Novi Sad J. Math.
Vol. 39, No. 1, 2009, 89-109

GENERAL COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE HYBRID MAPPINGS AND APPLICATIONS

Abdelkrim Aliouche[1], Valeriu Popa[2]

Abstract. We prove general common fixed point theorems for occasionally weakly compatible hybrid pairs of mappings in symmetric spaces satisfying implicit relations which generalize the theorems of [1]-[6], [8]-[14], [16]-[33], [36], [37], [39], [41], [44], [46]-[49], [51], [56], [58]-[63], [65]-[74], [76]-[85] and we correct the errors of [6], [21], [34], [47] and [84].

AMS Mathematics Subject Classification (2000): 54H25, 47H10

Key words and phrases: symmetric space, occasionally weakly compatible hybrid mappings, common fixed point

1. Introduction and preliminaries

It is well known that the Banach contraction principle is a fundamental result in fixed point theory which has been used and extended in many different directions. However, it has been observed by Hicks and Rhoades [35] that some of the defining properties of the metric are not needed in the proof of certain metric theorems. They established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structure admits a compatible symmetric or semi-metric.

Definition 1.1. Let X be a set. A symmetric on X is a mapping $d : X \times X \to [0, \infty)$ such that
$d(x, y) = 0$ iff $x = y$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

Let $B(X)$ be the set of all nonempty bounded subsets of X. As in [32], we define the functions $\delta(A, B)$ and $D(A, B)$ by
$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$,
$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ for all $A, B \in B(X)$.

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. If B consists also of a single point b, we write $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of δ that
$\delta(A, B) = \delta(B, A) \geq 0$,
$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$,
$\delta(A, B) = 0$ iff $A = B = \{a\}$,

[1]Department of Mathematics, University of Larbi Ben M'Hidi, Oum-El-Bouaghi, 04000, Algeria, e-mail: alioumath@yahoo.fr
[2]Department of Mathematics, University of Bacău Str. Spiru Haret nr. 8 600114 Bacău, Romania, e-mail: vpopa@ub.ro

$\delta(A, A) = diamA$ for all $A, B, C \in B(X)$.

Let A and S be self-mappings of a metric space (X, d) and $C(A, S)$ the set of coincidence points of A and S.

Jungck [40] defined A and S to be compatible if

$$\lim_{n\to\infty} d\left(SAx_n, ASx_n\right) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$ for some $t \in X$.

The same author [52] defined A and S to be R-weakly commuting if there exists an $R > 0$ such that

$$d\left(STx, TSx\right) \leq Rd\left(Tx, Sx\right) \text{ for all } x \in X. \tag{1.1}$$

Pant [53] defined A and S to be pointwise $R-$weakly commuting if for each $x \in X$, there exists an $R > 0$ such that (1.1) holds.

It was proved in [53] that pointwise $R-$weakly commuting is equivalent to commutativity at coincidence points. Thus, A and S are pointwise $R-$weakly commuting if and only if they are weakly compatible.

Definition 1.2. [42] A and S are said to be weakly compatible if $SAu = ASu$ for all $u \in C(A, S)$.

Thus, A and S are pointwise $R-$weakly commuting if and only if they are weakly compatible.

Definition 1.3. [15] A and S are said to be occasionally weakly compatible (owc) if $SAu = ASu$ for some $u \in C(A, S)$.

Remark 1.4. [15] If A and S are weakly compatible, then they are occasionally weakly compatible, but the following example shows that the converse is not true in general.

Example 1.5. Let $X = [1, \infty)$ with the usual metric. Define A, $S : X \to X$ by: $Ax = 3x - 2$ and $Sx = x^2$. We have $Ax = Sx$ iff $x = 1$ or $x = 2$ and $AS(1) = SA(1) = 1$, but $AS(2) \neq SA(2)$. Therefore, A and S are occasionally weakly compatible, but they are not weakly compatible.

Remark 1.6. Every mapping $A : X \to X$ and the identity mapping of X, id_X, are weakly compatible, while $A : X \to X$ and id_X are owc iff A has a fixed point.

Lemma 1.7. *[44] If A and S have a unique coincidence point $w = Ax = Sx$, then w is the unique common fixed point of A and S.*

Definition 1.8. 1) A point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$. We denote by $C(f, F)$ the set of all coincidence points of f and F.

2) A point $x \in X$ is a fixed point of F if $x \in Fx$.

3) A point $x \in X$ is a stationary point of F or a strict fixed point if $Fx = \{x\}$.

Definition 1.9. [41] The mappings $f : X \to X$ and $F : X \to B(X)$ are δ-compatible if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X)$, $fx_n \to t$ and $Fx_n \to \{t\}$ as $n \to \infty$ for some $t \in X$.

Definition 1.10. [43] The hybrid pair (f, F), $f : X \to X$ and $F : X \to B(X)$ is weakly compatible iff for all $x \in C(f, F)$, $fFx = Ffx$.

If the pair (f, F) is δ-compatible, then it is weakly compatible, but the converse is not true in general, see [43].

Recently, Abbas and Rhoades [5], extended the notion of owc mappings to hybrid pairs.

Definition 1.11. [5] The hybrid pair (f, F), $f : X \to X$ and $F : X \to CB(X)$ is owc iff there exists $x \in C(f, F)$ such that $fFx \subset Ffx$.

Example 1.12. Let $X = [0, 2]$ with usual metric. Define $f : X \to X$ and $F : X \to B(X)$ by:

$$fx = \begin{cases} 0 \text{ if } x = 0, \\ 2 - x \text{ if } x \neq 0 \end{cases} \text{ and } Fx = \begin{cases} [0, x] \text{ if } x \leq 1, \\ [0, 2x] \text{ if } x > 1 \end{cases}.$$

Clearly, $C(f, F) = \{0, 1\}$, $Ff0 = fF0 = \{0\}$ and $Ffx \neq fFx$ for all $x \in (0, 2]$. Hence, the pair (f, F) is owc, but it is not weakly compatible.

Remark 1.13. It is obvious that $0 \in F0 = \{0\}$ and $1 \in F1 = [0, 1]$. Therefore, 0 and 1 are fixed points for f and F and only 0 is a stationary fixed point for f and F.

In [64] and [65], the study of fixed points for mappings satisfying implicit relations was introduced and the study of a pair of hybrid mappings satisfying implicit relations was initiated in [66].

2. Implicit relations

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

(i) $\psi(t) < t$ for all $t > 0$

(ii) ψ is increasing.

Define $\Psi = \{\psi : \psi \text{ satisfies (i) and (ii) above}\}$.

Let G_6 denote the family of all real mappings $G(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(G_1) : G is increasing in variable t_1 and decreasing in variables t_2, t_5 and t_6.

(G_2) : $G(t, t, 0, 0, t, t) \geq 0$ for all $t > 0$.

Example 2.1. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, $0 \leq k \leq 1$.

(G_1) : Obviously.

(G_2) : $G(t, t, 0, 0, t, t) = 0$ for all $t > 0$.

Example 2.2. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^p - a t_2^p - (1-a) \max\{\alpha t_3^p, \beta t_4^p, t_3^{\frac{p}{2}} t_6^{\frac{p}{2}}, t_5^{\frac{p}{2}} t_6^{\frac{p}{2}}\}$, $0 < a, \alpha, \beta \leq 1$ and $p \geq 1$.

(G_1) and (G_2) as in Example 2.1

Example 2.3. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_2, t_3, t_4\} - d\max\{t_5, t_6\}$, $a, b, c > 0$, $d \geq 0$ and $a + d + c \leq 1$.

