Novi Sad J. Math.
Vol. 39, No. 1, 2009, 11-20

A COMMON FIXED POINT THEOREM IN COMPLETE FUZZY METRIC SPACES

D. Turkoglu[1], **S. Sedghi**[2], **N. Shobe**[3]

Abstract. In this paper, we establish a common fixed point theorem in complete fuzzy metric spaces.

AMS Mathematics Subject Classification (2000): 54E40; 54E35; 54H25

Key words and phrases: Fuzzy contractive mapping, Complete fuzzy metric space

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [11] in 1965. Since then, using this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [3] and Kramosil and Michalek [6] have introduced the concept of fuzzy topological spaces induced by fuzzy metric, which have very important applications in quantum particle physics, particularly in connections with both string and $\epsilon^{(\infty)}$ theory, given and studied by El Naschie [1, 2]. Many authors [4, 8, 9] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min(a, b)$.

Definition 1.2. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

[1]Department of Mathematics, Faculty of Science and Arts, Gazi University, 06500 Teknikokullar, Ankara, Turkey, e-mail: dturkoglu@gazi.edu.tr
[2]Department of Mathematics, Islamic Azad University-Ghaemshahr Branch Ghaemshahr P. O. Box 163, Iran, e-mail: sedghi_gh@yahoo.com
[3]Department of Mathematics, Islamic Azad University-Babol Branch, Iran, e-mail: nabi_shobe@yahoo.com

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.
6. $\lim_{t\to\infty} M(x, y, t) = 1$

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Example 1.3. Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Lemma 1.4. *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 1.5. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t).$$

Whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, i.e.

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t)$$

Lemma 1.6. *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Proof. see proposition 1 of [7] □

Definition 1.7. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.8. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Proposition 1.9. *[10]. Self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible, then they are weak compatible.*

The converse is not true as seen in the following example.

Example 1.10. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$, with t-norm defined $a * b = min\{a, b\}$, for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $t > 0$ and $x, y \in X$. Define self-maps A and S on X as follows:

$$Ax = \begin{cases} 2 & \text{if } 0 \le x \le 1, \\ \frac{x}{2} & \text{if } 1 < x \le 2, \end{cases} \qquad Sx = \begin{cases} 2 & \text{if } x = 1, \\ \frac{x+3}{5} & \text{otherwise,} \end{cases}$$

Then we have $S1 = A1{=}2$ and $S2 = A2 = 1$. Also $SA1 = AS1 = 1$ and $SA2 = AS2 = 2$. Thus (A, S) is weak compatible. Again,

$$Ax_n = 1 - \frac{1}{4n}, \qquad Sx_n = 1 - \frac{1}{10n}.$$

Thus,

$$Ax_n \to 1, \qquad Sx_n \to 1.$$

Further,

$$SAx_n = \frac{4}{5} - \frac{1}{20n}, \qquad ASx_n = 2.$$

Now,

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = \lim_{n\to\infty} M(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{t}{t + \frac{6}{5}} < 1, \quad \forall\, t > 0.$$

Hence (A, S) is not compatible.

Henceforth, we assume that $*$ is a continuous t-norm on X such that for every $\mu \in (0, 1)$, there is a $\lambda \in (0, 1)$ such that

$$\underbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}_{n} \ge 1 - \mu$$

Lemma 1.11. *Let $(X, M, *)$ be a fuzzy metric space. If we define $E_{\lambda,M} : X^2 \to^+ \cup \{0\}$ by*

$$E_{\lambda,M}(x, y) = \inf\{t > 0 \ : \ M(x, y, t) > 1 - \lambda\}$$

for each $\lambda \in (0, 1)$ and $x, y \in X$. Then we have
(i) *For any $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that*

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

for any $x_1, x_2, ..., x_n \in X$.
(ii) *The sequence $\{x_n\}_{n\in N}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also the sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence if and only if it is Cauchy with $E_{\lambda,M}$.*

