Novi Sad J. Math.
Vol. 39, No. 2, 2009, 103-109

SOME REMARKS ON sn-METRIZABLE SPACES[1]

Xun Ge[2], Jinjin Li[3]

Abstract. This paper shows that sn-metrizable spaces can not be characterized as sequence-covering, compact, σ-images of metric spaces or sn-open, π, σ-images of metric spaces. Also, a space with a locally countable sn-network need not to be an sn-metrizable space. These results correct some errors on sn-metrizable spaces.

AMS Mathematics Subject Classification (2000): 54C05, 54E35, 54E40

Key words and phrases: sn-metrizable space, sequence-covering mapping, sn-network, sn-open mapping, compact-mapping, π-mapping, σ-mapping

1. Introduction

sn-metrizable spaces are a class of important generalized metric spaces between g-metrizable spaces and $\aleph$-spaces [9]. To characterize sn-metrizable spaces by certain images of metric spaces is an interesting question, and many "nice" characterizations of sn-metrizable spaces have been obtained [3, 8, 9, 10, 11, 14, 15, 21, 24]. In the past years, the following results were given.

Proposition 1. *Let X be a space.*

(1) X is an sn-metrizable space iff X is a sequence-covering, compact, σ-image of a metric space [21, Theorem 3.2].

(2) X is an sn-metrizable space iff X is an sn-open, π, σ-image of a metric space [14, Theorem 2.7].

(3) If X has a locally countable sn-network, then X is an sn-metrizable space [21, Theorem 4.3].

Unfortunately, Proposition 1.1 is not true. In this paper, we give some examples to show that sn-metrizable spaces can not be characterized as sequence-covering, compact, σ-images of metric spaces or sn-open, π, σ-images of metric spaces, and a space with a locally countable sn-network need not to be an sn-metrizable space. These results correct Proposition 1.1.

Throughout this paper, all spaces are assumed to be regular T_1 and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers and ω_1 denotes the first uncountable ordinal. $\{x_n\}$ denotes a sequence, where the n-th term is x_n. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging

[1]This project is supported by NSFC(No. 10971185 and 10971186)

[2]College of Zhangjiagang, Jiangsu University of Science and Technology, Zhangjiagang 215600, P. R. China, e-mail: zhugexun@163.com

[3]Department of Mathematics, Zhangzhou Teachers College, Zhangzhou 363000, P. R. China, e-mail: jinjinli@fjzs.edu.cn

to x in X is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of X and $x \in X$. Then $(\mathcal{P})_x$ denotes the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$. For terms which are not defined here, we refer to [4].

2. Definitions and Remarks

Definition 2.1. *[5]. Let X be a space.*

(1) $P \subset X$ is called a sequential neighborhood of x in X, if each sequence $\{x_n\}$ converging to x is eventually in P.

(2) A subset U of X is called sequentially open if U is a sequential neighborhood of each of its points.

(3) X is called a sequential space if each sequential open subset of X is open in X.

(4) X is called a k-space if for each $A \subset X$, A is closed in X iff $A \bigcap K$ is closed in K for each compact subset K of X.

Remark 2.1. *(1) P is a sequential neighborhood of x iff each sequence $\{x_n\}$ converging to x is frequently in P.*

(2) The intersection of finite many sequential neighborhoods of x is a sequential neighborhood of x.

(3) Sequential spaces $\Longrightarrow$ k-spaces.

Definition 2.2. *Let $\mathcal{P}$ be a cover of a space X.*

(1) $\mathcal{P}$ is called a network for X [1], if whenever $x \in U$ with U open in X, there is $P \in \mathcal{P}$ such that $x \in P \subset U$.

(2) $\mathcal{P}$ is called a k-network of X [22], if whenever $K \subset U$ with K compact in X and U open in X, there is a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup\{F : F \in \mathcal{F}\} \subset U$.

