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A CLASS OF Z-METACYCLIC GROUPS INVOLVING THE LUCAS NUMBERS

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Abstract. The sequence $\{g_i\}_{i=1}^{\infty}$ is the sequence of Lucas numbers $g_1 = 2, g_2 = 1, g_{i+2} = g_{i+1} + g_i$, $(i \geq 1)$, and $\ell \geq 2$ is an integer. In this paper we consider the group $G(\ell)$ with an efficient presentation $\langle x, y \mid x^{\ell} = y^{\ell} = xyx^{[\frac{\ell}{2}]}y^{[\frac{3\ell}{2}]}\rangle$ where, $[x]$ is used for the integer part of a real x, and prove that $G(\ell)$ is finite of order

$$|G(\ell)| = \begin{cases} \frac{\ell(\ell+2)}{2}(1+3^{\frac{\ell}{2}}), & \ell \equiv 0 \text{ or } \pm 2 (\text{mod } 6), \\ 2\ell(\ell+1)g_{\ell+1}, & \ell \equiv 3 \ (\text{mod } 6), \\ \ell(\ell+1)g_{\ell+1}, & \ell \equiv \pm 1 (\text{mod } 6). \end{cases}$$

Moreover, if $\ell \equiv \pm 4$ or 8 or ± 12 or $20 (\text{mod } 40)$, or $\ell \equiv \pm 1 (\text{mod } 6)$ then, $G(\ell)$ is Z-metacyclic ($G'(\ell)$ and $\frac{G(\ell)}{G'(\ell)}$ are cyclic).

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1. Introduction

A finitely presented group G is said to have deficiency k if k is the largest integer such that G can be presented by m generators and $m - k$ relations. The deficiency zero groups are interesting to be considered for their finiteness and their structures. A considerable effort has, over the years, been put into presenting infinite classes of finite groups. For a short survey on these groups one may consider the articles [5, 16, 20, 21, 22] (for the cyclic, metacyclic and some related groups), the articles [4, 11, 19, 24] (for the linear groups), the articles [7, 8, 9, 12, 14, 15] (for the soluble groups) and the articles [3, 6, 10, 13, 17, 18] for some other classes of deficiency zero groups of interesting orders and various structures. In particular, Wiegold ([23]) considers the deficiency zero groups

$$G = \langle x, y \mid x^{\ell} = y^{m} = w(x, y)\rangle$$

where, w is a word on the generators x and y. Since the subgroup $\langle x^{\ell}\rangle$ is a central subgroup of G then G is finite if and only if the groups G/G' and

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$G/\langle x^\ell \rangle$ are finite. In this paper we consider the case when $w(x,y) = xyx^ny^{k+m}$ where ℓ, m, n and k are integers ≥ 2, that is, the groups

$$G(\ell,m,n,k) = \langle x, y \mid x^\ell = y^m = xyx^ny^{k+m} \rangle, \ (\ell, m, n, k \geq 2).$$

It follows readily that the group $G(\ell,m,n,k)$ is finite if and only if the group $\overline{G}(\ell,m,n,k) = G(\ell,m,n,k)/\langle x^\ell \rangle$ is finite, for, the commutator quotient of $G(\ell,m,n,k)$ is finite.

A simple calculation shows that the group $G(\ell,m,n,k)$ is cyclic if at least three of the four parameters ℓ, m, n and k, are equal, so we suppose that at most two parameters are equal and then there are the following cases:

$$G(\ell,\ell,n,k), G(\ell,m,\ell,k), G(\ell,m,m,k), G(\ell,m,n,\ell), G(\ell,m,n,m), G(\ell,m,n,n).$$

The aim of this paper is to study two subclasses of the groups $G(\ell,\ell,n,k)$. Some of them are Z-metacyclic groups (the groups with the cyclic commutator subgroup and cyclic commutator quotient group). Sections 2 and 3 are devoted to the study of the groups

$$G(\ell, \ell, [\frac{\ell}{2}], [\frac{\ell}{2}]),$$

for every integer $\ell \geq 2$.

