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THE UNIQUENESS AND UNIVERSALITY OF A GENERALIZED ORDERED SPACE[1]

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Abstract. If $\langle L, < \rangle$ is a dense linear order without end points, A and B disjoint and dense subsets of L and $\mathcal{O}_{AB}$ the topology on the set L generated by closed intervals $[a, b]$, where $a \in A$ and $b \in B$, then $\langle L, \mathcal{O}_{AB} \rangle$ is a generalized ordered space. We show that all spaces of the form $\langle \mathbb{R}, \mathcal{O}_{AB} \rangle$, where $A, B \subset \mathbb{R}$ are countable sets, are homeomorphic and universal in the class of second countable zero-dimensional spaces.

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1. Introduction

We remind the reader that, for a linear order $\langle L, < \rangle$, the standard topology $\mathcal{O}_<$ on the set L is generated by the family of all open intervals and then the space $\langle L, \mathcal{O}_< \rangle$ is called a *linearly ordered topological space* (LOTS). A topological space $\langle X, \mathcal{O} \rangle$ is called *linearly orderable* if there is a linear order $<$ on X such that $\mathcal{O} = \mathcal{O}_<$; *suborderable* if it is homeomorphic to a subspace of some LOTS; *generalized orderable (GO space)* if there is a linear order $<$ on X such that $\mathcal{O}_< \subset \mathcal{O}$ and each point has a neighborhood base consisting of intervals.

Čech [4] proved that the classes of suborderable and GO spaces coincide. Also it is known (see [4] or [8]) that, if $\langle L, < \rangle$ is a linear order and I, A and B are disjoint subsets of L, then

$$\mathcal{P}_{IAB} = \{x : x \in I\} \cup \{[a, \rightarrow) : a \in A\} \cup \{(\leftarrow, b] : b \in B\} \cup \mathcal{O}_<$$

is a subbase for a GO topology on L. So, if $\langle L, < \rangle$ is a linear order and $A, B \subset L$ are disjoint sets, then, clearly, the families $\mathcal{P}_{\emptyset AB}$ and

$$\mathcal{B}_{AB} = \{[a, b] : a \in A \wedge b \in B \wedge a < b\}$$

generate the same topology, let us denote it by $\mathcal{O}_{AB}$, on the set L and $\langle L, \mathcal{O}_{AB} \rangle$ is a GO space. Examples of such a construction are "the two arrows space"

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of Alexandroff and Urison ([1], see [5]) and some subspaces of the spaces constructed by Todorčević in [9].

The spaces of the form $\langle L, \mathcal{O}_{AB}\rangle$, where $\langle L, <\rangle$ is a dense linear order without end points and A and B are dense and disjoint subsets of L, were investigated in [6] and [7]. In the following theorem we collect some results from [7].

Theorem 1. *Let $\langle L, <\rangle$ be a dense linear order without end points and A and B dense, disjoint subsets of L. Then*

(a) The space $\langle L, \mathcal{O}_{AB}\rangle$ is zero-dimensional, non-compact, collectionwise normal, hereditarily normal and need not to be perfectly normal;

(b) For the cardinal functions on $\langle L, \mathcal{O}_{AB}\rangle$ we have: $e \leq l \leq c = hc = hl \leq d = hd \leq \min\{|A|, |B|\} \leq w = nw = \max\{|A|, |B|\} \leq |L|$, and $\chi = \psi = t \leq c$.

(c) $|A| = |B| = \aleph_0 \Rightarrow$ the space $\langle L, \mathcal{O}_{AB}\rangle$ is metrizable $\Rightarrow |A| = |B|$.

For the spaces of the form $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$, where $\mathbb{R}$ is the real line, we have

Fact 1. *If A and B are dense disjoint subsets of $\mathbb{R}$, the space $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$ is*

(a) zero-dimensional, non-compact, collectionwise normal, perfectly normal;

(b) hereditarily separable, hereditarily Lindelöf, first countable and $w(\mathbb{R}, \mathcal{O}_{AB}) = \max\{|A|, |B|\}$.

(c) second countable iff $|A| = |B| = \aleph_0$ iff it is metrizable.

