Novi Sad J. Math.
Vol. 39, No. 1, 2009, 47-56

ON A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS INVOLVING CHO-SRIVASTAVA OPERATOR

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Abstract. The authors introduce a new subclass $\mathcal{UH}(\alpha,\, \beta, \gamma,\, \delta,\, \lambda,\, k)$ of functions which are analytic in the open disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Various results studied include the coefficient estimates and distortion bounds, radii of close-to-convexity, starlikeness and convexity and integral means inequalities for functions belonging to the above class. Relevances of the main results are also briefly indicated.

AMS Mathematics Subject Classification (2000): 30C45

Key words and phrases: Analytic function, Starlike function, Convex function, Uniformly convex function, Convolution product, Cho-Srivastava operator

1. Introduction and Motivations

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad (z \in \Delta)$$

for $0 \leqq \beta < 1$. In particular, $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{K} = \mathcal{K}(0)$, respectively, are the well-known standard classes of starlike and convex functions. Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geqq 0), \tag{2}$$

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which are analytic in the open unit disc Δ, introduced and studied in [10]. Analogously to the subclasses $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of $\mathcal{S}$, respectively, the subclasses of $\mathcal{T}$ denoted by $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$, $0 \leqq \beta < 1$, have also been investigated in [10]. For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in \Delta). \tag{3}$$

Also, for functions $f \in \mathcal{A}$, we recall the multiplier transformation $I(\lambda, k)$ introduced by Cho and Srivastava [3] which is defined by

$$I(\lambda, k) f(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n \quad (\lambda \geqq 0;\ k \in \mathbb{Z}), \tag{4}$$

where

$$\Psi_n := \left(\frac{n + \lambda}{1 + \lambda} \right)^k \tag{5}$$

and, obviously it follows that

$$z\,(I(\lambda, k)\ f(z))' = (1 + \lambda) I(\lambda, k + 1)\ f(z) - \lambda I(\lambda, k)\ f(z). \tag{6}$$

In the special case when $\lambda = 1$, the operators $I(1, k)$ were studied earlier by Uralegaddi and Somanatha [15]. It may be observed that the operators $I(\lambda, k)$ are closely related to the multiplier transformations studied by Flett [4] and also to the differential and integral operators investigated by Sălăgean [8]. For comprehensive details of various convolution operators which are related to the multiplier transformations of Flett [4], one may refer to the paper of Li and Srivastava [5] (as well as the references cited by therein). For the purpose of this paper, we now define a unified class of analytic functions which is based on the Cho-Srivastava operator (1.4).

Definition 1. For $0 \leqq \delta \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$ and $\alpha \geqq 0$, and for all $z \in \Delta$, we let the class $\mathcal{UH}(\alpha, \beta, \gamma, \delta, \lambda, k)$ consist of functions $f \in \mathcal{T}$ which satisfy the condition

$$\Re \left(\frac{zF'(z)}{F(z)} - \beta \right) > \alpha \left| \frac{zF'(z)}{F(z)} - 1 \right|, \tag{7}$$

where

$$F(z) := F_1(z) + F_2(z) + F_3(z), \tag{8}$$

and

$$F_1(z) := \gamma\delta\,(1 + \lambda)^2\, I(\lambda, k + 2) f(z), \tag{9}$$

$$F_2(z) := \{\gamma - \delta - \gamma\delta(1 + 2\lambda)\}\,(1 + \lambda)\, I(\lambda, k + 1) f(z), \tag{10}$$

$$F_3(z) := \{1 - (\lambda + 1)(\gamma - \delta - \gamma\delta\lambda)\}\, I(\lambda, k) f(z), \tag{11}$$

and $I(\lambda, k) f(z)$ is the Cho-Srivastava operator defined by (4).