(G_1) : Obviously.

(G_2) : $G(t, t, 0, 0, t, t) = [1 - (a + d + c)]t \geq 0$ for all $t > 0$.

Example 2.4. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, t_3, t_4, t_5, t_6\})$, where $\psi \in \Psi$.

(G_1) : Obviously.

(G_2) : $G(t, t, 0, 0, t, t) = t - \psi(t) > 0$ for all $t > 0$.

Example 2.5.

$G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^p - \psi(at_2^p - (1-a)\max\{\alpha t_3^p, \beta t_4^p, t_3^{\frac{p}{2}} t_6^{\frac{p}{2}}, t_5^{\frac{p}{2}} t_6^{\frac{p}{2}}\})$, $0 < a, \alpha, \beta \leq 1$ and $\psi \in \Psi$.

(G_1) and (G_2) as in Example 2.4.

Example 2.6. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - at_2^3 - b\dfrac{t_5^2 t_6 + t_5 t_6^2}{t_3 + t_4 + 1}$, where $a, b > 0$ and $a + 2b \leq 1$.

(G_1) : Obviously.

(G_2) : $G(t, t, 0, 0, t, t) = [1 - (a + 2b)]t^3 \geq 0$ for all $t > 0$.

Example 2.7. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - c_1\max\{t_2^2, t_3^2, t_4^2\} - c_2\max\{t_3 t_6, t_4 t_5\} - c_3 t_5 t_6$,

$c_1, c_2, c_3 \geq 0$, $c_1 + 2c_2 + c_3 < 1$.

Example 2.8. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6)$, where $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is increasing in variables t_2, t_5 and t_6 and satisfies for all $t > 0$

$\phi(t, t, \alpha_1 t, \alpha_2 t, \alpha_3) < t$, where $\alpha_1 + \alpha_2 + \alpha_3 = 4$.

Example 2.9. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h\max\{t_2, t_3, t_4\} - (1-h)(at_5 + bt_6)\}$, $0 \leq h < 1$, $a, b \geq 0$ and $a + b \leq 1$.

Example 2.10. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^{2p} - a\psi_0(t_2^{2p}) -$

$(1-a)\max\{\psi_1({t_2}^{2p}), \psi_2(t_3^q t_4^{q'})\psi_3(t_5^r t_6^{r'}),$

$\psi_4(\frac{1}{2} t_3^s t_6^{s'}), \psi_5(\frac{1}{2} t_4^l t_5^{l'}\}$, where

$\psi_i \in \Psi$, $i = 0, 1, 2, 3, 4, 5$, $0 \leq a \leq 1$ and $0 < p, q, q', r, r', s, s', l, l' \leq 1$, such that $2p = q + q' = r + r' = s + s' = l + l'$.

Example 2.11.

$G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{\max\{t_3, t_4\}, t_5, t_6\} - \varphi(\max\{t_2, t_3, t_4, t_5, t_6\}$, $\varphi \in \Psi$.

(G_1) : Obvious.

(G_2) : $G(t, t, 0, 0, t, t) = t - \varphi(t) > 0$ for all $t > 0$.

Example 2.12. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, (t_3 + t_4)/2, (t_5 + t_6)/2\}$.

Example 2.13. $G(t_1, ...t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, k(t_5 + t_6)/2\})$, where $0 < k \leq 1$ and $\phi \in \Psi$.

Example 2.14. $G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{ct_2, ct_3, ct_4), at_5 + bt_6\}$, $0 \leq c < 1$, $a, b \geq 0$ and $a + b \leq 1$.

3. Main Results

Theorem 3.1. *Let (X,d) be a symmetric space, $f,h : X \to X$ and $F,H : X \to B(X)$ satisfying*

(3.1)

$$G(\delta(Fx,Hy),d(fx,hy),D(fx,Fx),D(hy,Hy),D(fx,Hy),D(hy,Fx)) < 0$$

for all $x,y \in X$, where $G \in G_6$. Then, f,h,G and H have at most a common fixed point in X which is a stationary point of F and H.

Proof. Let z be a common fixed point of f,h,G and H. Hence, $z = fz \in Fz$ and $z = hz \in Hz$. Using (3.1) and (G_1) we have successfully

$$\begin{aligned} 0 &> G(\delta(Fz,Hz),d(fz,hz),D(fz,Fz),D(hz,Hz),D(fz,Hz),D(hz,Fz)) \\ &\geq G(\delta(Fz,Hz),\delta(Fz,Hz),0,0,\delta(Fz,Hz),\delta(Fz,Hz)) \end{aligned}$$

which is a contradiction of (G_2). Therefore, $Fz = Hz = \{z\}$. Assume that $w \neq z$ is another common fixed point of f,h,G and H. Applying (3.1) and (G_1) we get

$$\begin{aligned} 0 &> G(\delta(Fz,Hw),d(fz,hw),D(fz,Fz),D(hw,Hw),D(fz,Hw),D(hw,Fz)) \\ &\geq G(d(z,w),d(z,w),0,0,d(z,w),d(z,w)). \end{aligned}$$

which is a contradiction of (G_2). Thus, z is unique. □

Theorem 3.2. *Let (X,d) be a symmetric space, $f,h : X \to X$ and $F,H : X \to B(X)$ satisfying* (3.1). *Suppose that the pairs (f,F) and (h,H) are owc. Then, f,h,G and H have a unique common fixed point in X which is a stationary point of F and H.*

Proof. Since the pairs (f,F) and (h,H) are owc, there exist $u,v \in X$ such that $fu \in Fu$ and $fFu \subset Ffu$ and $hv \in Hv$ and $hHv \subset Hhv$. It follows that $ffu \in Ffu$ and $hhv \in Hhv$. Let us show that $z = fu = hv$.

If $fu \neq hv$, using (3.1) and (G_1) we have successfully

$$\begin{aligned} 0 &> G(\delta(Fu,Hv),d(fu,hv),D(fu,Fu),D(hv,Hv),D(fu,Hv),D(hv,Fu)) \\ &\geq G(d(fu,hv),d(fu,hv),0,0,d(fu,hv),d(fu,hv)) \end{aligned}$$

which is a contradiction of (G_2). Hence $fu = hv$. We prove that z is a fixed point of f.

If $fz \neq z$, using (3.1) and (G_1), we get

$$\begin{aligned} 0 &> G(\delta(Fz,Hv),d(fz,hv),D(fz,Fz),D(hv,Hv),D(fz,Hv),D(hv,Fz)) \\ &\geq G(d(fz,hv),d(fz,hv),0,0,d(fz,hv),d(fz,hv)) \end{aligned}$$

which is a contradiction of (G_2). Hence, $z = fz$. Similarly, $z = hz = fz$. On the other hand, $z = fz \in Fz$ and $z = hz \in Hz$ and so z is a common fixed point of f, h, G and H. By Theorem 3.1, z is unique and is a stationary point of F and H. □

Theorem 3.2 generalizes the Theorems of [4, 8, 9, 17, 22, 27, 31, 32, 33, 41, 67, 72, 74, 79, 82, 83].

If we combine Examples 2.10 and 2.11 with Theorem 3.1, we obtain a generalization of the Theorems of [60] and [51].

Theorem 3.3. *Let (X, d) be a symmetric space and $f, g, S, T : X \to X$ satisfying*

$$G(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) < 0 \tag{3.2}$$

for all $x, y \in X$ with $fx \neq gy$, where G satisfies the condition (G_2). Suppose that the pairs (f, S) and (g, T) are owc. Then, f, g, S and T have a unique common fixed point in X.