If $x = \langle x_n : n \in \mathbb{N}\rangle$ is a sequence in the space $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$ then

x converges to a point $a \in A$ iff it converges to a in the standard topology and there is $n_0 \in \mathbb{N}$ such that $x_n \geq a$, for all $n \geq n_0$;

x converges to a point $b \in B$ iff it converges to b in the standard topology and there is $n_0 \in \mathbb{N}$ such that $x_n \leq b$, for all $n \geq n_0$;

x converges to a point $c \in \mathbb{R} \setminus (A \cup B)$ iff it converges to c in the standard topology.

Proof. (b) follows from Theorem 2(b) and the fact that the set of rationals $\mathbb{Q}$ is dense in the space $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$.

If the space $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$ is metrizable, then, since it is separable, it must be second countable and (c) is true.

Finally, if $O = \bigcup_{i\in I}[a_i, b_i] \in \mathcal{O}_{AB}$, then, since the space is hereditarily Lindelöf, there is a countable subset $C \subset I$ such that $O = \bigcup_{i\in C}[a_i, b_i]$. Thus O is a F_σ set, $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$ is a perfectly normal space and (a) is true.

The statements concerning the convergence of sequences are evident. □

In this paper we consider the spaces $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$, where A and B are countable dense disjoint subsets of $\mathbb{R}$.

2. Uniqueness and universality

If A and B are countable dense disjoint subsets of $\mathbb{R}$, then, by Fact 1, $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$ is a separable metrizable zero-dimensional space. First, using a variation of Cantor's "back and forth" method, we show that all spaces of this form are homeomorphic.

Theorem 2. *If for $i \in \{1,2\}$ the sets $A_i, B_i \subset \mathbb{R}$ are countable dense and disjoint, then the spaces $\langle \mathbb{R}, \mathcal{O}_{A_i B_i} \rangle$ are homeomorphic.*

Proof. Let $\mathbb{I}$ denote the set of all finite partial functions from $A_1 \cup B_1$ to $A_2 \cup B_2$ which are increasing and map elements of A_1 to elements of A_2 and elements of B_1 to elements of B_2. Since the sets A_1, A_2, B_1 and B_2 are dense we have

Claim 1. Let $f = \begin{pmatrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{pmatrix} \in \mathbb{I}$. Then

$$\begin{array}{l} \forall a^1 \in A_1 \setminus \{x_0, x_1, \dots, x_n\} \ \exists a^2 \in A_2 \ \ f \cup \{\langle a^1, a^2 \rangle\} \in \mathbb{I}; \\ \forall b^1 \in B_1 \setminus \{x_0, x_1, \dots, x_n\} \ \exists b^2 \in B_2 \ \ f \cup \{\langle b^1, b^2 \rangle\} \in \mathbb{I}; \\ \forall a^2 \in A_2 \setminus \{y_0, y_1, \dots, y_n\} \ \exists a^1 \in A_1 \ \ f \cup \{\langle a^1, a^2 \rangle\} \in \mathbb{I}; \\ \forall b^2 \in B_2 \setminus \{y_0, y_1, \dots, y_n\} \ \exists b^1 \in B_1 \ \ f \cup \{\langle b^1, b^2 \rangle\} \in \mathbb{I}. \end{array}$$

Claim 2. There is an order isomorphism $f : A_1 \cup B_1 \to A_2 \cup B_2$ such that $f[A_1] = A_2$ and $f[B_1] = B_2$.

Proof of Claim 2. Let $A_1 = \{a^1_k : k \in \omega\}$, $B_1 = \{b^1_l : l \in \omega\}$, $A_2 = \{a^2_m : m \in \omega\}$ and $B_2 = \{b^2_n : n \in \omega\}$ be fixed enumerations of the sets A_1, B_1, A_2 and B_2.

By recursion we construct four sequences of integers, $\langle k_i : i \in \omega \rangle$, $\langle l_i : i \in \omega \rangle$, $\langle m_i : i \in \omega \rangle$ and $\langle n_i : i \in \omega \rangle$, such that for each $j \in \omega$

$$f_j = \begin{pmatrix} a^1_{k_0} & a^1_{k_1} & \dots & a^1_{k_j} & b^1_{l_0} & b^1_{l_1} & \dots & b^1_{l_j} \\ a^2_{m_0} & a^2_{m_1} & \dots & a^2_{m_j} & b^2_{n_0} & b^2_{n_1} & \dots & b^2_{n_j} \end{pmatrix} \in \mathbb{I}\,. \tag{1}$$

Let $j \in \omega$ and suppose that the sequences $\langle k_i : i < j \rangle$, $\langle l_i : i < j \rangle$, $\langle m_i : i < j \rangle$ and $\langle n_i : i < j \rangle$ are defined such that $f_i \in \mathbb{I}$, for $i < j$. Using Claim 1 we define k_j, l_j, m_j and n_j such that $f_j = f_{j-1} \cup \{\langle a^1_{k_j}, a^2_{m_j} \rangle, \langle b^1_{l_j}, b^2_{n_j} \rangle\} \in \mathbb{I}\,$.