The function class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \lambda,\ k)$ unifies many well known classes of analytic univalent functions. To illustrate, we observe that the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ 0,\ 0)$ was studied by Kamali and Kadioglu [7] and the class $\mathcal{UH}(0,\ \beta, 0,\ \gamma,\ 0,\ 0)$ was studied by Altintas in [1]. Also, many other classes including $\mathcal{UH}(0,\ \beta, 0,\ 0,\ 0,\ 0)$ and $\mathcal{UH}(0,\ \beta, 1,\ 0,\ 0,\ 0)$ were investigated by Srivastava *et al.* [14]. We further note that the class $\mathcal{UH}(\alpha,\ \beta, 0,\ \gamma,\ 0,\ 0)$ is the known class of $\alpha-$uniformly convex functions of order β studied by Aqlan *et al.* [2] (also see [13]).

In the present paper we obtain a characterization property giving coefficients estimates, distortion theorem and covering theorem, extreme points and radii of close-to-convexity, starlikeness and convexity and integral means inequalities for functions belonging to the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \lambda,\ k)$.

2. Coefficient estimates and Distortion bounds

Theorem 1. *Let $f \in \mathcal{T}$ be given by* (2)*, then $f \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \lambda,\ k)$ if and only if*

$$\sum_{n=2}^{\infty} \left[\{n(\alpha+1)-(\alpha+\beta)\} \{(n-1)(n\gamma\delta+\gamma-\delta)+1\} \right] \Psi_n a_n \leqq 1-\beta, \tag{12}$$

where $0 \leqq \delta \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1$ *and* $\alpha \geqq 0$. *The result is sharp for the function*

$$f(z) = z - \frac{1-\beta}{\{n(\alpha+1)-(\alpha+\beta)\} \{(n-1)(n\gamma\delta+\gamma-\delta)+1\}} z^n \ (\ n \geq 2). \tag{13}$$

Proof. Following [2], we assert that $f \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \lambda,\ k)$ if and only if the condition (7) is satisfied and this is equivalent to

$$\Re \left\{ \frac{zF'(z)(1+\alpha e^{i\theta}) - F(z)\alpha e^{i\theta}}{F(z)} \right\} > \beta \qquad (-\pi \leqq \theta < \pi). \tag{14}$$

By putting $G(z) = zF'(z)(1+\alpha e^{i\theta}) - F(z)\alpha e^{i\theta}$, (14) is equivalent to

$$|G(z) + (1-\beta)F(z)| > |G(z) - (1+\beta)F(z)| \ (0 \leqq \beta < 1),$$

where $F(z)$ is as defined in (8). Simple computations readily give

$$|G(z) + (1-\beta)F(z)| \geqq (2-\beta)|z| - \\ - \sum_{n=2}^{\infty} \left[\left\{ n(\alpha+1)-(\alpha+\beta)+1 \right\} \left\{ (n-1)(n\gamma\delta+\gamma-\delta)+1 \right\} \right] \Psi_n a_n |z|^n$$

and

$$|G(z) - (1+\beta)F(z)| \leqq \beta|z| + \\ + \sum_{n=2}^{\infty} \left[\left\{ n(\alpha+1)-(\alpha+\beta) \right\} \left\{ (n-1)(n\gamma\delta+\gamma-\delta)+1 \right\} \right] \Psi_n a_n |z|^n.$$

It follows that

$$|G(z)+(1-\beta)F(z)|-|G(z)-(1+\beta)F(z)| \geqq 2(1-\beta)|z|-$$
$$-\quad 2\sum_{n=2}^{\infty}\left[\left\{ n(\alpha+1)-(\alpha+\beta)\right\}\left\{ (n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n a_n|z|^n \geqq 0,$$

which implies that $f\in\mathcal{UH}(\alpha,\ \beta,\gamma,\ \delta,\ \lambda,\ k)$.

On the other hand, for all $-\pi \leqq \theta < \pi$, we assume that

$$\Re\left\{\frac{zF'(z)}{F(z)}(1+\alpha e^{i\theta})-\alpha e^{i\theta}\right\} > \beta.$$

