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ON THE EXISTENCE OF NEARLY QUASI-EINSTEIN MANIFOLDS

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Abstract. The objective of the present paper is to establish the existence of nearly quasi-Einstein manifolds.

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1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = dimM \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n}g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([1], p. 432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], p. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation :

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1.2}$$

where $a, b \in \mathbb{R}$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \tag{1.3}$$

for all vector fields X. Moreover, different structures on Einstein manifolds have also been studied by several authors. In 1993 Tamassy and Binh [2] studied weakly symmetric structures on Einstein manifolds.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold [3] if its Ricci tensor S of type $(0, 2)$ is not identically zero

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and satisfies the condition (1.2). We shall call A the associated 1-form and U is called the generator of the manifold.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime is quasi-Einstein manifold [4]. Also quasi-Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [5]. So quasi-Einstein manifolds have some importance in the general theory of relativity.

It is to be noted that M. C. Chaki and R. K. Maity [6] also introduced the notion of quasi-Einstein manifolds which is different from that of R. Deszcz [3]. They took a and b as scalars and the generator U of the manifold as a unit vector field.

The notion of quasi-Einstein manifolds have been generalized by many authors in several ways such as generalized quasi-Einstein manifolds [7], [8].

In a recent paper [9], the authors introduced the notion of nearly quasi-Einstein manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a nearly quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bE(X,Y), \tag{1.4}$$

where a and b are non-zero scalars and E is a non-zero symmetric tensor of type $(0, 2)$. An n-dimensional nearly quasi-Einstein manifold was denoted by $N(QE)_n$. We shall call E the associated $(0, 2)$ tensor and a and b as associated scalars.

Remark 1. *It is known ([10], p.39) that the outer product of two covariant vectors is a covariant tensor of type (0,2) but the converse is not true, in general. Hence the manifolds which are quasi-Einstein are also nearly quasi-Einstein, but the converse is not true, in general. For this the name, nearly quasi-Einstein, was chosen.*

A concrete example of a nearly quasi-Einstein manifold was also given in [9] by the following theorem:

Theorem A. *Let* $(\mathbb{R}^4, g)$ *be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2,$$

$(i, j = 1, 2, 3, 4)$. *Then* $(\mathbb{R}^4, g)$ *is a* $N(QE)_4$ *with non-zero and non-constant scalar curvature which is not a quasi-Einstein manifold.*

In this paper we like to introduce another notion which generalizes the notion of a manifold of quasi-constant curvature [11]. A Riemannian manifold is called a manifold of quasi-constant curvature, if it is conformally flat and the curvature tensor $'R$ of type (0, 4) satisfies the condition

$$\begin{aligned} 'R(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &+ g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \end{aligned} \tag{1.5}$$

where $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$, R is the curvature tensor of type (1, 3), p, q are scalar functions and λ is a unit vector field defined by

$$g(X,\ \lambda) = T(X).$$

It can be easily seen that if the curvature tensor $'R$ is of the form (1.5), then the manifold is conformally flat. On the other hand, Gh. Vranceanu [12] defined the notion of almost constant curvature by the same expression (1.5). Later A. L. Mocanu [13] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Gh. Vranceanu are the same. Hence a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $'R$ satisfies the relation (1.5). If in (1.5) $q = 0$, then the manifold reduces to a manifold of constant curvature.

A Riemannian manifold is said to be a manifold of nearly quasi-constant curvature, if the curvature tensor $'R$ of type (0, 4) satisfies the condition

$$\begin{aligned} 'R(X, Y, Z, W) &= p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &+ q[g(X, W)B(Y, Z) - g(X, Z)B(Y, W) \\ &+ g(Y, Z)B(X, W) - g(Y, W)B(X, Z)], \end{aligned} \tag{1.6}$$

where $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$, R is the curvature tensor of type (1, 3), p, q are scalar functions and B is a non-zero symmetric tensor of type $(0, 2)$. An n-dimensional Riemannian manifold of nearly quasi-constant curvature shall be denoted by $N(QC)_n$. The name nearly quasi-constant curvature is chosen for the same reason as in **Remark 1**.