- If j is an odd number, let

$$\begin{aligned} k_j &= \min\Big\{k : a^1_k \notin \{a^1_{k_0}, a^1_{k_1}, \dots, a^1_{k_{j-1}}\}\Big\}, \\ m_j &= \min\Big\{m \in \omega : f_{j-1} \cup \{\langle a^1_{k_j}, a^2_m \rangle\} \in \mathbb{I}\Big\}, \\ l_j &= \min\Big\{l : b^1_l \notin \{b^1_{l_0}, b^1_{l_1}, \dots, b^1_{l_{j-1}}\}\Big\}, \\ n_j &= \min\Big\{n \in \omega : f_{j-1} \cup \{\langle a^1_{k_j}, a^2_{m_j} \rangle, \langle b^1_{l_j}, b^2_n \rangle\} \in \mathbb{I}\Big\}. \end{aligned}$$

- If j is an even number, let

$$\begin{aligned} m_j &= \min\Big\{m : a^2_m \notin \{a^2_{m_0}, a^2_{m_1}, \dots, a^2_{m_{j-1}}\}\Big\}, \\ k_j &= \min\Big\{k \in \omega : f_{j-1} \cup \{\langle a^1_k, a^2_{m_j} \rangle\} \in \mathbb{I}\Big\}, \\ n_j &= \min\Big\{n : b^2_n \notin \{b^2_{n_0}, b^2_{n_1}, \dots, b^2_{n_{j-1}}\}\Big\}, \\ l_j &= \min\Big\{l \in \omega : f_{j-1} \cup \{\langle a^1_{k_j}, a^2_{m_j} \rangle, \langle b^1_l, b^2_{n_j} \rangle\} \in \mathbb{I}\Big\}. \end{aligned}$$

So, the desired sequences are constructed. Clearly $f = \bigcup_{j\in\omega} f_j$ is a function which maps a subset of $A_1 \cup B_1$ onto a subset of $A_2 \cup B_2$.

In order to show that $\operatorname{dom} f = A_1 \cup B_1$ and $\operatorname{ran} f = A_2 \cup B_2$ we prove that $\{k_i : i \in \omega\} = \{l_i : i \in \omega\} = \{m_i : i \in \omega\} = \{n_i : i \in \omega\} = \omega$. Suppose that $\omega \setminus \{k_i : i \in \omega\} \neq \emptyset$ and $p = \min(\omega \setminus \{k_i : i \in \omega\})$. Then $k \in \{k_i : i \in \omega\}$, for each $k < p$, and, clearly, there is an odd number j such that $\{a^1_k : k < p\} \subset \{a^1_{k_0}, \ldots, a^1_{k_{j-1}}\}$ so $p = \min\{k \in \omega : a^1_k \notin \{a^1_{k_0}, a^1_{k_1}, \ldots, a^1_{k_{j-1}}\}\}$, which, by the construction, implies $p = k_j$. A contradiction. The proof of the other three equalities is similar.

We prove that the function f is increasing. If $x_1, x_2 \in \operatorname{dom} f$ and $x_1 < x_2$, then there is $j \in \omega$ such that $x_1, x_2 \in \operatorname{dom} f_j$ and, since $f_j \in \mathbb{I}$, we have $f(x_1) = f_j(x_1) < f_j(x_2) = f(x_2)$.

Finally we prove that $f[A_1] = A_2$ and $f[B_1] = B_2$. If $a \in A_1$, then there is $j \in \omega$ such that $a \in \operatorname{dom} f_j$ and, by the construction, $f(a) = f_j(a) \in A_2$, thus $f[A_1] \subset A_2$. The proof that $f[B_1] \subset B_2$ is similar and the equalities follow from the fact that $f : A_1 \cup B_1 \to A_2 \cup B_2$ is a bijection. Claim 2 is proved.

Claim 3. The mapping $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(z) = \sup\{f(x) : x \in A_1 \cup B_1 \wedge x \leq z\}$$

is an order isomorphism which extends f.