Choosing the values of z on the positive real axis such that $0\leqq|z|=r<1$, and using the fact that $\Re\{-e^{i\theta}\}\geqq -|e^{i\theta}|=-1$, the above inequality can be written as

$$\Re\left\{\frac{(1-\beta)-\sum_{n=2}^{\infty}\left[\left\{ n(\alpha+1)-(\alpha+\beta)\right\}\left\{(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n a_n r^{n-1}}{1-\sum_{n=2}^{\infty}\left[\left\{ (n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n a_n r^{n-1}}\right\}\geqq 0,$$

which on letting $r\to 1^-$ yields the desired inequality (2.1). □

Theorem 2. *If $f\in\mathcal{UH}(\alpha,\ \beta,\gamma,\ \delta,\ \lambda,\ k)$, then*

$$a_n \leqq \frac{1-\beta}{\left[\left\{ n(\alpha+1)-(\alpha+\beta)\right\}\left\{ (n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}\quad (n\geqq 2), \tag{15}$$

where $0\leqq\delta\leqq\gamma\leqq 1,\ \ 0\leqq\beta<1$ and $\alpha\geqq 0$. Equality in (15) holds for the function

$$f(z)=z-\frac{1-\beta}{\left[\left\{ n(\alpha+1)-(\alpha+\beta)\right\}\left\{ (n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}z^n. \tag{16}$$

It may be observed that for $\lambda=k=\delta=\gamma-1=\alpha=0$, Theorem 1 corresponds to the following results due to Silverman [10].

Corollary 1. *([10]) If $f\in\mathcal{T}$, then $f\in\mathcal{K}(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)a_n \leqq 1-\beta.$$

Corollary 2. *([10]) If $f\in\mathcal{K}(\beta)$, then $f\in\mathcal{T}^*\left(\frac{2}{3-\beta}\right)$. The result is sharp for the extremal function*

$$f(z)=z-\frac{1-\beta}{2(2-\beta)}z^2.$$

Theorem 3. *If* $f \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$, *then* $f \in \mathcal{T}^*(\eta)$, *where*

$$\eta = 1 - \frac{1-\beta}{\left[\left\{\ 2(\alpha+1)-(\alpha+\beta)\right\}\left\{\ 2\gamma\delta+\gamma-\delta+1\right\}\right]\Psi_2-(1-\beta)}.$$

This result is sharp with the extremal function given by

$$f(z) = z - \frac{1-\beta}{\left[\left\{\ 2(\alpha+1)-(\alpha+\beta)\right\}\left\{\ 2\gamma\delta+\gamma-\delta+1\right\}\right]\Psi_2}z^2.$$

Proof. It is sufficient to show that (12) implies

$$\sum_{n=2}^{\infty}(n-\eta)a_n \leqq 1-\eta.$$

In view of (2.4), we find that

(17)
$$\frac{n-\eta}{1-\eta} \leqq \frac{\left[\left\{\ n(\alpha+1)-(\alpha+\beta)\right\}\left\{\ (n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}{1-\beta} \quad (n \geqq 2).$$

For $n \geqq 2$, (17) is equivalent to

$$\eta \leqq 1 - \frac{(n-1)(1-\beta)}{\left[\left\{\ n(\alpha+1)-(\alpha+\beta)\right\}\left\{\ (n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n-(1-\beta)} = \Phi(n),$$

and since $\Phi(n) \leqq \Phi(2)(n \geqq 2)$, therefore, (17) holds true for any $0 \leqq \delta \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$ and $\alpha \geqq 0$. This completes the proof of Theorem 3. □

The following results give the growth and distortion bounds for the class of functions $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$ which can be established by adopting the well known methods of derivation and we omit the proof details.

Theorem 4. *If* $f \in \mathcal{UH}(\alpha,\ \beta, \gamma, \delta,\ \lambda,\ k)$, *then* ($z = re^{i\theta} \in \Delta$)*:*

(18) $$r - B(\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda)r^2 \leqq |f(z)| \leqq r + B(\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda)r^2,$$

where

(19) $$B(\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda) := \frac{1-\beta}{\left[\left\{\ 2(\alpha+1)-(\alpha+\beta)\right\}\left\{\ 2\gamma\delta+\gamma-\delta+1\right\}\right]\Psi_2}$$

and Ψ_2 *is given by* (5)

Theorem 5. *If* $f \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$, *then* ($|z| = r < 1$)*:*