Our notations are fairly standard, $[x]$ is used for the integer part of a real x, we denote $x^{-1}y^{-1}xy$ by $[x,y]$ and $y^{-1}xy$ by x^y, for elements x and y of a group. The main tools used in this investigation are the Todd-Coxeter coset enumeration algorithm (see [3] for example) and the modification to this algorithm described in [1] and [2].

2. The groups $G(\ell, \ell, \frac{\ell}{2}, \frac{\ell}{2}), (\ell$ is even)

For every even integer $\ell \geq 2$ let $G_1(\ell) = G(\ell, \ell, \frac{\ell}{2}, \frac{\ell}{2})$, then the subgroup $H_1 = \langle x^{i+1}yx^{-i} : i = 0, 1, \ldots, 2\ell - 1 \rangle$ of $G_1(\ell)$ is of index 2ℓ, for, we may define 2ℓ cosets as $1 = H_1$, and $ix = i + 1$, $(i = 1, 2, \ldots, 2\ell - 1)$ and a simple coset enumeration yields $\mid G_1(\ell) : H_1 \mid= 2\ell$. We now give the main results of this section:

Lemma 2.1. *For every even value of $\ell \geq 2$ the group H_1 has a presentation isomorphic to*

$$\langle a_1, a_2 \mid [a_1, a_2] = 1, a_1^\alpha = a_2^\alpha, a_2^\beta = a_1^\gamma \rangle,$$

where, $\alpha = \frac{1+3^{\frac{\ell}{2}}}{2}$, $\beta = \frac{\ell+3+3^{\frac{\ell}{2}}}{4}$ and $\gamma = \frac{3\ell+7+3^{\frac{\ell}{2}}}{4}$.

Proposition 2.2. *For every even integer $\ell \geq 2$, $G_1(\ell)$ is finite of order $\frac{\ell(\ell+2)}{2}(1+3^{\frac{\ell}{2}})$. Moreover, it is a Z-metacyclic group only if $\ell \equiv \pm 4$ or 8 or ± 12 or 20(mod 40).*

Proof of Lemma 2.1. Consider

$$G_1(\ell) = \langle x, y | x^\ell = y^\ell, xyx^{\frac{\ell}{2}}y^{\frac{\ell}{2}} \rangle$$

where, $\ell \geq 2$. Rename the generators of H_1 as $a_1 = xy,\ a_2 = x^2yx^{-1},\ \ldots, a_{2\ell} = x^{2\ell}yx^{-2\ell+1}$. By the above comments H_1 is of index 2ℓ in $G_1(\ell)$ and for using the Modified Todd-Coxeter algorithm (in the form given in [1]) we identify the number of a coset of H_1 and its representative. So, by defining the 2ℓ cosets as

$$1 = H_1,\ 1x = 2,\ 2x = 3, \ldots, (2\ell-1)x = 2\ell,$$

$$\begin{cases} 1xy = a_1.1 & \Rightarrow 2y = a_1.1 \\ 1x^2yx^{-1} = a_2.1 & \Rightarrow 3y = a_2.2 \\ \vdots & \\ 1x^{2\ell-1}yx^{-2\ell+2} = a_{2\ell}.1 & \Rightarrow (2\ell)y = a_{2\ell}.(2\ell-1). \\ 1x^{\ell}y^{-\ell} = 1 & \Rightarrow 1y = a_{\ell+1}^{-1}a_{\ell+2}^{-1}\ldots a_{2\ell-1}^{-1}\ .(2\ell). \end{cases}$$

Since then, the relation $1x^{2\ell}yx^{-2\ell+1} = 1$ yields

$$(2\ell)x = a_{2\ell-1}a_{2\ell-3}\ldots a_{\ell-1}\ .1\ .$$

We may now summarize our calculations in the following monitor table, for more clarity:

cosets	x	y
1	2	$a_{\ell+1}^{-1}a_{\ell+2}^{-1}\ldots a_{2\ell-1}^{-1}\ .(2\ell)$
2	3	$a_1.1$
3	4	$a_2.2$
$\vdots$		
$2\ell-1$	2ℓ	$a_{2\ell-1}.(2\ell-2)$
2ℓ	$a_{2\ell-1}a_{2\ell-3}\ldots a_{\ell-1}\ .1$	$a_{2\ell}.(2\ell-1)$