Proof of Claim 3. Since f is an increasing function, for $z \in A_1 \cup B_1$ we have $F(z) = f(z)$.

We prove that the function F is increasing. If $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$ then, by the density of A_1, there are $a_1, a_2 \in A_1$ such that $x_1 < a_1 < a_2 < x_2$. Since f is an increasing function, according to the definition of F we have $F(x_1) \leq F(a_1) = f(a_1) < f(a_2) = F(a_2) \leq F(x_2)$.

Finally we prove that F is a surjection. Let $y \in \mathbb{R}$ and $Y = \{w \in A_2 \cup B_2 : w \leq y\}$. Let $X = f^{-1}[Y]$ and let $x = \sup X$. Then it is easy to show that $F(x) = y$. Claim 3 is proved.

The mapping $F : \langle \mathbb{R}, \mathcal{O}_{A_1 B_1}\rangle \to \langle \mathbb{R}, \mathcal{O}_{A_2 B_2}\rangle$ is open because for $a_1 \in A_1$, $b_1 \in B_1$ satisfying $a_1 < b_1$, by Claims 2 and 3 we have $F(a_1) \in A_2$, $F(b_1) \in B_2$ and $F[[a_1, b_1]] = [F(a_1), F(b_1)]$.

F is continuous because for $a_2 \in A_2$ and $b_2 \in B_2$ satisfying $a_2 < b_2$ by Claims 2 and 3 we have $F^{-1}(a_2) \in A_1$, $F^{-1}(b_2) \in B_1$ and $F^{-1}[[a_2, b_2]] = [F^{-1}(a_2), F^{-1}(b_2)]$. Thus, the mapping F is a homeomorphism. □

Can the last result be extended for uncountable sets A and B? Since there are non-isomorphic uncountable dense subsets of $\mathbb{R}$, some kind of homogeneity of the sets A and B should be assumed. So, a subset $A \subset \mathbb{R}$ is called $\aleph_1$-dense iff it has $\aleph_1$-many elements in each interval. In [6], following the construction of Baumgartner from [2] modified by Todorčević (see [10]), the following consistency result is obtained.

Theorem 3. *Under the Proper Forcing Axiom, each two spaces of the form $\langle \mathbb{R}, \mathcal{O}_{AB}\rangle$, where A and B are disjoint $\aleph_1$-dense subsets of $\mathbb{R}$, are homeomorphic.*

More information concerning the Proper Forcing Axiom can be found in [3].

For countable $A, B \subset \mathbb{R}$ the spaces $\langle \mathbb{R}, \mathcal{O}_{AB} \rangle$ are second countable and zero-dimensional. Now we show that they are universal for all spaces with these two properties.

Theorem 4. *Each second countable zero-dimensional space can be embedded in the space $\langle \mathbb{R}, \mathcal{O}_{AB} \rangle$, where A and B are countable, disjoint, dense subsets of $\mathbb{R}$.*

Proof. Every zero-dimensional second countable space can be embedded in the Cantor cube 2^ω, which is homeomorphic to the Cantor set $C \subset \mathbb{R}$ with the standard topology. Thus, it is sufficient to embed the Cantor set C into the space $\langle \mathbb{R}, \mathcal{O}_{AB} \rangle$, for specially chosen sets A and B.

Let us define the sets A and B. Let $\{B_n : n \in \omega\}$ be an enumeration of the base $\{(p, q) : p, q \in \mathbb{Q},\ p < q\}$ for the standard topology on $\mathbb{R}$. Since the set $\mathbb{R} \setminus C$ is open and dense in the standard topology, from each set $B_n \setminus C$ we can choose two elements, a_n and b_n, such that $\{a_n : n \in \omega\} \cap \{b_n : n \in \omega\} = \emptyset$. Clearly, the sets $A = \{a_n : n \in \omega\}$ and $B = \{b_n : n \in \omega\}$ are dense and disjoint.

It remains to be proved that the standard topology on the Cantor set C coincides with the induced topology $(\mathcal{O}_{AB})_C$. Since $A, B \subset \mathbb{R} \setminus C$ we have $[a, b] \cap C = (a, b) \cap C$, for each $a \in A$ and $b \in B$, such that $a < b$, which is an open set in the standard topology on the Cantor set. Also, the topology $\mathcal{O}_{AB}$ is finer than the standard topology, which completes the proof. $\square$

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