(20) $$1 - B(\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda)r \leqq |f'(z)| \leqq 1 + B(\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda)r\,,$$

where $B(\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda)$ *is given by (2.8).*

The equality in Theorems 4 and 5 hold for the function given by

$$f(z) = z - \frac{1-\beta}{\left[\left\{\ 2(\alpha+1)-(\alpha+\beta)\right\}\left\{\ 2\gamma\delta+\gamma-\delta+1\right\}\right]\Psi_2}z^2.$$

3. Radii of close-to-convexity, starlikeness and convexity

The following results giving the radii of convexity, starlikeness and convexity for a function f to belong to the class $\mathcal{UH}(\alpha,\,\beta,\gamma,\,\delta,\;\lambda,\,k)$ can be established by following similar lines of proof as given in [13] and [14]. We merely state here these results and omit their proof details.

Theorem 6. *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{UH}(\alpha,\,\beta,\gamma,\,\delta,\;\lambda,\,k)$, then $f(z)$ is close-to-convex of order ρ $(0 \leqq \rho < 1)$ in $|z| < r_1(\alpha,\,\beta,\gamma,\delta,\rho)$, where*

$$r_1(\alpha,\,\beta,\gamma,\delta,\rho) = \\ = \inf_n \left[\frac{(1-\rho)\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}{n(1-\beta)}\right]^{\frac{1}{n-1}}$$

for $n \geqq 2$ with Ψ_n defined as in (5)*. The result is sharp for the function $f(z)$ given by* (13)*.*

Theorem 7. *Let the function $f(z)$ defined by* (2) *be in the class $\mathcal{UH}(\alpha,\,\beta,\gamma,\,\delta,\;\lambda,\,k)$, then $f(z)$ is starlike of order ρ $(0 \leqq \rho < 1)$ in $|z| < r_2(\alpha,\,\beta,\gamma,\delta,\rho)$, where*

$$r_2(\alpha,\,\beta,\gamma,\delta,\rho) = \\ = \inf_n \left[\frac{(1-\rho)\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}}$$

for $n \geqq 2$ with Ψ_n defined as in (5)*. The result is sharp for the function $f(z)$ given by* (13)*.*

Theorem 8. *Let the function $f(z)$ defined by* (2) *be in the class $\mathcal{UH}(\alpha,\,\beta,\gamma,\,\delta,\;\lambda,\,k)$, then $f(z)$ is convex of order ρ $(0 \leqq \rho < 1)$ in $|z| < r_3(\alpha,\,\beta,\gamma,\delta,\rho)$, where*

$$r_3(\alpha,\,\beta,\gamma,\delta,\rho) = \\ = \inf_n \left[\frac{(1-\rho)\left[\{\,n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}{n(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}}$$

for $n \geqq 2$ with Ψ_n defined as in (5)*. The result is sharp for the function $f(z)$ given by* (13)*.*

4. Extreme points of the class $\mathcal{UH}(\alpha,\,\beta,\,\gamma,\,\delta,\,\lambda,\,k)$

Theorem 9. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{1-\beta}{\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n} z^n, \tag{21}$$

for $n \geqq 2$ *and* Ψ_n *be as defined in* (5). *Then* $f \in \mathcal{UH}(\alpha,\, \beta, \gamma,\, \delta,\; \lambda,\, k)$ *if and only if it can be represented in the form*

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \quad (\mu_n \geqq 0), \quad \sum_{n=1}^{\infty} \mu_n = 1. \tag{22}$$

Proof. Suppose $f(z)$ is expressible in the form (22). Then

$$f(z) = z - \sum_{n=2}^{\infty} \mu_n \left\{ \frac{1-\beta}{\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n} \right\} z^n.$$

Since

$$\sum_{n=2}^{\infty} \mu_n \frac{\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n(1-\beta)}{(1-\beta)\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}$$

$$= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leqq 1,$$

which implies that $f \in \mathcal{UH}(\alpha,\, \beta, \gamma,\, \delta,\; \lambda,\, k)$. Conversely, suppose $f \in \mathcal{UH}(\alpha,\, \beta, \gamma,\, \delta,\; \lambda,\, k)$. Using (15), we may write

$$\mu_n = \frac{\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n}{1-\beta} a_n \quad (n \geqq 2)$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$. This gives $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, where $f_n(z)$ is given by (21). □

