequality in (5) holds, then $\{g_i\}_{i\in\mathbb{N}}$ is called an F-Bessel sequence for X_F with respect to Θ_F with bounds B_s, $s\in\mathbb{N}_0$.

When $X=X_F=X_s$ and $\Theta=\Theta_F=\Theta_s$, then one obtains the definitions of Θ-frame, Banach frame and Θ-Bessel sequence, respectively.

If $\{g_i\}_{i\in\mathbb{N}}$ is a pre-F-frame for X_F with respect to Θ_F with lower bounds A_s and upper bounds B_s , $s\in\mathbb{N}_0$, then for every $s\in\mathbb{N}_0$ we have

$$A_s\|f\|_s \leqslant |||\{g_i^s(f)\}_{i\in\mathbb{N}}|||_s \leqslant \lambda_s B_s\|f\|_s, \quad f\in X_s,$$

where g_i^s is the continuous extension of g_i on X_s . We will consider the following operators

$$U_s : X_s \to \Theta_s, \quad U_s f = \{g_i^s(f)\}_{i\in\mathbb{N}}, \quad s \in \mathbb{N}_0, \tag{6}$$

$$U : X_F \to \Theta_F, \quad Uf = \{g_i(f)\}_{i\in\mathbb{N}}, \tag{7}$$

and

$$U_s^{-1} : \mathcal{R}(U_s) \to X_s, \quad U^{-1} : \mathcal{R}(U) \to X_F. \tag{8}$$

The shift invariant spaces of the form

$$V(\varphi) = \Big\{ \sum_{j\in\mathbb{Z}^d} c_j\varphi(\cdot - j)\Big\},$$

where $c = \{c_j\}_{j\in\mathbb{Z}^d}$ is taken from some sequence space, are considered in [4]. φ is called generator of $V(\varphi)$. The space $V_p(\varphi)$ is the shift invariant space of the form $V_p(\varphi) = \{ \sum_{k\in\mathbb{Z}^d} c_k\varphi(\cdot - k) \mid c = \{c_k\}_{k\in\mathbb{Z}^d} \in \ell^p$. Let $V_0(\varphi)$ be the space of finite linear combination of integer translates of φ and $V_{0,p}(\varphi)$ be the L^p closure of $V_0(\varphi)$. Obviously, we have $V_0(\varphi) \subset V_p(\varphi) \subset V_{0,p}(\varphi)$. A function in $V_{0,p}(\varphi)$ is not necessarily generated by ℓ^p coefficients. If $V_p(\varphi)$ is itself closed, i.e. a Banach space, then $V_p(\varphi) = V_{0,p}(\varphi)$.

3. Preliminary result

Considering p-frames for shift invariant subspaces of L^p space, Aldroubi, Sun and Tung in [4] proved that when a sequence of translations of a finite set of appropriate functions $\varphi_1, \dots, \varphi_r$ forms an ℓ^p-frame for the shift-invariant space $V_p(\varphi_1, \dots, \varphi_r) \subseteq L^p$, for some $p > 1$, then this sequence is also an ℓ^r-frame for $V_r(\varphi_1, \dots, \varphi_r)$ for all values of $r > 1$.

In this paper we will consider weighted L^p_s, $s \geqslant 0$, spaces. A function f belongs to L^p_s with weight function $\omega_s(x) = (1 + |x|)^s$, $x \in \mathbb{R}^d$, $s \geqslant 0$, if $\omega_s f$ belongs to L^p. Equipped with the norm $\|f\|_{L^p_s} = \|\omega_s f\|_{L^p}$, the space L^p_s is a Banach space. Let $s \geqslant 0$, $1 \leqslant p < +\infty$ and

$$\mathcal{L}^p_s := \Big\{f \mid \|f\|_{\mathcal{L}^p_s} := \Big(\int\limits_{[0,1]^d} \Big(\sum_{j\in\mathbb{Z}^d} |f(x + j)|(1 + |x + j|)^s\Big)^p dx\Big)^{1/p} < +\infty\Big\},$$

$$\mathcal{L}^\infty_s := \Big\{f \mid \|f\|_{\mathcal{L}^\infty_s} := \sup_{x\in[0,1]^d} \sum_{j\in\mathbb{Z}^d} |f(x + j)|(1 + |x + j|)^s < +\infty\Big\};$$