Proof. *(i)* For every $\mu \in (0, 1)$, we can find a $\lambda \in (0, 1)$ such that

$$\underbrace{(1 - \lambda) * (1 - \lambda) * \cdots * (1 - \lambda)}_{n} \geq 1 - \mu$$

by definition

$$\begin{aligned} & M(x_1, x_n, E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n) + n\delta) \\ & \geq M(x_1, x_2, E_{\lambda,M}(x_1, x_2) + \delta) * \cdots * M(x_{n-1}, x_n, E_{\lambda,M}(x_{n-1}, x_n) + \delta) \\ & \geq \underbrace{(1 - \lambda) * (1 - \lambda) * \cdots * (1 - \lambda)}_{n} \geq 1 - \mu \end{aligned}$$

for very $\delta > 0$, which implies that

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n) + n\delta$$

.
Since $\delta > 0$ is arbitrary, we have

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n).$$

For (ii), note that since M is continuous in its third place and

$$E_{\lambda,M}(x, y) = \inf\{t > 0 \ : \ M(x, y, t) > 1 - \lambda\}.$$

Hence, we have

$$M(x_n, x, \eta) > 1 - \lambda \Longleftrightarrow E_{\lambda,M}(x_n, x) < \eta$$

for every $\eta > 0$. □

Lemma 1.12. *Let (X,M,*) be a fuzzy metric space. If there is a sequence* $\{x_n\}$ *in X, such that for every* $n \in \mathbb{N}$.

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$$

for every $k > 1$*, then the sequence* $\{x_n\}$ *is a Cauchy sequence.*

Proof. For every $\lambda \in (0,1)$ and $x_n, , x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda,M}(x_{n+1}, x_n) &= \inf\{t > 0 \ : \ M(x_{n+1}, x_n, t) > 1 - \lambda\} \\ &\leq \inf\{t > 0 \ : \ M(x_0, x_1, k^n t) > 1 - \lambda\} \\ &= \inf\{\frac{t}{k^n} \ : \ M(x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} \inf\{t > 0 \ : \ M(x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} E_{\lambda,M}(x_0, x_1). \end{aligned}$$

By Lemma (1.11), for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} E_{\mu,M}(x_n, x_m) &\leq E_{\lambda,M}(x_n, x_{n+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m) \\ &\leq \frac{1}{k^n} E_{\lambda,M}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,M}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda,M}(x_0, x_1) \\ &= E_{\lambda,M}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence, the sequence $\{x_n\}$ is a Cauchy sequence. □

2. THE MAIN RESULTS

A class of implicit relation

Let Φ be the set of all continuous functions $\phi : [0,1]^3 \longrightarrow [0,1]$, increasing in any coordinate and $\phi(t,t,t) > t$ for every $t \in [0,1)$.

Theorem 2.1. *Let* A, B, S *and* T *be self-mappings of a complete fuzzy metric space* $(X, M, *)$ *satisfying :*

(i)$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ *and* $A(X)$ *or* $B(X)$ *is a closed subset of* X*,*

(ii)

$$M(Ax, By, t) \geq \phi(M(Sx, Ty, kt), M(Ax, Sx, kt), M(By, Ty, kt)),$$

for every x, y *in* X*,*$k > 1$ *and* $\phi \in \Phi$*,*

(iii) *the pairs* (A, S) *and* (B, T) *are weak compatible. Then* A, B, S *and* T *have a unique common fixed point in* X*.*

Proof. Let $x_0 \in X$ be an arbitrary point as $A(X) \subseteq T(X), B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \cdots$.

Now, we prove that $\{y_n\}$ is a Cauchy sequence. Let $d_m(t) = M(y_m, y_{m+1}, t)$. Set $m = 2n$, we have

$$\begin{aligned} d_{2n}(t) &= M(y_{2n}, y_{2n+1}, t) = M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \phi(M(Sx_{2n}, Tx_{2n+1}, kt), M(Ax_{2n}, Sx_{2n}, kt), M(Bx_{2n+1}, Tx_{2n+1}, kt)) \\ &= \phi(M(y_{2n-1}, y_{2n}, kt), M(y_{2n}, y_{2n-1}, kt), M(y_{2n+1}, y_{2n}, kt)) \\ &= \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt)) \end{aligned}$$

We claim that for every $n \in \mathbb{N}$, $d_{2n}(kt) \geq d_{2n-1}(kt)$. For if $d_{2n}(kt) < d_{2n-1}(kt)$, for some $n \in \mathbb{N}$, since ϕ is an increasing function, then the last inequality above we get

$$d_{2n}(t) \geq \phi(d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)) > d_{2n}(kt).$$

That is, $d_{2n}(t) > d_{2n}(kt)$, a contradiction. Hence $d_{2n}(kt) \geq d_{2n-1}(kt)$ for every $n \in \mathbb{N}$ and $\forall t > 0$. Similarly for an odd integer $m = 2n+1$, we have $d_{2n+1}(kt) \geq d_{2n}(kt)$. Thus $\{d_n(t)\}$; is an increasing sequence in $[0, 1]$. Thus

$$d_{2n}(t) \geq \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt)) > d_{2n-1}(kt).$$