Proof. Since the pairs (f, S) and (g, T) are owc, there exist $u, v \in X$ such that $fu = Su$ and $fSu = Sfu$ and $gv = Tv$ and $gTv = Tgv$. Let us show that $z = fu = gv$.

If $fu \neq gv$, using (3.2) we have

$$\begin{aligned} & G(d(fu, gv), d(Su, Tv), d(fu, Su), d(gv, Tv), d(fu, Tv), d(Su, gv)) \\ = \; & G(d(fu, gv), d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) \end{aligned}$$

which is a contradiction of (G_2). Hence, $fu = gv$. In a similar manner, $z = fz = gz$. The uniqueness of z follows from (3.2) and (G_2). □

By Theorem 3.3 and Examples 2.1-2.23 we get the Theorems of [44].

The following Corollary generalizes Theorem 5 of Ciric et al. [25].

Colorallary 3.4. *Let (X, d) be a symmetric space and $P_1, P_2, ..., P_{2n}, Q_0, Q_1 : X \to X$ satisfying*

i) $P_2(P_4...P_{2n}) = (P_4...P_{2n})P_2$,
$P_2P_4(P_6...P_{2n}) = (P_6...P_{2n})P_2P_4$,
....
$P_2...P_{2n-2}(P_{2n}) = (P_{2n})P_2...P_{2n-2}$,
$Q_0(P_4...P_{2n}) = (P_4...P_{2n})Q_0$,
$Q_0(P_6...P_{2n}) = (P_6...P_{2n})Q_0$,
....
$Q_0P_{2n} = P_{2n}Q_0$,
$P_1(P_3...P_{2n-1}) = (P_3...P_{2n-1})P_1$,
$P_1P_3(P_5...P_{2n-1}) = (P_5...P_{2n-1})P_1P_3$,
....
$P_1...P_{2n-3}(P_{2n-1}) = P_{2n-1}(P_1...P_{2n-3})$,
$Q_1(P_3...P_{2n-1}) = (P_3...P_{2n-1})Q_1$,
$Q_1(P_5...P_{2n-1}) = (P_5...P_{2n-1})Q_1$,
....

$Q_1P_{2n-1} = P_{2n-1}Q_1$,
ii)

$$\begin{aligned} d(Q_0x, Q_1y) \quad < \quad & \max\{\psi(d(P_2P_4...P_{2n}x, P_1P_3...P_{2n-1}y)), \\ & \psi(d(P_2P_4...P_{2n}x, Q_0x)), \psi(d(P_1P_3...P_{2n-1}y, Q_1y)), \\ & \psi(d(Q_0x, P_1P_3...P_{2n-1}y)), \psi(d(P_2P_4...P_{2n}x, Q_1y))\} \end{aligned}$$

for all $x, y \in X$ with $Q_0x \neq Q_1y$ and ψ satisfies (i). Suppose that the pairs $(Q_0, P_2P_4...P_{2n})$ and $(Q_1, P_1P_3...P_{2n-1})$ are owc. Then, $P_1, P_2, ..., P_{2n}, Q_0$ and Q_1 have a unique common fixed point in X.

Proof. It follows from Theorem 3.3 and Example 2.4 by putting $f = Q_0$, $g = Q_1$, $S = P_2P_4...P_{2n}$, $T = P_1P_3...P_{2n-1}$. □

In the same manner we can generalize Theorem 6 of Ciric et al. [25].

Theorem 3.3 generalizes also the Theorems of [1, 2, 3, 10, 12, 13, 20, 26, 36, 37, 39, 44, 46, 49, 51, 52, 53, 55, 56, 58, 59, 60, 62, 63, 65, 76, 78, 80, 81].

If we combine Examples 2.10 and 2.11 with Theorem 3.3, we obtain generalizations of the Theorems of [60] and [51].

In the same manner, we can prove the following theorems.

Theorem 3.5. *Let (X, d) be a symmetric space, $f, h : X \to X$ and $F, H : X \to CB(X)$ satisfying*
(3.3)

$$G(\delta(Fx, Hy), d(fx, hy), D(fx, Fx), D(hy, Hy), \delta(fx, Hy), \delta(hy, Fx)) < 0$$

for all $x, y \in X$. Suppose that the pairs (f, F) and (h, H) are owc. Then, f, h, G and H have a unique common fixed point in X which is a stationary point of F and H.

If we combine Theorem 3.5 and Examples 2.1 and 2.2, we get corollaries which generalize Theorems 2.1 and 2.5 of [6].

Theorem 3.6. *Let (X, d) be a symmetric space and $f_n, S, T : X \to X, n \geq 1$ satisfying*

$$G(d(f_1x, f_ny), d(Sx, Ty), d(f_1x, Sx), d(f_ny, Ty), d(f_1x, Ty), d(Sx, f_ny)) < 0,$$

$n \geq 2$, for all $x, y \in X$ with $f_1x \neq f_ny$, $n \geq 2$, where G satisfies the condition (G_2). Suppose that the pairs (f_1, S) and (f_n, T), $n \geq 2$ are owc. Then, f_n, S and T have a unique common fixed point in X.

Theorem 3.6 generalizes Theorems of [13, 20, 28, 36, 52, 54, 65].

Theorem 3.7. *Let (X, d) be a symmetric space, $f, h : X \to X$ and $F_n : X \to B(X)$ satisfying*
(3.4)

$$G(\delta(F_1x, F_ny), d(fx, hy), D(fx, F_1x), D(hy, F_ny), D(fx, F_ny), D(hy, F_1x)) < 0,$$

$n \geq 2$, for all $x, y \in X$. Suppose that the pairs (f, F_1) and (h, F_n) are owc. Then, f, h and F_n have a unique common fixed point in X which is a stationary point of F_n, $n \geq 1$.

Theorem 3.8. *Let (X,d) be a symmetric space, $f,h : X \to X$ and $F_n : X \to CB(X)$, $n \geq 1$ satisfying*
(3.5)

$$G(\delta(F_1x, F_ny), d(fx, hy), D(fx, F_1x), D(hy, F_ny), \delta(fx, F_ny), \delta(hy, F_1x)) < 0$$

for all $x, y \in X$. Suppose that the pairs (f, F_1) and (h, F_n), $n \geq 1$ are owc. Then, f, h and F_n have a unique common fixed point in X which is a stationary point of F_n, $n \geq 1$.

Let L_6 denote the family of all real mappings $L(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}^5_+ \to \mathbb{R}$ satisfying the following conditions:

(L_1) : L is increasing in variable t_1 and decreasing in variables t_2, t_5 and t_6.

(L_2) : $L(t, t, 0, 0, 2t) \geq 0$ for all $t > 0$.

Similarly, we can prove the following theorems.

Theorem 3.9. *Let (X,d) be a symmetric space, $f,h : X \to X$ and $F, H : X \to B(X)$ satisfying*
(3.6)

$$L(\delta(Fx, Hy), d(fx, hy), D(fx, Fx), D(hy, Hy), D(fx, Hy) + D(hy, Fx)) < 0$$

for all $x, y \in X$, where $L \in L_6$. Suppose that the pairs (f, F) and (h, H) are owc. Then, f, h, G and H have a unique common fixed point in X which is a stationary point of F and H.

Theorem 3.9 generalizes a Theorem of [77].