Considering all of the relations

$$ix^{\ell} = iy^{\ell}, ixyx^{\frac{\ell}{2}}y^{\frac{\ell}{2}} = i, \quad i = 1, 2, \ldots, 2\ell$$

will give us a presentation for H_1. In details, the relations $ix^{\ell} = iy^{\ell},\ i = 2, \ldots, \ell$ yield

$$a_{\ell+j} = a_j, \quad (j = 1, 2, \ldots, \ell-1),$$

and using these results and the relations $ix^{\ell} = iy^{\ell}, \quad (i = \ell+1, \ldots, 2\ell)$, give us the new relations

$$R_i = [A, a_ia_{i-1}\ldots a_2a_1] = 1, \quad (i = 1, 2, \ldots, \ell-1),$$

where, $A = a_{\ell}a_{\ell-1}\ldots a_3a_2a_1$. To get the other relations of the subgroup H_1, consider $ixyx^{\frac{\ell}{2}}y^{\frac{\ell}{2}} = i$, for every $i = 1, 2, \ldots, \ell-1$. Then we get the relations:

$$S_i = a_i^2a_{i+\frac{\ell}{2}-1}a_{i+\frac{\ell}{2}-2}\ldots a_{i+1} = 1,\ (i = 1, 2, \ldots, \ell-1),$$

and finally, the relation $\ell xyx^{\frac{\ell}{2}}y^{\frac{\ell}{2}} = \ell$ gives us the relation

$$S_{\ell} = a_{\ell}^2a_{\frac{\ell}{2}-1}a_{\frac{\ell}{2}-2}\ldots a_2a_1 = 1$$

for the subgroup H_1 (the other derived relations by the relations $ixyx^{\frac{\ell}{2}}y^{\frac{\ell}{2}} = i,\ i = \ell+1,\ldots,2\ell$ are, indeed, the redundant or the trivial relations in H_1. So, H_1 has a preliminary presentation isomorphic to

$$H_1 = \langle a_1,\ldots,a_\ell \mid S_\ell = R_i = S_i = 1,\ i = 1,2,\ldots,\ell-1\rangle.$$

To simplify this presentation we show first that $[a_1,a_2] = 1$ holds in H_1. The relations $S_{\frac{\ell}{2}+1} = 1$ and $S_2 = 1$ may be rewritten as $a_\ell a_{\ell-1}\ldots a_{\frac{\ell}{2}+2}a_{\frac{\ell}{2}+1} = a^{-1}_{\frac{\ell}{2}+1}$ and $a_{\frac{\ell}{2}}a_{\frac{\ell}{2}-1}\ldots a_3a_2^2 = a^{-1}_{\frac{\ell}{2}+1}$, respectively. So

$$a_\ell a_{\ell-1}\ldots a_{\frac{\ell}{2}+2}a_{\frac{\ell}{2}+1} = a_{\frac{\ell}{2}}a_{\frac{\ell}{2}-1}\ldots a_3a_2^2.$$

This relation together with the relation $S_1 = 1$ yields $a_\ell a_{\ell-1}\ldots a_{\frac{\ell}{2}+1} = a_1^{-2}a_2$, a fairly simple calculation now gives us the relation $[a_1,a_2] = 1$, by considering $R_1 = 1$.

By adding this relation to those of H_1 we are now able to calculate the generators $a_3,\ a_4,\ldots,a_\ell$ in terms of a_1 and a_2, and then we can eliminate them. Indeed, a fairly tedious calculation yields

$$\begin{cases} a_k = a_1^{\frac{3}{2}-\frac{3^{k-1}}{2}}.a_2^{-\frac{1}{2}+\frac{3^{k-1}}{2}}, & k = 3,4,\ldots,\frac{\ell}{2}, \\ a_{k+\frac{\ell}{2}} = a_1^{\frac{3}{2}+\frac{3^{k-1}}{2}}.a_2^{-\frac{1}{2}-\frac{3^{k-1}}{2}}, & k = 1,2,\ldots,\frac{\ell}{2}, \end{cases}$$