$$W^p_s := \Big\{f \mid \|f\|_{W^p_s} := \Big(\sum_{j\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} |f(x + j)|^p (1 + |j|)^{ps}\Big)^{1/p} < +\infty\Big\};$$

$$\ell_s^p := \Big\{ c = \{c_i\}_{i\in\mathbb{N}} \mid \|c\|_{\ell_s^p} = \Big(\sum_{i\in\mathbb{N}} |c_i|^p (1+|i|)^{sp}\Big)^{1/p} < +\infty \Big\}.$$

Obviously we have $W_s^p \subset W_s^q \subset \mathcal{L}_s^\infty \subset \mathcal{L}_s^q \subset \mathcal{L}_s^p \subset L_s^p$, where $1 \leqslant p \leqslant q \leqslant +\infty$. For $p = 1$ and $s = 0$ we also have $\mathcal{L}^1 = L^1$.

Next we recall the inequalities from [3].

Lemma 1. *a) Let $f \in L_s^p$, $g \in L_s^1$ and $1 \leqslant p \leqslant +\infty$. Then*

$$\|f * g\|_{L_s^p} \leqslant \|f\|_{L_s^p} \|g\|_{L_s^1}. \tag{9}$$

b) If $f \in L_s^p$, $1 \leqslant p \leqslant +\infty$, and $g \in W_s^1$, then $f * g \in W_s^p$ and

$$\|f * g\|_{W_s^p} \leqslant \|f\|_{L_s^p} \|g\|_{W_s^1}. \tag{10}$$

c) If $c \in \ell_s^p$ and $d \in \ell_s^1$, then $c * d \in \ell_s^p$ and

$$\|c * d\|_{\ell_s^p} \leqslant \|c\|_{\ell_s^p} \|d\|_{\ell_s^1}. \tag{11}$$

For any sequence $c = \{c_i\}_{i\in\mathbb{N}} \in \ell_s^p$ and $f \in \mathcal{L}_s^p$, $1 \leqslant p \leqslant +\infty$, define, as in [4], their semi-convolution $f *' c$ by

$$(f *' c)(x) = \sum_{j\in\mathbb{Z}^d} c_j f(x-j), \quad x \in \mathbb{R}^d.$$

Lemma 2. *a) If $f \in W_s^p$, $1 \leqslant p \leqslant +\infty$, and $c \in \ell_s^1$, then the function $f *' c$ belongs to W_s^p and*

$$\|f *' c\|_{W_s^p} \leqslant \|c\|_{\ell_s^1} \|f\|_{W_s^p}, \tag{12}$$

and also if $f \in W_s^1$ and $c \in \ell_s^p$, $1 \leqslant p \leqslant +\infty$, then the function $f *' c$ belongs to W_s^p and

$$\|f *' c\|_{W_s^p} \leqslant \|c\|_{\ell_s^p} \|f\|_{W_s^1}. \tag{13}$$

b) If $f \in \mathcal{L}_s^p$ and $c \in \ell_s^1$, than $f *' c$ belongs to $f \in \mathcal{L}_s^p$ and

$$\|f *' c\|_{\mathcal{L}_s^p} \leqslant \|c\|_{\ell_s^1} \|f\|_{\mathcal{L}_s^p}. \tag{14}$$

c) $f *' \cdot$ is a continuous map from ℓ_s^p to L_s^p, and also from ℓ_s^1 to $\mathcal{L}_s^p$ if $f \in \mathcal{L}_s^p$, $1 \leqslant p \leqslant +\infty$.

We will give the proof of the next lemma since it is differently possed in [4].