Let Ψ_5 denote the set of all functions $\psi : [0, \infty)^5 \to [0, \infty)$ such that

(i) ψ is continuous,

(ii) ψ is increasing in all the variables,

(iii) $\psi(t_1, t_2, t_3, t_4, t_5) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

Theorem 3.10. *Let (X,d) be a symmetric space, $f, h : X \to X$ and $F, H : X \to B(X)$ satisfying*

$$\begin{aligned} &\phi_1(\delta(Fx, Hy)) \\ &\quad < \psi_1(d(fx, hy), D(fx, Fx), D(hy, Hy), D(fx, Hy), D(hy, Fx))) \quad (3.7) \\ &\quad\quad -\psi_2(d(fx, hy), D(fx, Fx), D(hy, Hy), D(fx, Hy), D(hy, Fx))) \end{aligned}$$

for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_5$ and $\phi_1(x) = \psi_1(x, x, x, x, x)$ for all $x \in [0, \infty)$. Suppose that the pairs (f, F) and (h, H) are owc. Then, f, h, G and H have a unique common fixed point in X which is a stationary point of F and H.

Proof. Since the pairs (f, F) and (h, H) are owc, there exist $u, v \in X$ such that $fu \in Fu$ and $fFu \subset Ffu$ and $hv \in Hv$ and $hHv \subset Hhv$. It follows that $ffu \in Ffu$ and $hhv \in Hhv$. Let us show that $z = fu = hv$.

If $fu \neq hv$, using (3.7), we have successfully

$$\begin{aligned}
&\phi_1(d(fu, hv)) \\
&\quad \leq \phi_1(\delta(Fu, Hv)) \\
&\quad < \psi_1(d(fu, hv), D(fu, Fu), D(hv, Hv), D(fu, Hv), D(hv, Fu))) \\
&\qquad -\psi_2(d(fu, hv), D(fu, Fu), D(hv, Hv), D(fu, Hv), D(hv, Fu))) \\
&\quad \leq \psi_1(d(fu, hv), 0, 0, d(fu, hv), d(fu, hv))) \\
&\qquad -\psi_2(d(fu, hv), 0, 0, d(fu, hv), d(fu, hv))) \\
&\quad \leq \psi_1(d(fu, hv), d(fu, hv), d(fu, hv), d(fu, hv), d(fu, hv))) \\
&\quad = \phi_1(d(fu, hv))
\end{aligned}$$

which is a contradiction. Hence, $fu = hv$. We prove that z is a fixed point of f. Similarly, $z = hz = fz$. On the other hand, $z = fz \in Fz$ and $z = hz \in Hz$ and so z is a common fixed point of f, h, G and H. As in Theorem 3.1, applying (3.7) we obtain that z is unique and is a stationary point of F and H. □

If f, g, F, H are single-valued in Theorem 3.10, we get a generalization of the Theorems of Rao et al. [70] and [71].

4. Applications

In this section we establish several common fixed point theorems for hybrid pairs.

I) Define $\Phi = \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue integrable mapping which is summable and satisfies $\int_0^{\epsilon} \varphi(t)dt > 0$ for all $\epsilon > 0\}$. Now, we give examples of mappings satisfying inequalities of integral type.

Example 4.1.

$$G(t_1, t_2, t_3, t_4, t_5, t_6) = \int_0^{t_1} \varphi(t)dt - \psi(\max\{\int_0^{t_2} \varphi(t)dt, \int_0^{t_3} \varphi(t)dt, \int_0^{t_4} \varphi(t)dt, \int_0^{t_5} \varphi(t)dt, \int_0^{t_6} \varphi(t)dt\}),$$

$\psi \in \Psi$.

Example 4.2.

$$G(t_1,t_2,t_3,t_4,t_5,t_6) =$$
$$= \Big(\int_0^{t_1}\varphi(t)dt\Big)^p - \psi\Big(a\Big(\int_0^{t_2}\varphi(t)dt\Big)^p - (1-a)\max\Big\{\alpha\Big(\int_0^{t_3}\varphi(t)dt\Big)^p, \beta\Big(\int_0^{t_4}\varphi(t)dt\Big)^p,$$
$$\Big(\int_0^{t_3}\varphi(t)dt\Big)^{\frac{p}{2}} \cdot \Big(\int_0^{t_6}\varphi(t)dt\Big)^{\frac{p}{2}}, \Big(\int_0^{t_5}\varphi(t)dt\Big)^{\frac{p}{2}} \cdot \Big(\int_0^{t_6}\varphi(t)dt\Big)^{\frac{p}{2}}\Big\}\Big),$$

$0 \leq a, \alpha, \beta \leq 1$, $p \geq 1$ and $\psi \in \Psi$.

Example 4.3.

$$G(t_1,t_2,t_3,t_4,t_5,t_6) = \int_0^{t_1}\varphi(t)dt - \alpha\max\Big\{\int_0^{t_2}\varphi(t)dt, \int_0^{t_3}\varphi(t)dt, \int_0^{t_4}\varphi(t)dt\Big\}$$
$$-(1-\alpha)\Big(a\int_0^{t_5}\varphi(t)dt + b\int_0^{t_6}\varphi(t)dt\Big),$$

$0 \leq \alpha < 1$, $a, b \geq 0$ and $a + b \leq 1$.

Example 4.4.

$$G(t_1,t_2,t_3,t_4,t_5,t_6) = \int_0^{t_1}\varphi(t)dt - \psi\Big(\max\Big\{\int_0^{t_2}\varphi(t)dt, \int_0^{t_3}\varphi(t)dt,$$
$$\int_0^{t_4}\varphi(t)dt, \frac{1}{2}\Big(\int_0^{t_5}\varphi(t)dt + \int_0^{t_6}\varphi(t)dt\Big)\Big\}\Big),$$

$\psi \in \Psi$.

Example 4.5. $G(t_1,t_2,t_3,t_4,t_5,t_6) = \displaystyle\int_0^{\phi(t_1,t_2,t_3,t_4,t_5,t_6)}\varphi(t)dt$, where $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ is increasing in variable t_1, decreasing in variables t_2, t_5 and t_6 and satisfies $\displaystyle\int_0^{\phi(u,u,0,0,u,u)}\varphi(t)dt \geq 0$ for all $u > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable.

For example $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - k\max\{t_2,t_3,t_4,t_5,t_6\}$, $0 \leq k \leq 1$ and $\varphi(t) = \dfrac{3\pi}{4(1+t)^2}\cos\Big(\dfrac{3\pi}{4(1+t)}\Big)$, $t \in \mathbb{R}_+$.

By Theorems 3.2-3.3 and Examples 4.1-4.5, we get generalizations of Theorems of [11, 16, 30], Theorem 2.1 of [8], Theorems of [29], Theorem 2.1 of [61], and Theorems of [18], [80].

II) Let $A \in (0, \infty]$, $R_A^+ = [0, A)$ and $F : R_A^+ \to \mathbb{R}$ satisfying

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$,

(ii) F is increasing on R_A^+,

Define $\mathcal{F}[0, A) = \{F : F$ satisfies (i) and (ii)$\}$.

The following examples were given in [85].

1) Let $F(t) = t$, then $F \in \mathcal{F}[0, A)$ for each $A \in (0, +\infty]$.

2) Suppose that φ is non-negative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, A).$$

Let $F(t) = \int_0^t \varphi(s)ds$, then $F \in [0, A)$.

3) Suppose that ψ is non-negative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^{\epsilon} \psi(t)dt > 0 \text{ for each } \epsilon \in (0, A)$$

and φ is non-negative, Lebesgue integrable on $[0, \int_0^A \psi(s)ds)$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, \int_0^A \psi(s)ds).$$

Let $F(t) = \int_0^{\int_0^t \psi(s)ds} \varphi(u)du$, then $F \in \mathcal{F}[0, A)$.