(3) $\mathcal{P}$ is called a cs-network of X [13], if each convergent sequence S converging to a point $x \in U$ with U open in X, then S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

Definition 2.3. *Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ is called an sn-network for X [9], if for each $x \in X$, the following hold.*

(1) $\mathcal{P}_x$ is a network at x in X, i.e., whenever $x \in U$ with U open in X, there is $P \in \mathcal{P}_x$ such that $x \in P \subset U$.

(2) $U, V \in \mathcal{P}_x$, then $W \subset U \bigcap V$ for some $W \in \mathcal{P}_x$.

(3) Each element of $\mathcal{P}_x$ is a sequential neighborhood of x.

Here, we call $\mathcal{P}_x$ is an sn-network at x in X for each $x \in X$.

Definition 2.4. *Let X be a space.*

(1) X is called an sn-metrizable space [9] if X has a σ-locally finite sn-network.

(2) X is called sn-first countable [9], if for each $x \in X$, there is a countable sn-network at x in X.

(3) X is called sn-second countable [7], if X has a countable sn-network.

(4) X is called an $\aleph$-space [22], if X has a σ-locally finite k-network.
(5) X is called an $\aleph_0$-space [22], if X has a countable k-network.

Remark 2.2. *The following implications hold.*

$$\begin{array}{ccccc} sn\text{-second countable space} & \Longrightarrow & sn\text{-metrizable space} & \Longrightarrow & sn\text{-first countable space} \\ \Downarrow & & \Downarrow & & \\ \aleph_0\text{-space} & \Longrightarrow & \aleph\text{-space} & & \end{array}$$

Definition 2.5. *Let X be a set and d be a non-negative real valued function defined on $X \times X$ such that $d(x, y) = 0$ iff $x = y$, and $d(x, y) = d(y, x)$ for all $x, y \in X$. d is abbreviated to be called a d-function on X.*

Definition 2.6. *Let d be a d-function on a space X. For each $x \in X, n \in \mathbb{N}$, put $S_n(x) = \{y \in X : d(x, y) < 1/n\}$. A space (X,d) is called a symmetric space [23], if $F \subset X$ is closed in X iff for each $x \notin F$, $S_n(x) \bigcap F = \emptyset$ for some $n \in \mathbb{N}$.*

Definition 2.7. *[23]. Let (X, d) be a symmetric space.*

(1) A sequence $\{x_n\}$ in X is called Cauchy if for any $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > k$.

(2) X is called a Cauchy symmetric space if each convergent sequence in X is Cauchy.

Definition 2.8. *Let $f : X \longrightarrow Y$ be a mapping.*

(1) f is called a 1-sequence-covering mapping [19], if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(2) f is called a sequence-covering mapping [19], if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$.

(3) f is called a sequentially-quotient mapping [2], if whenever S is a convergent sequence in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(4) f is called an sn-open mapping [14] if there is an sn-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y satisfying the condition: for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x, $P \subset f(U)$ for some $P \in \mathcal{P}_y$.

(5) f is called a σ-mapping [20], if there is a base $\mathcal{B}$ of X such that $f(\mathcal{B})$ is σ-locally-finite in Y.

(6) f is called a quotient mapping [4] if U is open in Y iff $f^{-1}(U)$ is open in X.

(7) f is called a compact mapping [19] if $f^{-1}(y)$ is a compact subset of X for each $y \in Y$.

(8) f is called a π-mapping if X is a metric space with a metric d and for each $y \in U$ with U open in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$.

Remark 2.3. *(1) "sn-open mapping" in Definition 2.8(4) is called "almost sn-open mapping" in [12]. By [12, Proposition 2.13], a mapping f from a metric space is an sn-open mapping iff f is a 1-sequence-covering mappings.*

(2) Quotient maps preserve sequential spaces [5].

(3) Quotient mappings from sequential spaces are sequentially-quotient [2].

(4) Sequentially-quotient mappings onto sequential spaces are quotient [2].

(5) Each compact mapping from a metric space is a π-mapping [12, Remark 1.4(1)].

3. The Main Results

Lemma 3.1. *[11, Theorem 2.7]. A space X is sn-second countable iff X is a sequentially-quotient, compact image of a separable metric space.*

Lemma 3.2. *Let X be a sequential space. Then the following are equivalent.*

(1) X is a Cauchy symmetric space.