Similarly, for an odd integer $m = 2n+1$, we have $d_{2n+1}(t) \geq d_{2n}(kt)$. Hence $d_n(t) \geq d_{n-1}(kt)$. That is,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, kt) \geq ... \geq M(y_0, y_1, k^n t).$$

Hence by Lemma 1.12 $\{y_n\}$ is Cauchy and the completeness of X, $\{y_n\}$ converges to y in X. That is,

$$\begin{aligned} \lim_{n\to\infty} y_n = y \Rightarrow \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y. \end{aligned}$$

As $B(X) \subseteq S(X)$, there exist $u \in X$ such that $Su = y$. So, for $\epsilon > 0$, we have

$$\begin{aligned} M(Au, y, t+\epsilon) &\geq M(Au, Bx_{2n+1}, t) * M(Bx_{2n+1}, y, \epsilon) \\ &\geq \phi(M(Su, Tx_{2n+1}, kt), M(Au, Su, kt), M(Bx_{2n+1}, Tx_{2n+1}, kt)) * \\ &\quad * M(Bx_{2n+1}, y, \epsilon). \end{aligned}$$

By continuous M and ϕ, on making $n \longrightarrow \infty$ the above inequality, we get

$$\begin{aligned} M(Au, y, t+\epsilon) &\geq \phi(M(y, y, kt), M(Au, y, kt), M(y, y, kt)) \\ &\geq \phi(M(Au, y, kt), M(Au, y, kt), M(Au, y, kt)). \end{aligned}$$

On making $\epsilon \longrightarrow 0$, we have

$$M(Au, y, t) \geq \phi(M(Au, y, kt), M(Au, y, kt), M(Au, y, kt)).$$

If $Au \neq y$, by above inequality we get $M(Au, y, t) > M(Au, y, kt)$, which is a contradiction. Hence $M(Au, y, t) = 1$, i.e $Au = y$. Thus $Au = Su = y$.
As $A(X) \subseteq T(X)$ there exist $v \in X$, such that $Tv = y$. So,

$$\begin{aligned} M(y, Bv, t) &= M(Au, Bv, t) \\ &\geq \phi(M(Su, Tv, kt), M(Au, Su, kt), M(Bv, Tv, kt)) \\ &= \phi(1, 1, M(Bv, y, kt)). \end{aligned}$$

we claim that $Bv = y$. For if $Bv \neq y$, then $M(Bv, y, t) < 1$.
On the above inequality we get

$$M(y, Bv, t) \geq \phi(M(y, Bv, kt), M(y, Bv, kt), M(y, Bv, kt)) > M(y, Bv, kt),$$

a contradiction. Hence $Tv = Bv = Au = Su = y$. Since (A, S) is weak compatible, we get that $ASu = SAu$, that is $Ay = Sy$.
Since (B, T) is weak compatible, we get that $TBv = BTv$, that is $Ty = By$. If $Ay \neq y$, then $M(Ay, y, t) < 1$. However

$$\begin{aligned} M(Ay, y, t) &= M(Ay, Bv, t) \\ &\geq \phi(M(Sy, Tv, kt), M(Ay, Sy, kt), M(Bv, Tv, kt)) \\ &\geq \phi(M(Ay, y, kt), 1, 1) \\ &\geq \phi(M(Ay, y, kt), M(Ay, y, kt), M(Ay, y, kt)) \\ &> M(Ay, y, kt) \end{aligned}$$

a contradiction. Thus $Ay = y$, hence $Ay = Sy = y$.
Similarly, we prove that $By = y$. For if $By \neq y$, then $M(By, y, t) < 1$, however

$$\begin{aligned} M(y, By, t) &= M(Ay, By, t) \\ &\geq \phi(M(Sy, Ty, kt), M(Ay, Sy, kt), M(By, Ty, kt)) > M(y, By, kt), \end{aligned}$$

a contradiction. Therefore, $Ay = By = Sy = Ty = y$, that is, y is a common fixed point of A, B, S and T.
Uniqueness, let x be another common fixed point of A, B, S and T.
Then $x = Ax = Bx = Sx = Tx$ and $M(x, y, t) < 1$, hence

$$\begin{aligned} M(y, x, t) &= M(Ay, Bx, t) \\ &\geq \phi(M(Sy, Tx, kt), M(Ay, Sy, kt), M(Bx, Tx, kt)) \\ &= \phi(M(y, x, kt), 1, 1) > M(y, x, kt), \end{aligned}$$

a contradiction. Therefore, y is the unique common fixed point of self-maps A, B, S and T. □