4) If $G \in [0, A)$ and $F \in \mathcal{F}[0, G(A - 0))$, then a composition mapping $F \circ G \in \mathcal{F}[0, A)$. For instance, let $H(t) = \int_0^{F(t)} \varphi(s)ds$, then $H \in \mathcal{F}[0, A)$ whenever $F \in \mathcal{F}[0, A)$ and φ is non-negative, Lebesgue integrable on $\mathcal{F}[0, F(A - 0))$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, F(A - 0)).$$

Example 4.6.

$$G(t_1, t_2, t_3, t_4, t_5, t_6) = F(t_1) - \psi(\max\{F(t_2), F(t_3), F(t_4), F(t_5), F(t_6)\}),$$

$\psi \in \Psi$.

(G_1) : Obviously.

(G_2) : $G(t, t, 0, 0, t, t) = 0$ for all $t > 0$.

Example 4.7.

$$\begin{aligned} & G(t_1, t_2, t_3, t_4, t_5, t_6) \\ = \;& (F(t_1))^p - \psi(a(F(t_2))^p - \\ & (1-a)\max\{\alpha(F(t_3))^p, \beta(F(t_4))^p, (F(t_3))^{\frac{p}{2}} \cdot (F(t_6))^{\frac{p}{2}}, \\ & (F(t_5))^{\frac{p}{2}} \cdot (F(t_6))^{\frac{p}{2}}\}), \end{aligned}$$

where $0 \leq \alpha, a, b \leq 1$ and $\psi \in \Psi$.

Example 4.8. $G(t_1, t_2, t_3, t_4, t_5, t_6) = F(t_1) - a(t_2)F(t_2) + b(t_2)(F(t_3) + F(t_4))$ $-c(t_2)\min\{F(t_5), F(t_6)\}$, where $a, b, c : [0, \infty) \to [0, 1)$ are increasing functions satisfying the condition $a(t) + 2b(t) + c(t) < 1$ for all $t > 0$.

Let $A = D = \sup_{x,y \in A} d(x, y)$ if $D = \infty$ and $A > D$ if $D < \infty$

By Theorems 3.2-3.3 and Examples 4.6-4.8, we get generalizations of the Theorems of [11], [16], [85], [14] and [22].

Remark 4.9. In the Theorems of [5] and [21], to prove that $z = Tz$, the authors used the inequality: $d(fx, gy) \leq H(Tx, Sy)$" which is false because "$a \in A$ and $b \in B$ implies $d(a, b) \leq H(A, B)$" is not true in general, as shown by the following example

Remark 4.10. Let $d(x, y) = |x - y|$, $A = [0, \frac{1}{2}]$ and $B = [\frac{1}{4}, 1]$. We have $0 \in A$ and $1 \in B$, but $d(0, 1) = 1 > H(A, B) = \frac{1}{2}$. Therefore, Theorems of [5] and [21] are false as it is proved by the following example.

Example 4.11. Let $X = \{0, 1\}, Sx = Tx = 1 - x$ and $Fx = Gx = \{0, 1\}$ for all $x \in X$.

We have $T(0) \in F(0)$ and $T(1) \in F(1)$; i.e., T and F have coincidence points. As $TF(0) = \{0, 1\} = FT(0)$ and $TF(1) = \{0, 1\} = FT(1)$, it follows that the pair (T, F) is weakly compatible and so it is owc. Since $T^2(0) \neq T(0)$ and $T^2(1) \neq T(1)$, T and F have no common fixed point.

To correct these errors, the function H in [5] and [21] should be replaced by the function δ.

There are also some errors in [5].

1) In the abstract of [5], the authors said: We obtain several fixed point theorems for hybrid pairs of single-valued and multivalued occasionally weakly compatible maps defined on a symmetric space satisfying a contractive condition of integral type, but their theorems were proved in metric spaces except for

Corollary 2.4. Therefore, in Theorems 2.1, 2.2, 2.6, 2.7 and Corollaries 2.3, 2.5, metric space should be replaced by symmetric space.

2) The condition (g_1) should be: g is nondecreasing in the 1st 4th and 5th variables.

3) The condition (g_2) is not needed in the proof of Theorem 2.6.

4) The condition (g_3) should be only: if $u \in \mathbb{R}_+$ is such that $u \leq g(u,0,0,u,u)$, then $u = 0$.

5) The condition $\phi(2t) \leq 2\phi(t)$ is not needed in the proof of Theorem 2.7.

There are also some errors in the paper [6].

1) In the abstract of [6], the authors said: We obtain several fixed point theorems for hybrid pairs of single valued and multivalued occasionally weakly compatible maps defined on a symmetric space, using, the δ distance, but their theorems were proved in metric spaces except for Corollary 2.4. Therefore, in Theorems 2.1, 2.6, 2.7 and Corollaries 2.2, 2.4, metric space should be replaced by symmetric space.

The same errors 2), 3), 4) and 5) of [6] are in [6].

Remark 4.12. In the proof of Lemma 1 of [84] and Theorem 2.1 of [34], the authors applied the inequality

$$a \leq b + c \Longrightarrow \int_0^a \varphi(t)dt \leq \int_0^b \varphi(t)dt + \int_0^c \varphi(t)dt$$

which is false in general, as shown by the following example.

Example 4.13. Let $\varphi(t) = t$, $a = 1$, $b = \dfrac{1}{2}$ and $c = \dfrac{3}{4}$. Then

$$\begin{aligned} \int_0^1 \varphi(t)dt &= \frac{1}{2} > \int_0^{\frac{1}{2}} \varphi(t)dt + \int_0^{\frac{3}{4}} \varphi(t)dt \\ &= \frac{1}{8} + \frac{9}{32} = \frac{13}{32}. \end{aligned}$$

To correct these errors, the authors should follow the proof of Theorem 2 of [73].

Remark 4.14. In the proof of Theorem 1 of [47], the authors applied the inequality

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \Longrightarrow \{x_n\} \text{ is a Cauchy sequence}$$

which is false in general. It suffices to take $x_n = \dfrac{1}{n}$, $n \in \mathbb{N}^*$. Thus, to correct this error, the authors should follow the proof of Theorem 2 of [69].

The following Examples support our Theorem 3.3 and 3.2 respectively.

Example 4.15. Let $X = [0,10]$ be endowed with the symmetric $d(x,y) = (x-y)^2$ and

$$Sx = \begin{cases} 3 \text{ if } x \in \,]0,2]\,, \\ 0 \text{ if } x \in \{0\} \cup \,]2,10] \end{cases}, \quad fx = \begin{cases} 0 \text{ if } x = 0, \\ x+2 \text{ if } x \in \,]0,2]\,, \\ x-2 \text{ if } x \in \,]2,10] \end{cases},$$

$$Tx = \begin{cases} 0 \text{ if } x \in [0,2]\,, \\ 4 \text{ if } x \in \,]2,10] \end{cases}, \quad gx = \begin{cases} 0 \text{ if } x = 0, \\ x+5 \text{ if } x \in \,]0,2]\,, \\ x-2 \text{ if } x \in \,]2,10] \end{cases}.$$

The pairs (S,f) and (T,g) are owc because $Sf(0) = fS(0) = Tg(0) = gT(0) = 0$, but $Sf(1) = 0 \neq fS(1) = 1$ and $Tg(6) = 4 \neq gT(6) = 2$.