Corollary 3. *The extreme points of* $f \in \mathcal{UH}(\alpha,\, \beta, \gamma,\, \delta,\; \lambda,\, k)$ *are the functions* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\beta}{\left[\{n(\alpha+1)-(\alpha+\beta)\}\left\{\,(n-1)(n\gamma\delta+\gamma-\delta)+1\right\}\right]\Psi_n} z^n \quad (n \geqq 2).$$

Theorem 10. *The class* $\mathcal{UH}(\alpha,\, \beta, \gamma,\, \delta,\; \lambda,\, k)$ *is a convex set.*

Proof. Suppose the functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,\,j} z^n \quad (a_{n,\,j} \geqq 0; \quad j = 1, 2) \tag{23}$$

be in the class $\mathcal{UH}(\alpha,\, \beta, \gamma,\, \delta,\; \lambda,\, k)$. It sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \leqq \mu \leqq 1)$$

is in the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$. Since

$$g(z) = z - \sum_{n=2}^{\infty}[\mu a_{n,1} + (1-\mu)a_{n,2}]z^n,$$

and applying Theorem 1, we get

$$\sum_{n=2}^{\infty}\Big[\{n(\alpha+1)-(\alpha+\beta)\}\Big\{(\,(n-1)(n\gamma\delta+\gamma-\delta)+1\Big\}\Big]\,\Psi_n[\mu a_{n,1}+(1-\mu)a_{n,2}]$$
$$\leqq \mu(1-\beta)+(1-\mu)(1-\beta) \leqq 1-\beta,$$

which asserts that $g \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$. Hence $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$ is convex. □

5. Integral Means Inequalities

Lemma 1. *([6]) If the functions f and g are analytic in Δ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$:*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta. \tag{24}$$

In [10], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to obtain the following integral means inequality (which was conjectured in [11] and settled in [12]):

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\eta} d\theta$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [12], he also proved his conjecture for the subclasses $T^*(\beta)$ and $C(\beta)$ of $\mathcal{T}$.

In this section, we obtain integral means inequalities for the functions in the family $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$. By assigning appropriate values to the parameters $\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k$, we can deduce various integral means inequalities for various known as well as new subclasses. We prove the following result.

Theorem 11. *Suppose $f(z) \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \delta,\ \ \lambda,\ k)$ and $\eta > 0$. If $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\beta}{\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,2)}z^2,$$

where

(25)

$$\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,2) = (2-\beta)\Big[\Big\{2(\alpha+1)-(\alpha+\beta)\Big\}\Big\{\,(2\gamma\delta+\gamma-\delta)+1\Big\}\Big]\,\Psi_2$$

$$\Psi_2 := \left(\frac{2+\lambda}{1+\lambda}\right)^k,$$

then for $z = re^{i\theta}$ $(0 < r < 1)$*:*

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{26}$$

Proof. For

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,$$

(26) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{(1-\gamma)}{\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,2)} z\right|^\eta d\theta.$$

By Lemma 1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,2)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\gamma}{\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,2)} w(z), \tag{27}$$

and using (12), we obtain

$$\begin{aligned} |w(z)| &= \left|\sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,n)}{1-\gamma} |a_n| z^{n-1}\right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,n)}{1-\gamma} |a_n| \\ &\leq |z|, \end{aligned}$$

where

$$\Phi(\alpha,\beta,\delta,\lambda,\gamma,k,n) = \left[\{n(\alpha+1) - (\alpha+\beta)\}\{(n-1)(n\gamma\delta+\gamma-\delta)+1\}\right]\Psi_n$$

and Ψ_n is given by (5). This completes the proof of Theorem 11. □

Finally, we conclude this paper by remarking that by suitably specializing the values of the parameters $\lambda,\ k, \delta, \gamma, and\ \ \alpha$ in the various results mentioned in this paper, we would be led to some interesting results including those which were obtained in [7], [9], [10] and [12].

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Received by the editors June 30, 2008