(2) X is a quotient, sequence-covering, π-image of a metric space.

(3) X is a sequence-covering, π-image of a metric space.

Proof. (1) $\Longleftrightarrow$ (2): It holds by [23, Theorem 2.3].

(2) $\Longrightarrow$ (3): It is clear.

(3) $\Longrightarrow$ (2): It holds by Remark 2.3(4).

Example 3.1. *There is an sn-metrizable space which is not any sequence-covering, π-image of a metric space.*

Proof. For each $n \in \mathbb{N}$, put $\mathcal{C}_n$ be a convergent sequence containing its limit point p_n, where $C_n \bigcap C_m = \emptyset$ if $n \neq m$. Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be the set of all rational numbers of real line $\mathbb{R}$. Put $M = (\oplus\{C_n : n \in \mathbb{N}\}) \oplus \mathbb{R}$, and let X be the quotient space obtained from M by identifying each p_n in C_n with q_n in $\mathbb{R}$. Then we have the following two facts by [23, Example 2.14(3)] and [17, Example 3.1.13(2)].

Fact 1. X is a quotient, compact image of a separable metric space.

Fact 2. X is not a Cauchy symmetric space.

(1) X is an *sn*-metrizable space: By Fact 1, X is a quotient, compact image of a separable metric space. So X is a Sequentially-quotient, compact image of a separable metric space by Remark 2.3(3), hence X is *sn*-second countable by Lemma 3.1. It follows that X is an *sn*-metrizable space by Remark 2.2.

(2) X is not any sequence-covering, π-image of a metric space: By Fact 2, X is not a Cauchy symmetric space. So X is not any sequence-covering, π-image of a metric space by Lemma 3.2

Remark 3.1. *(1) By Remark 2.3(5), the space X in Example 3.1 is an sn-metrizable space which is not any sequence-covering, compact, σ-image of a metric space.*

(2) It is clear that 1-sequence-covering mappings are sequence-covering mappings. So sn-open mappings defined on metric spaces are sequence-covering mappings by Remark 2.3(1). Thus, the space X in Example 3.1 is an sn-metrizable space which is not any sn-open, π, σ-image of a metric space.

Lemma 3.3. *[18, Theorem 2.8.6]. A space X has a locally countable k-network iff X has a locally countable cs-network.*

Lemma 3.4. *Let X be an sn-first countable space. Then X has a locally countable sn-network iff X has a locally countable cs-network.*

Proof. Necessity: It is clear because each sn-network of X is a cs-network of X.

Sufficiency: Let $\mathcal{P}$ be a locally countable cs-network of X. Without loss of generality, we can assume that $\mathcal{P}$ is closed under finite intersections. For each $x \in X$, let $\{B_n(x) : n \in \mathbb{N}\}$ be a countable sn-network at x in X, and let $\mathcal{P}_x = \{P \in \mathcal{P} : B_n(x) \subset P \ for\ some\ n \in \mathbb{N}\}$. Then each element of $\mathcal{P}_x$ is a sequential neighborhood of X. Put $\mathcal{P}' = \bigcup\{\mathcal{P}_x : x \in X\}$, then $\mathcal{P}' \subset \mathcal{P}$ is locally countable. It suffices to prove that $\mathcal{P}_x$ is a network at x in X for each $x \in X$. If not, there is an open neighborhood U of x such that $P \not\subset U$ for each $P \in \mathcal{P}_x$. Let $\{P \in \mathcal{P} : x \in P \subset U\} = \{P_m(x) : m \in \mathbb{N}\}$. Then $B_n(x) \not\subset P_m(x)$ for each $n, m \in \mathbb{N}$. Choose $x_{n,m} \in B_n(x) - P_m(x)$. For $n \geq m$, let $x_{n,m} = y_k$, where $k = m + n(n-1)/2$. Then the sequence $\{y_k : k \in \mathbb{N}\}$ converges to x. Thus, there is $m, i \in \mathbb{N}$ such that $\{y_k : k \geq i\} \bigcup \{x\} \subset P_m(x) \subset U$. Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P_m(x)$. This is a contradiction.