and we get the desired presentation for H_1. □

Proof of Proposition 2.2. Let $\alpha = \frac{1+3^{\frac{\ell}{2}}}{2}$, $\beta = \frac{\ell+3+3^{\frac{\ell}{2}}}{4}$ and $\gamma = \frac{3\ell+7+3^{\frac{\ell}{2}}}{4}$. For every even integer ℓ there are two cases: $\ell-2$ is not divisible by 8 or $\ell-2$ is divisible by 8. In the first case α, β and γ are pairwise co-primes, however the highest common factor of every pair of them is 2 in the second case. In the first case H_1 is a cyclic group (one may consider the subgroup $K_1 = \langle a_1\rangle$ of H_1 to show that $\mid H_1 : K_1 \mid= 1$) and then, $\mid H_1 \mid= \frac{1}{2}(1+\frac{\ell}{2})(1+3^{\frac{\ell}{2}}$. For the second case we consider the subgroup $L_1 = \langle a_1, a_2^2\rangle$ of H_1 which is of index 2 in H_1 and will be presented as follows, by letting $X = a_1$ and $Y = a_2^2$:

$$L_1 = \langle X,Y \mid [X,Y] = 1, X^\alpha = Y^{\frac{\alpha}{2}}, X^\gamma = Y^{\frac{\beta}{2}}\rangle.$$

Since $\ell \equiv 2 \pmod 8$ then $h.c.f.(\frac{\alpha}{2},\frac{\beta}{2}) = h.c.f.(\gamma,\frac{\beta}{2}) = 1$ and L_1 is a cyclic group of order $\frac{1}{4}(1+\frac{\ell}{2})(1+3^{\frac{\ell}{2}})$, i.e.; $\mid H_1 \mid= \frac{1}{2}(1+\frac{\ell}{2})(1+3^{\frac{\ell}{2}})$. Consequently, $\mid G_1(\ell) \mid= 2\ell\times\frac{1}{2}(1+\frac{\ell}{2})(1+3^{\frac{\ell}{2}})$, as desired.

To complete the proof let $\ell = 40q\pm 4$. Then by the above results $\mid G_1'(\ell) \mid= \frac{1+3^{\frac{\ell}{2}}}{2}$ and also

$$\frac{G_1(\ell)}{G_1'(\ell)} = \begin{cases} Z_2\times Z_{\frac{\ell(\ell+2)}{2}}, & \ell\equiv 2 \pmod 4, \\ Z_{\ell(\ell+2)}, & \ell\equiv 0 \pmod 4, \end{cases}$$

and $G_1'(\ell) \cong Z_5 \times Z_t$ where, $t = 9^{\frac{\ell}{4}-1} - 9^{\frac{\ell}{4}-2} + 9^{\frac{\ell}{4}-3} - \cdots - 9 + 1$. However, $h.c.f.(t,5) = h.c.f.(\frac{\ell}{4},5) = h.c.f.(10q \pm 1, 5) = 1$ shows that $G_1'(\ell)$ is cyclic. The same proof carries over the cases $\ell = 40q+8$, $\ell = 40q \pm 12$ and $\ell = 40q+20$. $\square$

3. The groups $G(\ell,\ell,\frac{\ell-1}{2},\frac{\ell-1}{2}),(\ell$ is odd$)$

For every odd integer $\ell \geq 3$ let $G_2(\ell) = G(\ell,\ell,\frac{\ell-1}{2},\frac{\ell-1}{2})$. The sequences of Fibonacci and Lucas numbers $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ will be used in this section, which are defined as follows:

$$\begin{array}{ll} f_2 = 1,\ f_{i+2} = f_{i+1} + f_i, & (i \geq 1), \\ g_1 = 2, g_2 = 1, g_{i+2} = g_{i+1} + g_i, & (i \geq 1), \end{array}$$

and the main result of this section is:

Proposition 3.1. *For every odd integer $\ell \geq 3$, the group $G_2(\ell)$ is finite and*

$$\mid G_2(\ell) \mid = \begin{cases} 2\ell(\ell+1)g_{\ell+1}, & \ell \equiv 3 \ (mod\ 6), \\ \ell(\ell+1)g_{\ell+1}, & \ell \equiv \pm 1 (mod\ 6), \end{cases}$$

Moreover, this group is Z-metacyclic only if $\ell \equiv \pm 1 (mod\ 6)$.