In 1956 S. S. Chern [14] studied a type of Riemannian manifold whose curvature tensor $'R$ of type $(0, 4)$ satisfies the condition

$$'R(X, Y, Z, W) = F(X, Z)F(Y, W) - F(Y, Z)F(X, W), \tag{1.7}$$

where F is a non-zero symmetric tensor of type $(0, 2)$. Such an n-dimensional manifold was called a special manifold with the associated symmetric tensor F and was denoted by $\psi(F)_n$.

Such a manifold is important for the following reasons:

Firstly, for possessing some remarkable properties relating to curvature and characterstic classes and secondly, for containing a manifold of quasi-constant curvature [11] as a subclass.

The paper is organized as follows:

Section 2 contains the proof of the theorem for the existence of a $N(QE)_n$. In section 3 we prove that a nearly quasi-umbilical hypersurface of a special manifold, $\psi(F)_n$, is a $N(QC)_n$. Finally, we have studied the relations between a $N(QC)_n$ and a $N(QE)_n$.

2. Existence Theorem of a $N(QE)_n$

In this section we prove the following theorem:

Theorem 2.1. *If the non-zero Ricci tensor S of a Riemannian manifold with non-zero scalar curvature satisfies the relation*

$$\begin{aligned} S(Y,Z)S(X,W) - S(X,Z)S(Y,W) \\ = \mu[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \end{aligned} \tag{2.1}$$

where μ is a non-zero scalar, then the manifold is a nearly quasi-Eistein manifold.

Proof. Contracting X and W in (2.1) we get

$$rS(Y,Z) - g(Q^2Y,Z) = \mu(n-1)g(Y,Z) \tag{2.2}$$

where Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$. Now, since, $r \neq 0$ we get from (2.2) that

$$S(Y,Z) = \frac{\mu(n-1)}{r}g(Y,Z) + \frac{1}{r}E(Y,Z) \tag{2.3}$$

where E is a $(0,2)$ type non-zero symmetric tensor defined $E(Y,Z) = g(Q^2Y,Z)$, which shows that the manifold is a $N(QE)_n$. □

3. Existence of a Manifold of nearly Quasi-Constant Curvature

In this section we prove the following.

Theorem 3.1. *A nearly quasi-umbilical hypersurface of a manifold of special curvature $\psi(F)_n$ is a manifold of nearly quasi-constant curvature.*

Proof. Let $(M^{n-1},\tilde{g})$ be a hypersurface of (M^n,g). If A is the (1,1) tensor corresponding to the normal valued second fundamental tensor H, then we have ([15], p.41)

$$\tilde{g}(A_\xi(X),Y) = g(H(X,Y),\xi) \tag{3.1}$$

where ξ is the unit normal vector field and X, Y are tangent vector fields.

Let H_ξ be the symmetric (0,2) tensor associated with A_ξ in the hypersurface defined by

$$\tilde{g}(A_\xi(X),Y) = H_\xi(X,Y). \tag{3.2}$$

A hypersurface of a Riemannian manifold (M^n,g) shall be called nearly quasi-umbilical if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha g(X,Y) + F(X,Y) \tag{3.3}$$

where F is a symmetric (0,2) tensor and α is a scalar. If $\alpha = 0$ (*resp.* $F = 0$ *or* $\alpha = F = 0$) holds, then it is called nearly cylindrical (resp. umbilical or geodesic). The name nearly quasi-umbilical is chosen for the same reason as in **Remark 1**.

Now from (3.1), (3.2) and (3.3) we obtain

$$g(H(X,Y),\xi) = \alpha g(X,Y)g(\xi,\xi) + F(X,Y)g(\xi,\xi)$$

which implies that

$$H(X,Y) = \alpha g(X,Y)\xi + F(X,Y)\xi, \tag{3.4}$$

since ξ is the only unit normal vector field.

We have the following equation of Gauss ([15], p.45) for any vector fields X, Y, Z, W tangent to the hypersurface

$$\begin{aligned} g(R(X,Y)Z,W) &= \tilde{g}(\tilde{R}(X,Y)Z,W) - g(H(X,W),H(Y,Z)) \\ &+ g(H(Y,W),H(X,Z)) \end{aligned} \tag{3.5}$$

where $\tilde{R}$ is the curvature tensor of the hypersurface.