Lemma 3. *Let $f \in L_s^p$ and $g \in W_s^1$, $1 \leqslant p \leqslant +\infty$, $s \geqslant 0$. Then the sequence $\Big\{ \int_{\mathbb{R}^d} f(x) g(x-j)\,dx \Big\}_{j\in\mathbb{Z}^d}$ belongs to ℓ_s^p and we have*

$$\Big\| \Big\{ \int_{\mathbb{R}^d} f(x) \overline{g(x-j)}\,dx \Big\}_{j\in\mathbb{Z}^d} \Big\|_{\ell_s^p} \leqslant \|f\|_{L_s^p} \|g\|_{W_s^1}. \tag{15}$$

Proof. Using inequality (11) for fixed $x \in \mathbb{R}^d$, we obtain

$$\Big\|\Big\{\int_{\mathbb{R}^d} f(x)\overline{g(x-j)}dx\Big\}_{j\in\mathbb{Z}^d}\Big\|_{\ell^p_s} = \Big(\sum_{j\in\mathbb{Z}^d}\Big|\int_{\mathbb{R}^d} f(x)\overline{g(x-j)}dx\Big|^p(1+|j|)^{sp}\Big)^{1/p}$$

$$\leqslant \Big(\sum_{j\in\mathbb{Z}^d}\Big(\int_{\mathbb{R}^d}|f(x)||g(x-j)|dx\Big)^p(1+|j|)^{sp}\Big)^{1/p}$$

$$= \Big(\sum_{j\in\mathbb{Z}^d}\Big(\int_{[0,1]^d}\sum_{k\in\mathbb{Z}^d}|f(x+k)||g(x+k-j)|dx\Big)^p(1+|j|)^{sp}\Big)^{1/p}$$

$$\leqslant \Big(\sum_{j\in\mathbb{Z}^d}\int_{[0,1]^d}\Big(\sum_{k\in\mathbb{Z}^d}|f(x+k)||g(x+k-j)|dx\Big)^p(1+|j|)^{sp}dx\Big)^{1/p}$$

$$= \Big(\sum_{j\in\mathbb{Z}^d}\int_{[0,1]^d}\Big(\sum_{k\in\mathbb{Z}^d}|f(x+k)||g(x+k-j)|(1+|k|)^s\Big)^p dx\Big)^{1/p}$$

$$\leqslant \Big(\int_{[0,1]^d}\sum_{j\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d}|f(x+k)||g(x+k-j)|(1+|k|)^s\Big)^p dx\Big)^{1/p}$$

$$\leqslant \Big(\int_{[0,1]^d}\sum_{j\in\mathbb{Z}^d}|f(x+j)|^p(1+|j|)^{sp}\Big(\sum_{k\in\mathbb{Z}^d}|g(x-k)|(1+|k|)^s\Big)^p dx\Big)^{1/p}$$

$$\leqslant \|f\|_{L^p_s}\Big(\sup_{x\in[0,1]^d}\Big(\sum_{k\in\mathbb{Z}^d}|g(x-k)|(1+|k|)^s\Big)^p\Big)^{1/p} \leqslant \|f\|_{L^p_s}\|g\|_{W^1_s}. \qquad \square$$

4. Main result

Our main result is related to Theorem 1 in [4].
Let $\varphi \in \mathcal{L}^p_s$, $1 \leqslant p \leqslant \infty$. We consider shift-invariant spaces of the form

$$V^p_s(\varphi) = \Big\{\sum_{j\in\mathbb{Z}^d} c_j\varphi(\cdot - j) \;\Big|\; c \in \ell^p_s\Big\}. \tag{16}$$

Note, if $s = 0$, then we have space $V^p(\varphi)$ considered in [4].

Theorem 1. *Let $\varphi \in \bigcap_{s\geqslant 0} W^1_s$. Then the following statements are equivalent to each other.*

i) $V^p_s(\varphi)$ is closed in L^p_s for all $s \geqslant 0$ and for all $1 \leqslant p \leqslant +\infty$.

ii) For all $s \geqslant 0$ and $1 \leqslant p \leqslant +\infty$, the family $\{\varphi(\cdot - j) \mid j \in \mathbb{Z}^d\}$ is a p-frame for $V^p_s(\varphi)$, i.e. there exist positive constants A_s, B_s (depending on φ and *s) such that*