To prove this proposition we first prove some preliminaries.

Lemma 3.2. *For every odd value of $\ell \geq 3$, the relation $x^{2\ell(\ell+1)} = 1$ holds in $G_2(\ell)$.*

Proof. The second relation of

$$G_2(\ell) = \langle x, y \mid x^{\ell} = y^{\ell}, xyx^{\frac{\ell-1}{2}}y^{\frac{\ell-1}{2}} = 1 \rangle$$

is equivalent to $x^{-(\frac{\ell-1}{2})} = y^{\frac{\ell-1}{2}}xy$ and squaring both sides yields $y^{\frac{\ell-1}{2}}xy^{\frac{\ell+1}{2}}xy = x^{-\ell+1}$, or

$$xyx^{-1} = y^{-(\frac{\ell+1}{2})}x^{-1}y^{-(\frac{\ell-1}{2})}x^{-\ell}.$$

This may be reduced to

$$xyx^{-1} = y^{-(\frac{\ell+1}{2})}x^{-1-2\ell}x^{\ell}y^{-(\frac{\ell-1}{2})},$$

for, x^{ℓ} and y^{ℓ} are central elements (because of the relation $x^{\ell} = y^{\ell}$.) The last relation will be reduced to

$$xyx^{-1} = y^{-(\frac{\ell+1}{2})}x^{-1-2\ell}y^{\frac{\ell+1}{2}}$$

(by substituting y^{ℓ} for x^{ℓ}.)

Raising both sides of the last relation to the power ℓ and considering $x^{\ell} = y^{\ell}$ once again, we get $y^{\ell} = x^{-\ell(2\ell+1)}$, or $x^{2\ell(\ell+1)} = 1$ as desired. $\square$

Finding the order of $G_2(\ell)$ is possible by getting a suitable quotient group of $G_2(\ell)$. To do this we proceed as follows:

First we show that x^ℓ, as an element of $G_2(\ell)$ has order $\ell+1$ or $2(\ell+1)$, if $\ell \equiv \pm 1 (\mathrm{mod}\ 6)$ or $\ell \equiv 3 (\mathrm{mod}\ 6)$, respectively. Let $\ell \equiv \pm 1 (\mathrm{mod}\ 6)$ and consider the subgroup $\langle x^\ell, x^i y x^{-i+1} : i = 1, 2, \ldots, \ell - 1 \rangle$. An easy coset enumeration shows that this subgroup is of index ℓ in $G_2(\ell)$ and by letting $c_i = x^i y x^{-i+1}$, $(i = 1, 2, \ldots, \ell - 1)$ and $c_\ell = x^\ell$, we can get a presentation for this subgroup. Simplifying the relations of this subgroup gives us two interesting relations: $c_1^{(\frac{\ell+1}{2})g_{\ell+1}} = 1$ and $c_\ell^2 = c_1^{g_{\ell+1}}$. On the other hand a numerical result concerning the Lucas numbers shows that $h.c.f.(\ell+1, g_{\ell+1}) = 1$ and then the relation $c_1^{2(\frac{\ell+1}{2})g_{\ell+1}} = 1$ holds in this subgroup. Consequently, the equation $x^{\ell(\ell+1)} = 1$ holds in $G_2(\ell)$. When $\ell \equiv 3 (\mathrm{mod}\ 6)$ we may proceed in a similar way to prove that the equation $x^{2\ell(\ell+1)} = 1$ holds in $G_2(\ell)$.