Now, we begin to verify the rest of conditions of Theorem 3.3.

Let $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - h\max\{t_2,t_3,t_4,t_5,t_6)\}$, $0 < h \le 1$.

Now, we verify that (A,S) and (B,T) satisfy all the conditions of Theorem 3.2. Set

$$\begin{aligned} R &= d(Sx,Ty) - h\max\{d(fx,gy), d(fx,fx), \\ &\quad d(gy,Ty), d(gy,Sx), d(fx,Ty)\},\ 0 < h \le 1. \end{aligned}$$

We have the following cases.

1) If $x = 0$ and $y \in (0,2]$, we get $R = -h(y+5)^2 < 0$ for all $0 < h \le 1$.

2) If $x = 0$ and $y \in (2,10]$, we get

$$R = 16 - h \max_{y\in(2,10]} \{(y-2)^2, (y-6)^2, 16\} < 0$$

for $h > \frac{16}{64} = \frac{1}{4}$ and so there exists $0 < h \le 1$.

3) If $x \in (0,2]$ and $y = 0$, we get

$$R = 9 - h \max_{x\in(0,2]} \left\{(x+2)^2, (x-1)^2, 9\right\} < 0$$

for $h > \frac{9}{16}$ and so there exists $0 < h \le 1$.

4) If $x, y \in (0,2]$ we get

$$R = 9 - h\max\left\{ \begin{array}{c} (x-y-3)^2, (x-1)^2, (y+5)^2, \\ (y+2)^2, (x+2)^2 \end{array} \right\} < 0$$

for $h > \frac{9}{49}$ and so there exists $0 \le h \le 1$.

5) If $x \in (0,2]$ and $y \in (2,10]$, we get

$$R = 1 - h\max\left\{ \begin{array}{c} (x-y+4)^2, (x-1)^2, (y-6)^2, \\ (y-5)^2, (x-1)^2 \end{array} \right\} < 0$$

for $h > \frac{1}{36}$ and so there exists $0 \le h \le 1$.

6) If $x \in (2,10]$ and $y = 0$, we get $R < 0$ for all $0 < h \le 1$.

In the same manner, if $x \in (2,10]$ and $y \in (0,2]$, we get $R < 0$ for all $0 < h \leq 1$.

7) If $x, y \in (2,10]$ we get

$$R = 16 - h \max_{x,y\in(2,10]} \left\{ \begin{array}{c} (x-y)^2, (x-2)^2, (y-6)^2, \\ (y-2)^2, (x-6)^2 \end{array} \right\} < 0$$

for $h > \frac{16}{64} = \frac{1}{4}$ and so there exists $0 < h \leq 1$. All conditions of Theorem 3.4 are verified and 0 is the unique common fixed point of f, g, S and T.

Example 4.16. Let $X = [0,10]$ be endowed with the symmetric $d(x,y) = (x-y)^2$ and

$$Sx = \left\{ \begin{array}{c} [1,3] \text{ if } x \in \left]0,2\right], \\ \{0\} \text{ if } x \in \{0\} \cup \left]2,10\right] \end{array} \right., \ fx = \left\{ \begin{array}{c} 0 \text{ if } x = 0, \\ x+2 \text{ if } x \in \left]0,2\right], \\ x-2 \text{ if } x \in \left]2,10\right] \end{array} \right.,$$

$$Tx = \left\{ \begin{array}{c} \{0\} \text{ if } x \in [0,2], \\ [1,4] \text{ if } x \in \left]2,10\right] \end{array} \right., \ gx = \left\{ \begin{array}{c} 0 \text{ if } x = 0, \\ x+5 \text{ if } x \in \left]0,2\right], \\ x-2 \text{ if } x \in \left]2,10\right] \end{array} \right..$$

The pairs (S,f) and (T,g) are owc because $Sf(0) = fS(0) = Tg(0) = gT(0) = \{0\}$, but $Sf(1) = \{0\} \neq fS(1) = f([1,3]) = (0,1] \cup [3,4]$ and $Tg(3) = \{0\} \neq gT(3) = g([1,4]) = [6,7] \cup (0,2]$.

Now, we begin to verify the rest of conditions of Theorem 3.2.

Let $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h \max\{t_2, t_3, t_4, t_5, t_6)\}$, $0 < h \leq 1$.

Now, we verify that (A,S) and (B,T) satisfy all the conditions of Theorem 4.2. Set

$$\begin{aligned} R &= \delta(Sx,Ty) - h\max\{d(fx,gy), D(fx,Sx), \\ &\quad D(gy,Ty), D(gy,Sx), D(fx,Ty)\} \end{aligned}$$

We have the following cases.

1) If $x = 0$ and $y \in (0,2]$, we get $R = -h(y+5)^2 < 0$ for all $0 < h \leq 1$.

2) If $x = 0$ and $y \in (2,10]$, we get

$$\begin{aligned} \delta(Sx,Ty) &= 16 = \frac{1}{4} \max_{y\in(2,10]} (y-2)^2 \\ &= \frac{1}{4} \max_{y\in(2,10]} d(fx,gy) \\ &< \frac{1}{4} \max\{d(fx,gy), D(fx,Sx), D(gy,Ty), \\ &\quad D(gy,Sx), D(fx,Ty)\} \end{aligned}$$

4) If $x \in (0,2]$ and $y = 0$, we get

$$\begin{aligned}\delta(Sx,Ty) &= 9 = \frac{9}{16}\max_{x\in(0,2]}(x+2)^2 \\ &= \frac{9}{16}\max_{x\in(0,2]} d(fx,gy) \\ &\leq \frac{9}{16}\max\{d(fx,gy), D(fx,Sx), D(gy,Ty), \\ & D(gy,Sx), D(fx,Ty)\}\end{aligned}$$

5) If $x, y \in (0,2]$ we get

$$\begin{aligned}\delta(Sx,Ty) &= 9 = \frac{9}{49}\max_{y\in(0,2]}(y+5)^2 \\ &= \frac{9}{49}\max_{y\in(0,2]} D(gy,Ty). \\ &< \frac{9}{49}\max\{d(fx,gy), D(fx,Sx), D(gy,Ty), \\ & D(gy,Sx), D(fx,Ty)\}\end{aligned}$$

6) If $x \in (0,2]$ and $y \in (2,10]$, we get

$$\begin{aligned}\delta(Sx,Ty) &= 9 = \frac{9}{25}\max_{y\in(0,2]}(y-5)^2 \\ &= \frac{9}{25}\max_{y\in(0,2]} D(gy,Sx). \\ &< \frac{9}{25}\max\{d(fx,gy), D(fx,Sx), D(gy,Ty), \\ & D(gy,Sx), D(fx,Ty)\}\end{aligned}$$

7) If $x \in (2,10]$ and $y = 0$, we get $R < 0$ for all $0 < h \leq 1$.

In the same manner, if $x \in (2,10]$ and $y \in (0,2]$, we get $R < 0$ for all $0 < h \leq 1$.

8) If $x, y \in (2,10]$ we get

$$\begin{aligned}\delta(Sx,Ty) &= 16 = \frac{16}{64}\max_{y\in(0,2]}(y-2)^2 \\ &= \frac{1}{4}\max_{y\in(0,2]} D(gy,Sx). \\ &< \frac{1}{4}\max\{d(fx,gy), D(fx,Sx), D(gy,Ty), \\ & D(gy,Sx), D(fx,Ty)\}\end{aligned}$$

All the conditions of Theorem 3.2 are satisfied and 0 is the unique common fixed point of f, g, S and T which is a stationary point of S and T.