(17)
$$A_s\|f\|_{L^p_s} \leqslant \Big\|\Big\{\int\limits_{\mathbb{R}^d} f(x)\overline{\varphi(x-j)}dx\Big\}_{j\in\mathbb{Z}^d}\Big\|_{\ell^p_s} \leqslant B_s\|f\|_{L^p_s}, \quad \forall f \in V^p_s(\varphi).$$

iii) There exist positive constants C_1 *and* C_2 *such that*

$$0 < C_1 \leqslant \sum_{j\in\mathbb{Z}^d} |\widehat{\varphi}(x+j)|^2 \leqslant C_2 < +\infty, \quad a.e.\ x \in \mathbb{R}^d. \tag{18}$$

iv) There exist positive constants K^1_s *and* K^2_s *(depending on* φ *and* s*) such that for all* $1 \leqslant p \leqslant +\infty$ *we have*

$$K^1_s\|f\|_{L^p_s} \leqslant \inf_{c\in M} \|c\|_{\ell^p_s} \leqslant K^2_s\|f\|_{L^p_s}, \quad \forall f \in V^p_s(\varphi),\ s \geqslant 0, \tag{19}$$

where

$$M = \Big\{c = \{c_k\}_{k\in\mathbb{Z}^d} \in \ell^p_s \mid f(\cdot) = \sum_{k\in\mathbb{Z}^d} c_k\varphi(\cdot - k)\Big\}. \tag{20}$$

v) There exists $\psi \in \bigcap\limits_{s\geqslant 0} W^1_s$ *such that*

(21)
$$f = \sum_{j\in\mathbb{Z}^d} \langle f, \psi(\cdot - j)\rangle\varphi(\cdot - j) = \sum_{j\in\mathbb{Z}^d} \langle f, \varphi(\cdot - j)\rangle\psi(\cdot - j), \quad \forall f \in V^p_s(\varphi).$$

Proof.

$v) \Rightarrow iv)$

Let $f = \sum\limits_{j\in\mathbb{Z}^d} \langle f, \psi(\cdot - j)\rangle\varphi(\cdot - j)$ and let M be given by (20). Using (15) we have
$$\inf_{c\in M} \|c\|_{\ell^p_s} \leqslant \Big\|\Big\{\int\limits_{\mathbb{R}^d} f(x)\overline{\psi(x-j)}dx\Big\}_{j\in\mathbb{Z}^d}\Big\|_{\ell^p_s} \leqslant \|f\|_{L^p_s}\|\psi\|_{W^1_s}.$$
For $K^2_s = \|\psi\|_{W^1_s}$ we have the right-hand side of the inequality.
Using (13), we have
$$\|f\|_{L^p_s} \leqslant \|f\|_{W^p_s} = \|\varphi *' c\|_{W^p_s} \leqslant \|\varphi\|_{W^1_s}\|c\|_{\ell^p_s},$$
and for $K^1_s = \frac{1}{\|\varphi\|_{W^1_s}}$ we prove the left-hand side of the inequality (19).

Assertions $v) \Rightarrow ii)$, $ii) \Leftrightarrow iv)$, and $iv) \Rightarrow i)$ are simple and their proofs are omitted.

$iii) \Rightarrow iv)$

We have already seen that for $\varphi \in W^1_s$ and $c \in \ell^p_s$, $1 \leqslant p \leqslant +\infty$, the inequality
$$\|\varphi *' c\|_{W^p_s} \leqslant \|c\|_{\ell^p_s}\|\varphi\|_{W^1_s},$$

holds. With $\|\varphi *' c\|_{L^p_s} \leqslant \|\varphi *' c\|_{W^p_s}$ for all $1 \leqslant p \leqslant +\infty$, and $K^1_s = \|\varphi\|^{-1}_{W^1_s}$, we have that the left side of the inequality (17).

The family $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ with the condition (18) is a Riesz basis of $V^2(\varphi)$ (see [3]), so there exists a unique function $\psi \in V^2(\varphi)$ such that $\{\psi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ is also a Riesz basis for $V^2(\varphi)$ and such that it satisfies the biorthogonality relations

$$\langle \psi(x), \varphi(x)\rangle = 1, \quad \langle \psi(x), \varphi(x-k)\rangle = 0, \;\; k \neq 0.$$