Example 3.2. *There is a space X with a locally countable sn-network such that X is not an sn-metrizable space.*

Proof. Let $X = \omega_1 \bigcup (\omega_1 \times \{1/n : n \in \mathbb{N}\})$. Define a neighborhood base $\mathcal{B}_x$ for each $x \in X$ for the desired topology on X as follows.

(1) If $x \in X - \omega_1$, then $\mathcal{B}_x = \{\{x\}\}$.

(2) If $x \in \omega_1$, then $\mathcal{B}_x = \{\{x\} \bigcup (\bigcup\{V(n,x) \times \{1/n\} : n \geq m\}) : m \in \mathbb{N}\ and\ V(n,x)\ is\ a\ neighborhood\ of\ x\ in\ \omega_1\ with\ the\ order\ topology\}$.

By [16, Example 1], X has a locally countable k-network and X is not an $\aleph$-space. By Lemma 3.3 and Remark 2.2, X has a locally countable cs-network and X is not an sn-metrizable space. It suffices to prove that X is sn-first countable by Lemma 3.4.

Let $x \in X$. If $x \in X - \omega_1$, then $\{\{x\}\}$ is a countable sn-network at x in X. If $x \in \omega_1$, put $\mathcal{P}_x = \{P_{x,m} : m \in \mathbb{N}\}$, where $P_{x,m} = \{x\} \bigcup \{(x, 1/n) : n \geq m\}$. Then $\mathcal{P}_x$ is a countable network at x in X. We only need to prove that each $P_{x,m}$ is a sequential neighborhood of x.

Let $\{x_n\}$ be a sequence converging to x. Put $K = \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$, then K is a compact subset of X. By [16, Example 1], we have the following facts.

Fact 1. $K \bigcap \omega_1$ is finite.

Fact 2. $K - \bigcup\{\{y\} \bigcup \{(y, 1/n) : n \in \mathbb{N}\} : y \in K \bigcap \omega_1\}$ is finite.

Case 1. If there is $y \in K \bigcap \omega_1$ such that $y = x_n$ for infinite many $n \in \mathbb{N}$, i.e., there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $y = x_{n_k}$ for each $k \in \mathbb{N}$, then $y = x$, So $\{x_n\}$ is frequently in $P_{x,m}$.

Case 2. If Case 1 does not hold, without loss of the generality, we may assume $K \bigcap \omega_1 = \{x\}$ by Fact 1. By Fact 2, $K - \{x\} \bigcup \{(x, 1/n) : n \in \mathbb{N}\}$ is finite. If there is $y \in K - \{x\} \bigcup \{(x, 1/n) : n \in \mathbb{N}\}$ such that $y = x_n$ for infinite many $n \in \mathbb{N}$, then $\{x_n\}$ is frequently in $P_{x,m}$ by a similar way in the proof of Case 1. Conversely, there is $k_0 \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : n \geq k_0\} \subset \{x\} \bigcup \{(x, 1/n) : n \in \mathbb{N}\}$. So $\{x_n\}$ is eventually in $P_{x,m}$.

By the above Case 1 and Case 2, $P_{x,m}$ is a sequential neighborhood of x by Remark 2.1(1).

As a further investigation, we give for Example 3.2 the following result.

Lemma 3.5. *[16, Theorem 1]. A k-space with a locally countable k-network is a topological sum of $\aleph_0$-spaces.*

Lemma 3.6. *[11, Theorem 2.1]. A space X is sn-second countable iff X is an sn-first countable, $\aleph_0$-space.*

Proposition 2. *If X is a k-space X with a locally countable sn-network, then X is an sn-metrizable space.*