Secondly, since $\langle x^\ell \rangle$ is a central subgroup of $G_2(\ell)$ then adding the relation $x^\ell = 1$ to those of $G_2(\ell)$ gives the group

$$H_2(\ell) = \langle x, y \mid x^\ell = y^\ell = 1, xyx^{\frac{\ell-1}{2}} y^{\frac{\ell-1}{2}} = 1 \rangle$$

which is $G_2(\ell)$ factored by the cyclic group $Z_{\ell+1}$ or $Z_{2(\ell+1)}$ if $\ell \equiv \pm 1 (\mathrm{mod}\ 6)$ either $\ell \equiv 3 (\mathrm{mod}\ 6)$. We are now going to identify the group $H_2(\ell)$ as follows:

Lemma 3.3. *For every odd value of $\ell \geq 3$, the group $H_2(\ell)$ is a metabelian group of order $\ell \times g_{\ell+1}$.*

Proof. Abelianising the relations of $H_2(\ell)$ shows that $xy \in H_2'(\ell)$, so, the subgroup $K_2(\ell) = \langle xy, x^2 y x^{-1}, \ldots, x^{\ell-1} y x^{-\ell+2}, yx \rangle$ is contained in $H_2'(\ell)$. Showing that $\mid H_2(\ell) : K_2(\ell) \mid = \ell$ is easy by defining ℓ cosets as $1 = K_2(\ell)$, $ix = i + 1$, ($i = 1, 2, \ldots, \ell - 1$). Consequently, $H_2'(\ell) = K_2(\ell)$. We now use the Modified algorithm to find a presentation for $K_2(\ell)$. Let $a_i = x^i y x^{-i+1}$, $(i = 1, 2, \ldots, \ell - 1)$ and $a_\ell = yx$. For every i, $(i = 1, 2, \ldots, \ell)$, the relations $iy^\ell = i$ yield only one relation for the group $K_2(\ell)$, and this is indeed, the relation:

$$r = a_\ell a_{\ell-1} \ldots a_2 a_1 = 1,$$

and the relations $ixyx^{\frac{\ell-1}{2}} y^{\frac{\ell-1}{2}} = i$ yield the following relations for $K_2(\ell)$:

$$\begin{cases} s_1 = a_1^2 a_{\frac{\ell-1}{2}} a_{\frac{\ell-3}{2}} \ldots a_4 a_3 a_2 = 1, \\ s_2 = a_2^2 a_{\frac{\ell+1}{2}} a_{\frac{\ell-1}{2}} \ldots a_5 a_4 a_3 = 1, \\ s_3 = a_3^2 a_{\frac{\ell+3}{2}} a_{\frac{\ell+1}{2}} \ldots a_6 a_5 a_4 = 1, \\ \vdots \\ s_{\frac{\ell+3}{2}} = a_{\frac{\ell+3}{2}}^2 a_\ell a_{\ell-1} \ldots a_{\frac{\ell+9}{2}} a_{\frac{\ell+7}{2}} a_{\frac{\ell+5}{2}} = 1, \\ s_{\frac{\ell+5}{2}} = a_{\frac{\ell+5}{2}}^2 a_1 a_\ell \ldots a_{\frac{\ell+11}{2}} a_{\frac{\ell+9}{2}} a_{\frac{\ell+7}{2}} = 1, \\ s_{\frac{\ell+7}{2}} = a_{\frac{\ell+7}{2}}^2 a_2 a_1 a_\ell \ldots a_{\frac{\ell+13}{2}} a_{\frac{\ell+11}{2}} a_{\frac{\ell+9}{2}} = 1, \\ \vdots \\ s_{\ell-1} = a_{\ell-1}^2 a_{\frac{\ell-5}{2}} a_{\frac{\ell-7}{2}} \ldots a_2 a_1 a_\ell = 1, \\ s_\ell = a_\ell^2 a_{\frac{\ell-3}{2}} a_{\frac{\ell-5}{2}} \ldots a_3 a_2 a_1 = 1. \end{cases}$$