Theorem 2.3 in [3] says that if $\varphi \in W^1_s$ and the family $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ is a Riesz basis for $V^2(\varphi)$, then the dual generator ψ is in W^1_s. Since we have that $\varphi \in W^1_s$ for all $s \geqslant 0$, then we have that $\psi \in \bigcap\limits_{s\geqslant 0} W^1_s$. Since

$$(\varphi *' c)(x) = \sum_{k\in\mathbb{Z}^d} c_k \varphi(x-k) \in V^p_s(\varphi),$$

then c_k, $k \in \mathbb{Z}^d$, can be expressed in the form

$$c_k = \int\limits_{\mathbb{R}^d} (\varphi *' c)(x)\overline{\psi(x-k)}dx.$$

For $1 \leqslant p \leqslant +\infty$ (with usual changes for $p = \infty$), we have

$$\begin{aligned}
|c_k(1+|k|)^s|^p &= \Big|\int\limits_{\mathbb{R}^d} (\varphi *' c)(x)\overline{\psi(x-k)}(1+|k|)^s dx\Big|^p \\
&\leqslant \Big(\int\limits_{[0,1]^d} \sum_{j\in\mathbb{Z}^d} |\varphi *' c|(x+j)|\psi(x+j-k)|(1+|k|)^s dx\Big)^p \\
&\leqslant \int\limits_{[0,1]^d} \Big(\sum_{j\in\mathbb{Z}^d} |\varphi *' c|(|\psi(x+j-k)|(1+|k|)^s\Big)^p dx.
\end{aligned}$$

We sum over $k \in \mathbb{Z}^d$ and obtain

$$\sum_{k\in\mathbb{Z}^d} |c_k|^p(1+|k|)^{sp} \leqslant \int\limits_{[0,1]^d} \sum_{j\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} |\varphi *' c|(x+j)|\psi(x+j-k)|(1+|k|)^s\Big)^p dx$$

$$\leqslant \int\limits_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} |\varphi *' c|^p(x+k)|(1+|k|)^{sp}\Big(\sum_{k\in\mathbb{Z}^d} |\psi(x+k)|(1+|k|)^s\Big)^p dx$$

$$\leqslant \|\psi\|^p_{W^1_s}\|\varphi *' c\|^p_{L^p_s}.$$

It follows

$$\|c\|_{\ell^p_s} \leqslant \|\psi\|_{W^1_s}\|\varphi *' c\|_{L^p_s}.$$

For the lower bound in the inequality (19) one may choose $K^2_s = \|\psi\|_{W^1_s}$. Finally,

$$\|c\|_{\ell^p_s} \leqslant K^2_s\|f\|_{L^p_s}.$$

$i) \Rightarrow iii)$

Since $V_s^p(\varphi)$ is closed in L_s^p for all $1 \leqslant p \leqslant +\infty$, $s \geqslant 0$, then for $p = 2$ and $s = 1$ we have the standard assumption on the generator φ, i.e. there exist two constants C_1 and C_2 such that

$$0 < C_1 \leqslant \sum_{j\in\mathbb{Z}^d} |\widehat{\varphi}(x+j)|^2 \leqslant C_2 < +\infty, \quad \text{a.e. } x \in \mathbb{R}^d.$$

□

Corollary 1. *Let* $\varphi \in \bigcap_{s\geqslant 0} W_s^1$. *Then* $V_s^p(\varphi) \subset V_s^q(\varphi)$, *for all* $1 \leqslant p \leqslant q \leqslant +\infty$ *and* $s \geqslant 0$.

Proof. Let $f(x) = \sum_{k\in\mathbb{Z}^d} c_k\varphi(x-k)$, for some $c = \{c_k\}_{k\in\mathbb{Z}^d} \in \ell_s^p$, $1 \leqslant p \leqslant +\infty$. Since $\ell_s^p \subset \ell_s^q$, $1 \leqslant p \leqslant q \leqslant +\infty$, Theorem 1 implies the inequalities

$$\|f\|_{L_s^q} \leqslant B_s\|c\|_{\ell_s^q} \leqslant B_s'\|c\|_{\ell_s^p} \leqslant \|f\|_{L_s^p}, \quad \forall s \geqslant 0,\ 1 \leqslant p \leqslant q \leqslant +\infty.$$

□

Remark 1. *From the inequalities* (19) *and* (17) *we can conclude that* ℓ_s^p *and* V_s^p *are isomorphic Banach spaces for all* $s \geqslant 0$ *and* $1 \leqslant p \leqslant +\infty$, *and for* $f \in V_s^p(\varphi)$ *we have the equivalence between* $\inf_{c\in M}\{\|c\|_{\ell_s^p}\}$ *and the* L_s^p*-norm of* f.