Theorem 2.2. *Let S and T be aself-mappings of a complete fuzzy metric space $(X, M, *)$. If F, G are two mappings of Y into X and A, B are two mappings of X into Y, where Y is a nonempty set, such that it satisfies the following conditions:*

(i)$FA(X) \subseteq T(X)$, $GB(X) \subseteq S(X)$ *and* $A(X)$ *or* $B(X)$ *is a complete subset of* X,

(ii)$M(FAx, GBy, t) \geq \phi(M(Sx, Ty, kt), M(FAx, Sx, kt), M(GBy, Ty, kt))$, *for every* x, y *in* X,$k > 1$ *and* $\phi \in \Phi$,

(iii) *the pairs* (FA, S) *and* (GB, T) *are weak compatible.*
Then FA, GB, S *and* T *have a unique common fixed point in* X.

Proof. By Theorem 2.1 it suffices to set $FA = A$ and $GB = B$. □

Theorem 2.3. *Let S and T be self-mappings of a complete fuzzy metric space $(X, M, *)$, satisfying*

$$\text{(i)} \qquad \begin{aligned} M(Sx, Ty, t) \geq\ & a(t)M(x, Sy, kt) + b(t)M(x, Sx, kt) \\ & + c(t)M(Sy, TSy, kt) \\ & + h(t)\max\{M(x, TSy, kt), M(Sx, Sy, kt)\} \end{aligned}$$

for every $x, y \in X$ and some $k > 1$, where a, b and c, h are functions of $[0, \infty)$ into $(0, 1)$ such that

$$a(t) + b(t) + c(t) + h(t) = 1, \quad for \quad any \quad t > 0$$

Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X, defined as

$$\begin{aligned} x_{2n+1} &= Sx_{2n} \qquad n = 0, 1, 2, \cdots \\ x_{2n} &= Tx_{2n-1} \qquad n = 1, 2, \cdots . \end{aligned}$$

For simplicity, we set

$$d_n(t) = M(x_n, x_{n+1}, t), \quad n = 0, 1, 2, \cdots$$

Now, we prove that the sequence $d_n(t) = M(x_n, x_{n+1}, t)$ is an increasing sequence in $[0,1]$.

$$\begin{aligned} d_{2n}(t) &= M(x_{2n}, x_{2n+1}, t) = M(Sx_{2n}, Tx_{2n-1}, t) = M(Sx_{2n}, TSx_{2n-2}, t) \\ &\geq a(t)M(x_{2n}, Sx_{2n-2}, kt) + b(t)M(x_{2n}, Sx_{2n}, kt) \\ &\quad +c(t)M(Sx_{2n-2}, TSx_{2n-2}, kt) \\ &\quad +h(t)\max\{M(x_{2n}, TSx_{2n-2}, kt), \\ &\quad M(Sx_{2n}, Sx_{2n-2}, kt)\} \\ &= a(t)M(x_{2n}, x_{2n-1}, kt) + b(t)M(x_{2n}, x_{2n+1}, kt) \\ &\quad +c(t)M(x_{2n-1}, x_{2n}, kt) \\ &\quad +h(t)\max\{M(x_{2n}, x_{2n}, kt), M(x_{2n+1}, x_{2n-1}, kt)\} \\ &= a(t)d_{2n-1}(kt) + b(t)d_{2n}(kt) + c(t)d_{2n-1}(kt) + h(t) \end{aligned}$$

Let $d_{2n}(kt) < d_{2n-1}(kt)$ in the above inequality we have

$$d_{2n}(t) > a(t)d_{2n}(kt) + b(t)d_{2n}(kt) + c(t)d_{2n}(kt) + h(t)d_{2n}(kt) = d_{2n}(kt)$$

which is a contradiction. Thus, $d_{2n}(kt) \geq d_{2n-1}(kt)$. Similarly, we have $d_{2n+1}(kt) \geq d_{2n}(kt)$. Hence in the above equality we get $d_n(t) > d_{n-1}(kt)$. That is

$$M(x_n, x_{n+1}, t) = M(x_{n-1}, x_n, kt) \geq \cdots \geq M(x_0, x_1, k^n t).$$