So, $K_2(\ell) = \langle a_1, a_2, \ldots, a_\ell \mid r = s_i = 1,\ i = 1, 2, \ldots, \ell\rangle$. Two classes of the new relations are acceptable by the relations of $K_2(\ell)$, and they are:

$$a_i = a_{i+1}a_{i+\frac{\ell+1}{2}}, \quad a_i^2 = a_{i+1}a_{i+\frac{\ell-1}{2}}, \quad (i = 1, 2, \ldots, \ell),$$

where, indices are reduced modulo ℓ (the proofs are easy, for, rewriting the relation $r = 1$ as $(a_{i-1}a_{i-2}\ldots a_2a_1a_\ell a_{\ell-1}\ldots a_{i+\frac{\ell+1}{2}})(a_{i+\frac{\ell-1}{2}}\ldots a_{i+1})a_i = 1$ and using $s_{i+1} = s_{i+\frac{\ell+1}{2}} = 1$ give us $a_i = a_{i+1}a_{i+\frac{\ell+1}{2}}$, and the second relation may be derived by considering $s_i = s_{i+1} = 1$, for every i.) We use now these new relations to prove that $K_2(\ell)$ is abelian. By the relations

$$\begin{aligned}&a_1 = a_2a_{\frac{\ell+3}{2}} = (a_3a_{\frac{\ell+5}{2}})a_{\frac{\ell+3}{2}} = \cdots = a_{\frac{\ell+1}{2}}a_\ell a_{\ell-1}\ldots a_{\frac{\ell+5}{2}}a_{\frac{\ell+3}{2}},\\ &a_{\frac{\ell-1}{2}}\ldots a_3a_2a_1 = a_1^{-1},\end{aligned}$$

and using $r = 1$ we get $[a_{\frac{\ell+1}{2}}, a_1] = 1$. Since $a_1^2 = a_2a_{\frac{\ell+1}{2}}$ then $[a_1, a_2] = 1$ holds, and we will get the relation $[a_i, a_{i+1}] = 1$, for every i. This proves that $K_2(\ell)$ is abelian, and a hand calculation yields:

$$\begin{cases} a_{\frac{\ell+3}{2}} = a_1a_2^{-1}, & \\ a_i = a_1^{-f_{2(i-2)}}\,.\,a_2^{f_{2(i-1)}}, & i = 3, 4, \ldots, \frac{\ell+1}{2}, \\ a_{i+\frac{\ell+1}{2}} = a_1^{f_{2i-3}}\,.\,a_2^{-f_{2i-1}}, & i = 2, 3, \ldots, \frac{\ell-1}{2}. \end{cases}$$

Showing that $K_2(\ell)$ can be generated by a_1 and A_2. If $\ell = 3$, $K_2(\ell)$ is of order 4, and if $\ell \geq 4$,

$$K_2(\ell) = \langle a_1, a_2 \mid a_1^{-1+f_{\ell-2}} = a_2^{f_\ell}, a_1^{2+f_{\ell-3}} = a_2^{1+f_{\ell-1}}, [a_1, a_2] = 1\rangle.$$

Then, the order of this group is equal to $(-1 + f_{\ell-2})(1 + f_{\ell-1}) - f_\ell(2 + f_{\ell-3}) = g_{\ell+1}$. So, $\mid H_2(\ell) \mid = \ell \times g_{\ell+1}$. $\square$

Proof of Proposition 3.1. By Lemma 3.3 and the comments after Lemma 3.2 we conclude that

$$\mid G_2(\ell) \mid = \begin{cases} 2\ell(\ell+1)g_{\ell+1}, & \ell \equiv 3 \pmod 6, \\ \ell(\ell+1)g_{\ell+1}, & \ell \equiv \pm 1 (\mathrm{mod}\ 6). \end{cases}$$

To complete the proof let us consider the result of Lemma 3.3, concerning the derived subgroup of $H_2(\ell)$, i.e.; if $\ell \equiv \pm1(\mathrm{mod}\ 6)$ then $h.c.f.(-1 + f_{\ell-2}, 2 + f_{\ell-3}) = 1$ and $H_2'(\ell)$ is cyclic of order $g_{\ell+1}$, however, if $\ell \equiv 3(\mathrm{mod}\ 6)$, $H_2'(\ell)$ is not cyclic (because $g_{\ell+1}$ is divisible by 4 in this case). On the other hand, $\frac{G_2(\ell)}{G_2'(\ell)}$ is a cyclic group of order $\ell(\ell+1)$, for every odd values of ℓ and then, $\mid G_2'(\ell) \mid = g_{\ell+1}$. Consequently, $G_2(\ell)$ is a Z-metacyclic group only if $\ell \equiv \pm1(\mathrm{mod}\ 6)$, for, $H_2(\ell)$ is a central homomorphic image of $G_2(\ell)$. $\square$

References

[1] Beetham, M.J., Campbell, C.M., A note on the Todd-Coxeter coset enumeration algorithm. Proc. Edinburgh Math. Soc. 20 (Series 2) (1976), 73-79.