As a consequence of Theorem 1 and from [3, Theorem 1], and since $\ell_{s_1}^p \subset \ell_{s_2}^p$, for $0 \leqslant s_2 \leqslant s_1$, we have the following corollary.

Corollary 2. *Let* $\varphi \in \bigcap_{s\geqslant 0} W_s^1$. *Then* $V_{s_1}^p(\varphi) \subset V_{s_2}^p(\varphi)$ *for* $0 \leqslant s_2 \leqslant s_1$ *and every* $1 \leqslant p \leqslant +\infty$.

We construct Fréchet spaces $X_{F,p}$, $p \geqslant 1$, as the intersection of translator invariant spaces $V_s^p(\varphi)$, $s \in \mathbb{N}$. Note that, for $1 \leqslant p \leqslant +\infty$,

$$\{\mathbf{0}\} \neq \bigcap_{s\in\mathbb{N}_0} V_s^p(\varphi) \subseteq \cdots \subseteq V_2^p(\varphi) \subseteq V_1^p(\varphi) \subseteq V_0^p(\varphi) = V^p(\varphi).$$

Also, we have that $X_{F,p} = \bigcap_{s\in\mathbb{N}_0} V_s^p(\varphi)$ is dense in $V_s^p(\varphi)$ for all $s \in \mathbb{N}_0$. The corresponding sequence space $Q_{F,p}$, $p \geqslant 1$, is the intersection of the weighted sequence space ℓ_s^p, $s \in \mathbb{N}_0$. Note that $\bigcap_{s\in\mathbb{N}_0} \ell_s^p$, for every $p \geqslant 1$, is actually the space of rapidly decreasing sequences s. We proved that if $\varphi \in W_s^1$, then a sequence $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ is a p-frame for $V_s^p(\varphi)$ as well as $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ is an r-frame for $V_s^r(\varphi)$, for all $1 \leqslant r \leqslant +\infty$. So we have that the definition of $X_{F,p}$ does not depend on $p \geqslant 1$, so $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ is a pre-F-frame for $X_{F,p}$ as well as that $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ is a pre-F-frame for $X_{F,r}$, for all $1 \leqslant r \leqslant +\infty$.

Since the corresponding function space for s is the space of rapidly increasing functions

$$\mathcal{S} = \{f \mid \|f\|_m = \sup_{n \leqslant m} (1 + |x|^2)^{m/2} |f^{(n)}(x)| < +\infty\},$$

and its dual is $\mathcal{S}'$- the space of slowly decreasing distributions, we obtain that dual space of Fréchet space $X_F = X_{F,p}$, for any p, is isomorphic to (a complemented subspace of) the space $\mathcal{S}'$.

Denote by $\mathcal{P}(-\pi, \pi)$ the space of smooth 2π-periodic functions with the family of norms $|\theta|_k = \sup\{|\theta^{(k)}(t)|; t \in (-\pi, \pi)\}$, $k \in \mathbb{N}_0$. It is a Fréchet space and its dual is the space of 2π-periodic tempered distributions. Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transformation and its inverse transformation, respectively. We have

Theorem 2. *Let* $\varphi \in \bigcap_{s \geqslant 0} W_s^1$ *and* $X_F = \bigcap_{s \in \mathbb{N}_0} V_s^p(\varphi)$ *for some* $1 \leqslant p \leqslant +\infty$. *Then*

$$X_F = \mathcal{F}^{-1}(\hat{\varphi} \cdot \mathcal{P}(-\pi, \pi)),$$

in the topological sense.

Proof. For $f \in X_F$ we have $f = \sum_{j \in \mathbb{Z}^d} c_j \varphi(\cdot - j)$, for some sequence $c = \{c_j\}_{j \in \mathbb{Z}^d} \in s$. Then

$$\widehat{f} = \sum_{j \in \mathbb{Z}^d} \widehat{c_j \varphi(\cdot - j)} = \Big(\sum_{j \in \mathbb{Z}^d} c_j e^{ij\cdot} \Big) \widehat{\varphi}.$$

This implies the assertion. □

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