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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A SCALAR DELAY DIFFERENCE EQUATIONS WITH CONTINUOUS TIME[1]

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Abstract. In this work we study the asymptotic behavior of solutions of scalar delay difference equation with continuous time of the form

$$x(t) = a(t)x(t-1) + b(t)x(p(t)),$$

where a, b, p are given real functions such that $p(t) < t$ and p is monotone increasing.

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1. Introduction

Huang, Yu and Dai in [4], [5] presented conditions when every solution of discrete difference equation $x_n - x_{n-1} = -F(x_n) + F(x_{n-k})$ is bounded and tends to a constant, where $k \in \mathbf{N}$, F is a continuous, increasing real function. Similar problems for differential equation $x'(t) = \beta(t)[x(t) - x(t - \tau(t))]$, where τ and β are positive continuous real functions, was investigated in papers by Atkinson and Haddock [1], by Diblik [2], [3] and in the references therein. The authors developed conditions which ensure that all solutions of considered equation are asymptotically constant.

In the following we give conditions for the existence of bounded solutions of a class of delay difference equations with continuous time. For the special case of considered equation we show the existence of asymptotically constant solution. The obtained results are analogous to the parts of results given by Huang, Yu and Dai for the special case of function F.

Assume that t_0 is a positive real number and $a,\ b : [t_0, \infty) \to \mathbf{R}$ are given real functions such that $0 < a(t) < 1$. Let the delay function $p : [t_0, \infty) \to \mathbf{R}$ be given such that, $p(t) < t$ for every $t \in [t_0, \infty)$ and p is monotone increasing. Consider the delay difference equation with continuous time of the form

$$x(t) = a(t)x(t-1) + b(t)x(p(t)), \tag{1}$$

and the special case of the above equation for $b(t) = 1 - a(t)$.

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2. Notations

For given $m \in \mathbf{N}$, $t \in \mathbf{R}_+$ and a function $f : \mathbf{R} \to \mathbf{R}$ we use the standard notation

$$\prod_{\ell=t}^{t-1} f(\ell) = 1, \qquad \prod_{\ell=t-m}^{t} f(\ell) = f(t-m)f(t-m+1)\dots f(t),$$

$$\sum_{\tau=t}^{t-1} f(\tau) = 0, \qquad \sum_{\tau=t-m}^{t} f(\tau) = f(t-m) + f(t-m+1) + \ldots + f(t).$$

The difference operator Δ is defined by $\Delta f(t) = f(t+1) - f(t)$. For a function $g : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$, the difference operator Δ_t is given by $\Delta_t g(t,a) = g(t+1,a) - g(t,a)$. Set $t_{-1} = \min\{\inf\{p(s) : s \geq t_0\}, t_0 - 1\}$, $t_n = \inf\{s : p(s) > t_{n-1}\}$ for all $n = 1, 2, \ldots$ Then $\{t_n\}_{n=-1}^{\infty}$ is an increasing sequence such that

$$\lim_{n\to\infty} t_n = \infty, \qquad \bigcup_{n=1}^{\infty} [t_{n-1}, t_n) = [t_0, \infty) \quad \text{and} \quad p(t) \in [t_{n-1}, t_n)$$

for all $t_n \leq t < t_{n+1}$, $n = 0, 1, 2, \ldots$ For a given function $\varphi : [t_{-1}, t_0) \to \mathbf{R}$, Equation (1) has the unique solution x^{φ} satisfying the *initial condition*

$$x^{\varphi}(t) = \varphi(t) \quad \text{for} \quad t_{-1} \leq t < t_0. \tag{2}$$

For a given nonnegative integer n, fix a point $t \geq t_0$, and define the natural numbers $k_n(t)$ such that $k_n(t) := [t - t_n]$, $n = 0, 1, 2, \ldots$, where $[t]$ is the integer part of t for all $t \in \mathbf{R}$. Then, for $t \in [t_n, t_{n+1})$, we have $t - k_n(t) - 1 < t_n$ and $t - k_n(t) \geq t_n$, $n = 0, 1, 2, \ldots$, and for the arbitrary $t \geq t_0$, $m = 0$ we have $t - k_0(t) - 1 < t_0$ and $t - k_0(t) \geq t_0$. Set $T_n(t) := \{t - k_n(t) - 1, t - k_n(t), t - k_n(t) + 1, \ldots, t - 1, t\}$.