[2] Benson, C.T., Mendelson N.S., A calculus for a certain classes of word problem in groups. J. Combinatorial Theory 1 (1966), 200-208.

[3] Campbell, C.M., Coxeter, H.S.M., Robertson, E.F., Some families of finite groups having two generators and two relations. Proc. R. Soc. London 357 (Series A) (1977), 423-438.

[4] Campbell, C.M., Robertson, E.F., A deficiency zero presentation for SL(2,p). Bull. London Math. Soc. 12 (1980), 17-20.

[5] Campbell, C.M., Robertson, E.F., The orders of certain metacyclic groups. Bull. London Math. Soc. 6 (1974), 312-314.

[6] Campbell, C.M., Robertson, E.F., Remarks on a class of 2-generator groups of deficiency zero. J. Austral. Math. Soc. 19 (Series A) (1975), 297-305.

[7] Campbell, C.M., Robertson, E.F., Deficiency zero groups involving Fibonacci and Lucas numbers. Proc. Royal Soc. Edinburgh 81 (Series A) (1978), 273-286.

[8] Campbell, C.M., Robertson, E.F., On 2-generator 2-relation soluble groups. Proc. Edinburgh Math. Soc. 23 (1980), 269-273.

[9] Campbell, C.M., Robertson, E.F., Thomas, R.M., Finite groups of deficiency zero involving the Lucas numbers. Proc. Edinburgh Math. Soc. 33 (1990), 1-10.

[10] Campbell, C.M., Robertson, E.F., Thomas, R.M., On groups related to Fibonacci groups. Group Theory. Proceedings of 1987, Singapore Conference, 323-331.

[11] Campbell, C.M., Robertson, E.F., Williams, P.D., On presentations of $PSL(2,p^n)$. J. Austral. Math. Soc. 48 (Series A) (1990), 333-346.

[12] Doostie, H., Non-metabelian soluble groups involving the Lucas numbers. J. Sci. I.R.Iran 3 (Nos. 1, 2) (1992), 63-68.

[13] Doostie, H., Gholamie, R., Thomas, R.M., Certain extensions of $(\ell, m \mid n, k)$-groups. Southeast Asian Bull. Math. 27 (2003), 21-34.

[14] Doostie, H., Jamali, A.R., A class of deficiency zero soluble groups of derived length 4. Proc. Royal Soc. Edinburgh 121 (Series A) (1992), 163-168.

[15] Kenne, P.E., Some new efficient soluble groups. Communication in Algebra 18 (No. 8) (1990), 2747-2753.

[16] MacDonald, I.D., On a class of finitely presented groups. Canad. J. Math. 14 (1962), 602-613.

[17] Perkel, M., Groups of type $F^{a,b,-c}$. Israeal J. Math. 52 (Nos. 1-2) (1985), 167-177.

[18] Schenkman, E., Some two-generator groups with two relations. Arch. Math. 18 (1967), 362-363.

[19] Sunday, J.G., Presentaions of groups $SL(2,m)$ and $PSL(2,m)$. Canad. J. Math. 6 (No. 1) (1972), 1129-1131.

[20] Wamsley, J.W., A class of two generator two relation finite groups. J. Austral. Math. Soc. 14 (1970), 38-39.

[21] Wamsley, J.W., The deficiency of metacyclic groups. Proc. Amer. Math. Soc. 24 (1970), 724-726.

[22] Wamsley, J.W., Some finite groups with zero deficiency. J. Austral. Math. Soc. 18 (1974), 73-75.

[23] Wiegold, J., On some groups with trivial multiplicator. Bull. Austral. Math. Soc. 40 (1989), 331-332.

[24] Zassenhauss, H.J., A presentation of the group $PSL(2, p)$ with three defining relations. Canad. J. Math. 21 (1969), 310-411.

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