Acknowledgements

The authors would like to thank the referee for his valuable comments and suggestions.

References

[1] Aamri, M., El Moutawakil, D., Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. 270 (2002), 181-188.

[2] Aamri, M., El Moutawakil, D., Common Fixed Points Under Contractive Conditions In Symmetric Spaces. Applied Mathematics E-Notes 3 (2003), 156-162.

[3] Aamri, M., Bassou, A., EL Moutawakil, D., Common fixed points for weakly compatible maps in symmetric spaces with application to probabilistic spaces. Applied Mathematics E-Notes 5 (2005), 171-175.

[4] Aamri, M., Bassou, A., Bennani, S. El Moutawakil, D., Common Fixed Points of Mappings and Set-Valued Mappings in Symmetric Spaces with Application to Probabilistic Spaces. Journal of Applied Mathematics and Stochastic Analysis, Volume 2007 (2007), Article ID 59204, 7 pages.

[5] Abbas, M., Rhoades, B. E., Common Fixed Point Theorems for Hybrid Pairs of Occasionally Weakly Compatible Mappings Satisfying Generalized Contractive Condition of Integral Type. Fixed Point Theory and Applications, Volume 2007 (2007), Article ID 54101, 9 pages.

[6] Abbas, M., Rhoades, B. E., Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings defined on symmetric spaces. PanAmerican Mathematical Journal 18 (1) (2008), 55–62.

[7] M. Abbas and B. E Rhoades, Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type revisited, erratum, Fixed Point Theory and Appl. (2008), ID274793, 2 pages.

[8] Ahmed, M. A., Common fixed point theorems for weakly compatible mappings. Rocky Mountain J. Math. 33 (2) (2003), 1189-1203.

[9] Ahmed, M. A., Common fixed point theorems under contractive conditions of skof type. PU.M.A. 15 (1) (2004), 17-27.

[10] Akkouchi, M., A Common Fixed Point Theorem Connected to a Result of V. Popa and H. K. Pathak. Georgian Math. Journal. 13 (1) (2006), 1-6.

[11] Aliouche, A., A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type. J. Math. Anal. Appl. 322 (2) (2006), 796-802.

[12] Aliouche, A., A common fixed point theorem for weakly compatible mappings in compact metric spaces satisfying an implicit relation. Sarajevo J. Math. 3 (1) (2007), 1-8.

[13] Aliouche, A., Djoudi, A., Common fixed point theorems for mappings satisfying an implicit relation without decreasing assumption. Hacettepe J. Math. Stat. 36 (1) (2007), 11-18

[14] Aliouche, A., Common fixed point theorems of Gregus type for weakly compatible mappings satisfying generalized contractive conditions. J. Math Anal. Appl. 341 (1) (2008), 707-719.

[15] Al-Thagafi, M. A., Shahzad, N., Generalized I-nonexpansive self maps and invariant approximations. Acta. Math. Sinica. 24 (5) (2008), 867-876.

[16] Altun, I., Turkoglu, D., Rhoades, B. E., Fixed points of weakly compatible mappings satisfying a general contractive condition of integral type. Fixed Point Theory And Applications, Volume 2007 (2007), Article ID 17301, 9 pages.

[17] Altun, İ., Türkoğlu, D., Some fixed point theorems for weakly compatible multivalued mappings satisfying an implicit relation. Filomat 22 (1) (2008), 13-23.

[18] Altun, İ., Türkoğlu, D., Some fixed point theorems for weakly compatible mappings satisfying an implicit relation. Taiwanese J. Math., to appear..

[19] Altun, İ., Türkoğlu, D., A fixed point theorem for weakly compatible mappings satisfying a general contractive condition of operator type. to appear.

[20] Bouhadjera, H., General common fixed point theorems for compatible mappings of type (C). Sarajevo J. Math. 1 (2) (2005), 261-270.

[21] Bouhadjera, H., Godet-Thobie, C., Common fixed point theorems for occasionally weakly compatible maps. HAL: hal-00273238, version 1.

[22] Bouhadjera, H., Djoudi, A., Common fixed point theorems for single and set-valued maps. Analele Universităţii Oradea Fasc. Matematica, Tom XV (2008), 109-127.

[23] Branciari, A., A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. 29 (2002), 531-536.

[24] Ciric, L., Nikoli, N. T., Ume, J. S., Common fixed point theorems for families of weakly compatible quasi contraction mappings. Acta Math. Hungar. 113 (4) (2006), 257-267.

[25] Ciric, L., Razani, A., Radenovic, S., Ume, J. S., Common fixed point theorems for families of weakly compatible maps. Computers and Mathematics with Applications 55 (2008), 2533-2543.

[26] Djoudi, A., Nisse, L., Gregus type fixed points for weakly compatible mappings. Bull. Belg. Math. Soc. 10 (2003), 369-378.

[27] Djoudi, A., Khemis, R., Fixed points for set and single valued maps without continuity. Demonstratio Math. 38 (3) (2005), 739-751.

[28] Djoudi, A., Aliouche, A., A general common fixed point theorem for reciprocally continuous mappings satisfying an implicit relation. The Austral. J. Math. Anal. Appl. 3 (2006), 1-7.

[29] Djoudi, A., Aliouche, A., Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type. J. Math Anal. Appl. 329 (1) (2007), 31-45.

[30] Djoudi, A., Merghadi, F., Common fixed point theorems for maps under a contractive condition of integral type. J. Math. Anal. Appl. 341 (2) (2008), 953-960.

[31] Elamrani, M., Mehdaoui, B., Common fixed point theorems for compatible and weakly compatible mappings. Revista Colombiana de Matematicas 34 (2000), 25-33

[32] Fisher, B., Common fixed points of mappings and set-valued mappings. Rostock. Math. Kolloq. 18 (1981) 69-77.

[33] Fisher, B., Common fixed points of mappings and set-valued mappings on a metric spaces. Kyungpook Math. J. 25 (1985), 35 42.

[34] Gairola, U. C., Rawat, A. S., A fixed point theorem for integral type inequality. Int. Journal of Math. Analysis, Vol. 2 no. 15 (2008), 709-712.

[35] Hicks, T. L., Rhoades, B. E., Fixed point theory in symmetric spaces with applications to probabilistic spaces. Nonlinear Analysis 36 (1999), 331-344.

[36] Imdad, M., Kumar, S., Khan, M. S., Remarks on some fixed point theorems satisfying implicit relations. Radovi Mat. 1 (2002), 135-143.

[37] Imdad, M., Ali, Javid, Khan, Ladlay, Coincidence and fixed points in symmetric spaces under strict contractions, J. Math. Anal. Appl. 320 (1) (2006), 352-360.

[38] M. Imdad, Javid Ali and Ladlay Khan, Corrigendum to "Coincidence and fixed points. in symmetric spaces under strict contractions". J. Math. Anal. Appl. 320 (2006) 352–360.

[39] Imdad, M., Ali, Javid, Jungck's common fixed point theorem and E.A property. Acta Math. Sinica 24 (1) (2008), 87-94

[40] Jungck, G., Compatible mappings and common fixed points. Int. J. Math. Math. Sci. 9 (1986), 771-779.

[41] Jungck, G., Rhoades, B. E., Some fixed points theorems for compatible mappings. Int. J. Math. Math. Sci. 16 (1993), 417-428.

[42] Jungck, G., Common fixed points for non-continuous non-self maps on non-metric spaces. Far East J. Math. Sci. 4 (2) (1996), 199-215.