Let us assume that the hypersurface is nearly quasi-umbilical. Then from (3.4) and (3.5) it follows that

$$\begin{aligned} & g(R(X,Y)Z,W) = \tilde{g}(\tilde{R}(X,Y)Z,W) + \alpha^2[g(Y,W)g(X,Z) \\ & -g(X,W)g(Y,Z)] + \alpha[g(Y,W)F(X,Z) + g(X,Z)F(Y,W) \\ & -g(X,W)F(Y,Z) - g(Y,Z)F(X,W)] \\ & +[F(Y,W)F(X,Z) - F(X,W)F(Y,Z)]. \end{aligned} \tag{3.6}$$

Since $g(R(X,Y)Z,W) = {'R}(X,Y,Z,W)$, using (1.7) in (3.6) we have

$$\begin{aligned} \tilde{g}(\tilde{R}(X,Y)Z,W) &= \alpha^2[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)] \\ &+\alpha[g(X,W)F(Y,Z) + g(Y,Z)F(X,W) \\ &-g(Y,W)F(X,Z) - g(X,Z)F(Y,W)]. \end{aligned} \tag{3.7}$$

Hence the nearly quasi-umbilical hypersurface of a manifold of special curvature $\psi(F)_n$ is a manifold of nearly quasi-constant curvature. □

4. Relations Between $N(QC)_n$ and $N(QE)_n$

Theorem 4.1. *A manifold of nearly quasi-constant curvature is a nearly quasi-Einstein manifold.*

Proof. Putting $X = W = e_i$ in (1.6) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$S(Y,Z) = [p(n-1) + q\tilde{B}]g(Y,Z) + q(n-2)B(Y,Z) \tag{4.1}$$

where $\tilde{B}$ is the trace of B. Hence the manifold is a nearly quasi-Einstein manifold. □

From **Theorem**3.1 and **Theorem**4.1 we can state the following proposition:

Proposition 1. *A nearly quasi-umbilical hypersurface of a manifold of special curvature* $\psi(F)_n$ *is a manifold of nearly quasi-Einstein manifold.*

Now contracting (4.1) with respect to Y, *and* Z we get

$$r = n(n-1)p + 2(n-1)q\tilde{B}\,. \tag{4.2}$$

In a Riemannian manifold (M^n, g) $(n > 3)$ the conformal curvature tensor $'C$ of type $(0,4)$ has the following form:

$$\begin{aligned} 'C(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) \\ -S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \\ +\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{4.3}$$

Using (1.6), (4.1) and (4.2) in (4.3) we see that

$$'C(X,Y,Z,W) = 0,$$

that is, the manifold under consideration is conformally flat. Hence we can state the following:

Remark 2. *Every* $N(QC)_n$ $(n > 3)$ *is a conformally flat* $N(QE)_n$.

In this section we have proved that every $N(QC)_n$ $(n > 3)$ is a conformally flat $N(QE)_n$. Now we shall prove that the converse is also true, that is, every conformally flat $N(QE)_n$ $(n > 3)$ is a $N(QC)_n$.

Theorem 4.2. *Every conformally flat* $N(QE)_n$ $(n > 3)$ *is a manifold of nearly quasi-constant curvature.*

Proof. Since the manifold is conformally flat, we have

$$\begin{aligned} 'R(X,Y,Z,W) &= \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \\ &+ \frac{r}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \end{aligned} \tag{4.4}$$

Using (1.4) we have

$$\begin{aligned} 'R(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ q[g(X,W)E(Y,Z) - g(X,Z)E(Y,W) \\ &+ g(Y,Z)E(X,W) - g(Y,W)E(X,Z)] \end{aligned} \tag{4.5}$$

where $p = -\frac{a+b\tilde{E}}{(n-1)(n-2)}$ and $q = \frac{b}{n-2}$. Here $\tilde{E}$ is the trce of E. This shows that the manifold is a manifold of nearly quasi-constant curvature. □

Remark 3. *If the dimension of a* $N(QE)_n$ *is three, then the conformal curvature tensor vanishes identically and such a three dimensional* $N(QE)_3$ *is a manifold of nearly quasi-constant curvature.*

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