Proof. Let X be a k-space with a locally countable sn-network. Then X is sn-first countable. So X has a locally countable cs-network by Lemma 3.4, hence X has a locally countable k-network by Lemma 3.3. By Lemma 3.5, X is a topological sum of $\aleph_0$-spaces. Put $X = \oplus\{X_\alpha : \alpha \in \Gamma\}$, where each X_α is an $\aleph_0$-space. Since sn-first countability is hereditary to subspace, each X_α is sn-second countable by Lemma 3.6. For each $\alpha \in \Gamma$, let $\{P_{\alpha,n} : n \in \mathbb{N}\}$ be a countable sn-network of X_α. Put $\mathcal{P}_n = \{P_{\alpha,n} : \alpha \in \Gamma\}$ for each $n \in \mathbb{N}$, and put $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, then $\mathcal{P}$ is a σ-locally finite sn-network of X. So X is an sn-metrizable space.

References

[1] Arhangel'skii, A.V., An addition theorem for the weight of sets lying in bicompacta. Dokl. Akad. Nauk. SSSR. 126 (1959), 239-241.

[2] Boone, J.R., Siwiec, F., Sequentially quotient mappings. Czech. Math. J. 26 (1976), 174-182.

[3] Cai C., Li, J., Notes on sn-metrizable spaces. J. of Mthematical Study 39 (2006), 282-285. (in Chinese)

[4] Engelking, R., General Topology (revised and completed edition). Heldermann, Berlin, 1989.

[5] Franklin, S.P., Spaces in which sequences suffice. Fund. Math. 57 (1965), 107-115.

[6] Ge, X., Notes on almost open mappings. Matematicki Vesnik 60 (2008), 181-186.

[7] Ge, X., Shen, J., Ge, Y., Spaces with σ-weakly hereditarily closure-preserving sn-networks. Novi Sad J. Math. 37 (2007), 33-37.

[8] Ge, Y., On spaces with a σ-locally-finite universal cs-network. Questions Answers in General Topology 18 (2000), 93-96.

[9] Ge, Y., On sn-metrizable spaces. Acta Math. Sinica 45 (2002), 355-360. (in Chinese)

[10] Ge, Y., Characterizations of sn-metrizable spaces. Publ. Inst. Math. Nouv. Ser. 74(88) (2003), 121-128.

[11] Ge, Y., Spaces with countable sn-networks. Comment Math. Univ. Carolinae 45 (2004), 169-176.

[12] Ge, Y., Weak forms of open mappings and strong forms of sequence-covering mappings. Matematički Vesnik 59 (2007), 1-8.

[13] Guthrie, J.A., A characterization of $\aleph_0$-spaces. General Topology Appl. 1 (1971), 105-110.

[14] Kang, B.F., On the *sn*-open images of metric spaces. Journal of Zhangzhou Normal University (Nat. Sci.), Volume 2007, Issue 1, 13-16. (in Chinese)

[15] Li, J., A note on g-metrizable spaces. Czech. Math. J. 53 (2003), 491-495.

[16] Lin, S., Spaces with a locally countable k-network. Northeastern Math. J. 6 (1990), 39-44.

[17] Lin, S., Point-Countable Covers and Sequence-Covering Mappings. Chinese Science Press, Beijing, 2002. (in Chinese)

[18] Lin, S., Generalized Metric Spaces and Mappings (the second edition). Chinese Science Press, Beijing, 2007. (in Chinese)

[19] Lin, S., Yan, P., Sequence-Covering Maps of Metric Spaces. Topology Appl. 109 (2001), 301-314.

[20] Lin, S., Yan, P., Notes on cfp-covers. Comment. Math. Univ. Carolinae 44 (2003), 295-306.

[21] Luo, Z., *sn*-Metrizable spaces and related matters. International Journal of Mathematics and Mathematical Sciences, Volume 2005, Issue 16, 2523-2531.

[22] O'Meara, P., On paracompactness in function spaces with the compact-open topology. Proc. Amer. Math. Soc. 29 (1971), 183-189.

[23] Tanaka, Y., Symmetric spaces, g-developable spaces and g-metrizable spaces. Math. Japonica 36 (1991), 71-84.

[24] Tanaka, Y., Li, Z., Certain covering-maps and k-networks, and related matters. Topology Proc. 27 (2003), 317-334

Received by the editors April 15, 2009