[43] Jungck, G., Rhoades, B. E., Fixed points for set valued functions without continuity. Indian J. Pure Appl. Math. 29 (3) (1998), 227–238.

[44] Jungck, G., Rhoades, B. E., Fixed point theorems for occasionally weakly compatible mappings. Fixed Point Theory 7 (2) (2006), 287-296.

[45] Jungck, G., Rhoades, B. E., Fixed point theorems for occasionally weakly compatible mappings, erratum. Fixed Point Theory 9 (1) (2008), 383-384.

[46] Beg, I., Abbas, M., Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition. Fixed Point Theory and Applications, vol. 2006, Article ID 74503, 7 pages, 2006

[47] Kohli, J. K., Vashistha, S., Common fixed point theorems for compatible and weakly compatible mappings satisfying general contractive type conditions. Stud. Cerc. Şt, Seria Matematică, Univ. Bacău, 16 (2006), 33-42.

[48] Kumar, S., Chugh, R., Kumar, R., Fixed point theorem for compatible mappings satisfying a contractive condition of integral type. Soochow J. Math. 33 (2) (2007), 181-185.

[49] Li, J., Fu, M., Liu, Z., Kang, S. M., A Common Fixed Point Theorem and its Application in Dynamic Programming. Applied Mathematical Sciences, Vol. 2 no. 17 (2008), 829-842.

[50] Liu, Z., Liu, Q., Stationary points for set-valued mappings on two metric spaces. Rostock. Math. Kolloq. 55 (2001), 23-29.

[51] Liu, Z., Guo, Z., Kang, S. M., Shim, S. H., Common fixed point theorems for compatible mappings of type (P) and an application to dynamic programming. J. Appl. Math. & Informatics Vol. 26 (1-2) (2008), 61-73.

[52] Pant, R. P., Common fixed points of noncommuting mappings. J. Math. Anal. Appl. 188 (1994), 436-440.

[53] Pant, R. P., Common fixed points for four mappings. Bull. Calcutta Math. Soc. 9 (1998), 281-286.

[54] Pant, R. P., A Common fixed point theorem under a new condition. Indian J. Pure. Appl. Math. 30 (2) (1999), 147-152.

[55] Pant, R. P., R-weakly commutativity and common fixed points. Soochow J. Math. 25 (1) (1999), 29-36.

[56] Pant, R. P., Pant, V., Common fixed points under strict contractive conditions. J. Math. Anal. Appl. 248 (1) (2000), 327-332.

[57] Pant, R. P., Pant, V., Jha, K., Note on common fixed points under strict contractive conditions. J. Math. Anal. Appl. 274 (2) (2002), 879-880.

[58] Pathak, H. K., Khan, M. S., Compatible mappings of type (B) and common fixed point theorems of Gregus type. Czechoslovak Math. J. 45 (120) (1995), 685-698.

[59] Pathak, H. K., Cho, Y. J., Kang, S. M., Lee, B. S., Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming. Le Matematiche. 1 (1995), 15-33.

[60] Pathak, H. K., Khan, M. S., Rakesh, T., A common fixed point theorem and its application to nonlinear integral equations. Computers Math. Appl. 53 (2007), 961-971.

[61] Pathak, H. K., Tiwari, R., Khan, M. S., A Common Fixed Point Theorem Satisfying Integral Type Implicit Relations. Applied Mathematics E-Notes 7 (2007), 222-228.

[62] Pathak, H. K., Rodriguez-Lopez, Rosana, Verma, R. K., A Common Fixed Point Theorem using implicit relation and property (E.A) in metric spaces. Filomat 21 (2) (2007), 211–234.

[63] Pathak, H. K., Verma, R. K., Coincidence and common fixed points in symmetric spaces under implicit relation and application. International Mathematical Forum 3 no. 30 (2008), 1489-1499.

[64] Popa, V., Fixed point theorems for implicit contractive mappings. Stud. Cerc. St. Ser. Mat. Univ. Bacau 7 (1997), 127-133.

[65] Popa, V., Some fixed point theorems for compatible mappings satisfying an implicit relation. Demonsratio Math. 32 (1) (1999), 157-163.

[66] Popa, V., A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation. Demonstratio Math. 33 (1) (2000), 159-164.

[67] Popa, V., A general fixed point theorem for weakly δ-compatible mappings in compact metric spaces. Stud. Cerc. St. Ser. Mat. Univ. Bacau 14 (2004), 129-134.

[68] Popa, V., Mocanu, M., A new view point in the study of fixed points for mappings satisfying a contractive condition of integal type. Bul. Inst. Politehnic Iasi Sec. Mat. Mec. Teor. Fiz. 5 (2007), 269-282.

[69] Popa, V., Mocanu, M., Some fixed point theorems for mappings satisfying implicit relations in symmetric spaces (to appear).

[70] Rao, K. P. R., Som Babu, A., Vasu Babu, D., Common Fixed Points Through Generalized Altering Distance Function. International Mathematical Forum 2 no. 65 (2007), 3233-3239.

[71] Rao, K. P. R., Ravi Babu, G., Vasu Babu, D., Common fixed point theorems through generalized altering distance functions. Mathematical Communications 13 (2008), 67-73.

[72] Rashwan, R. A., Ahmed, M. A., Common fixed points for weakly δ-compatible mappings. Italian J. Pure Apl. Math. 8 (2000), 33-44.

[73] Rhoades, B. E., Two fixed point theorems for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. 63 (2003), 4007-4013.

[74] Rhoades, B. E., Common fixed points of compatible set-valued mappings. Publicationes Mathematicae Debrecen, 48 (3-4) (1996), 237-240.

[75] Rus, A., Fixed and strict fixed points for multi-valued mappings. Univ. Cluj, preprint, 3 (1985).

[76] Sahu, D. R., Imdad, M., Kumar, S., Fixed point theorems in symmetrizable topological spaces. Nonlinear Analysis Forum 9 (2004), 97-107.

[77] Sedghi, S., Altun, I., Shobe, N., A fixed point theorem for multi-maps satisfying an implicit relation on metric spaces. Appl. Anal. Discrete Math. 2 (2008), 189-196.

[78] Sessa, S., On a weak commutativity condition of mappings in fixed point considerations. Publ. Inst. Math. Beograd. 32 (46) (1982), 149-153.

[79] Sessa, S., Fisher, B., On common fixed point of weakly commuting mappings and set-valued mappings. Int. J. Math. & Math. Sci. 9 (2) (1986), 323-329

[80] Türkoğlu, D., Altun, İ., A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation. Bul. de la Soc. Math. Mexicana 13 (1) (2007).

[81] Tas, K., Telci, M., Fisher, B., Common fixed point theorems for compatible mappings. Int. J. Math. Math. Sci. 19 (1996), 451-456.

[82] Türkoğlu, D., Altun, İ., Fisher, B., Common fixed point theorems for four mappings with some weak conditions of commutativity. Novi Sad J. Math. 36 (1) (2006), 75-86.

[83] Türkoğlu, D., Altun, İ., Fisher, B., Fixed point theorem for sequences of maps. Demonstratio Math. 38 (2) (2005), 461-468.

[84] Vijayaraju, P., Rhoades, B. E., Mohanraj, R., A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. 15 (2005), 2359-2364.

[85] Zhang, X., Common fixed point theorems for some new generalized contractive type mappings. J. Math. Anal. Appl. 333 (2) (2007), 780-786.

Received by the editors September 19, 2008