Hence by Lemma 1.12, the sequence $\{x_n\}$ is Cauchy and by completeness of X, $\{x_n\}$ converges to x in X. That is,
$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Sx_{2n-1} = x,$ and $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Tx_{2n} = x.$
Now, we prove that $Sx = x$. If $Sx \neq x$ by (i),

$$\begin{aligned} M(Sx, x_{2n}, t) &= M(Sx, TSx_{2n-2}, t) \\ &\geq a(t)M(x, Sx_{2n-2}, kt) + b(t)M(x, Sx, kt) \\ &+ c(t)M(Sx_{2n-2}, Tx_{2n-2}, kt) + h(t)\max\{M(x, TSx_{2n-2}, kt), \\ &\quad M(Sx, Sx_{2n-2}, kt)\}. \end{aligned}$$

Taking limit as $n \to \infty$ we get

$$\begin{aligned} M(Sx, x, t) &\geq a(t)M(x, x, kt) + b(t)M(x, Sx, kt) \\ &+ c(t)M(x, x, kt) + h(t)\max\{M(x, x, kt), M(Sx, x, kt)\} \\ &> M(x, Sx, kt) \end{aligned}$$

is a contradiction. Thus $M(x, Sx, t) = 1$ that is $Sx = x$. Now, we prove that $Tx = x$. If $Tx \neq x$ then by (ii) we have,

$$\begin{aligned} M(x, Tx, t) &= M(Sx, TSx, t) \\ &\geq a(t)M(x, Sx, kt) + b(t)M(x, Sx, kt) \\ &+ c(t)M(Sx, Tx, kt) + h(t)\max\{M(x, TSx, kt), M(Sx, Sx, kt)\} \\ &> M(x, Tx, kt) \end{aligned}$$

is a contradiction. Hence $Sx = Tx = x$, that is x is a common fixed point of S and T. Now to prove uniqueness let, if possible, $y \neq x$ be another common fixed point of S and T. Then there exists $t > 0$ such that $M(x, y, t) < 1$ and

$$\begin{aligned} M(x,y,t) &= M(Sx,Ty,t) = M(Sx,TSy,t) \\ &\geq a(t)M(x,Sy,kt) + b(t)M(x,Sx,kt) \\ &+ c(t)M(Sy,TSy,kt) + h(t)\max\{M(x,TSy,kt), M(Sx,Sy,kt)\} \\ &= a(t)M(x,y,kt) + b(t) + c(t) + h(t)M(x,y,kt) \\ &> [(a(t) + b(t) + c(t)) + h(t)]M(x,y,kt) = M(x,y,kt), \end{aligned}$$

which is a contradiction. Therefore, $x = y$, i.e., x is a unique common fixed point of S and T. □

References

[1] El Naschie, MS., On the uncertainty of Cantorian geometry and two-slit experiment. Chaos, Solitons and Fractals 9 (1998), 517-529.

[2] El Naschie MS., The idealized quantum two-slit gedanken experiment revisited - Criticism and reinterpretation. Chaos, Solitons and Fractals 27 (2006), 9-13.

[3] George, A, Veeramani, P., On some result in fuzzy metric space. Fuzzy Sets Syst. 64 (1994), 395-399.

[4] Gregori, V, Sapena, A., On fixed-point theorem in fuzzy metric spaces. Fuzzy Sets and Syst. 125 (2002), 245-252.

[5] Jungck, G., Rhoades B. E., Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. **29** no. 3 (1998), 227-238.

[6] Kramosilk, I., Michalek, J., Fuzzy metric and statistical metric spaces. Kybernetica 11 (1975), 326-334.

[7] Rodríguez López, J., Ramaguera, S,. The Hausdorff fuzzy metric on compact sets. Fuzzy Sets Syst. 147 (2004), 273-283.

[8] Miheţ, D., A Banach contraction theorem in fuzzy metric spaces. Fuzzy Sets Syst. 144 (2004), 431-439.

[9] Schweizer, B., Sherwood, H., Tardiff RM. Contractions on PM-space examples and counterexamples. Stochastica 1 (1988) 1, 5-17.

[10] Singh, B., and Jain, S., A fixed point theorem in Menger space through weak compatibility. J. Math. Anal. Appl. 301 no. 2 (2005), 439-448.

[11] Zadeh, LA., Fuzzy sets. Inform and Control 8, (1965), 338-353.

Received by the editors January 31, 2007