3. Main Results

In the paper we will use the following hypotheses:

- (H_1) $p : [t_0, \infty) \to \mathbf{R}$ is a given function such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \leq t - \delta$ for every $t \in [t_0, T]$, and p is monotone increasing;
- (H_2) $a : [t_0, \infty) \to \mathbf{R}$ is a given real function such that $0 < a(t) < 1$ for all $t \geq t_0$;
- (H_3) $b : [t_0, \infty) \to \mathbf{R}$ is a given real function such that $|b(t)| \leq 1 - a(t)$, for all $t \geq t_0$;
- (H_4) $b : [t_0, \infty) \to \mathbf{R}$ is a given real function such that $b(t) \geq 1 - a(t)$, for all $t \geq t_0$;

Lemma 3.1. *The initial value problem* (1), (2) *for* $t \in [t_n, t_{n+1})$ *is equivalent to the functional equation of the form*

$$x(t) = x(t - k_n(t) - 1) \prod_{\ell=t-k_n(t)}^{t} a(\ell) + \sum_{\tau=t-k_n(t)}^{t} b(\tau)x(p(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell). \tag{3}$$

Proof. Multiply both sides of Equation (1) by the term $\displaystyle\prod_{\ell=t-k_n(t)}^{t} \frac{1}{a(\ell)}$.
Let $t \in [t_n, t_{n+1})$ and $\tau \in T_n(t)$. Then Equation (1) is equivalent to

$$\Delta_\tau \left(x(\tau - 1) \prod_{\ell=t-k_n(t)}^{\tau-1} \frac{1}{a(\ell)} \right) = b(\tau)x(p(\tau)) \prod_{\ell=t-k_n(t)}^{\tau} \frac{1}{a(\ell)} .$$

Summing up both sides of the above equation from $t - k_n(t)$ to t we obtain

$$x(t) \prod_{\ell=t-k_n(t)}^{t} \frac{1}{a(\ell)} - x(t - k_n(t) - 1) = \sum_{\tau=t-k_n(t)}^{t} b(\tau)x(p(\tau)) \prod_{\ell=t-k_n(t)}^{\tau} \frac{1}{a(\ell)} ,$$

or the equivalent form (3). □

Theorem 3.1. *Suppose that conditions* (H_1), (H_2) *and* (H_3) *hold. Let* $x = x^\varphi$ *be the solution of the initial value problem* (1), (2), *and set*

$$A_n := \sup_{t_{n-1} \le t < t_n} |x(t)|, \quad n = 0, 1, 2, \ldots$$

Then $A_{n+1} \le A_n$ *for all nonnegative integer* n.

Proof. For all natural numbers n and for the arbitrary number $\epsilon > 0$ there is a time $t \in [t_n, t_{n+1})$ such that $|x(t)| \ge A_{n+1} - \epsilon$. Then we obtain

$$\begin{aligned} A_{n+1} - \epsilon &\le |x(t)| \\ &\le |x(t - k_n(t) - 1)| \prod_{\ell=t-k_n(t)}^{t} a(\ell) + \sum_{\tau=t-k_n(t)}^{t} |b(\tau)||x(p(\tau))| \prod_{\ell=\tau+1}^{t} a(\ell) \\ &\le A_n \left(\prod_{\ell=t-k_n(t)}^{t} a(\ell) + \sum_{\tau=t-k_n(t)}^{t} (1 - a(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell) \right) \\ &= A_n \left(\prod_{\ell=t-k_n(t)}^{t} a(\ell) + \sum_{\tau=t-k_n(t)}^{t} \Delta_\tau \left(\prod_{\ell=\tau}^{t} a(\ell) \right) \right) = A_n. \end{aligned}$$

If $\epsilon \to 0$ we will obtain that $A_{n+1} \le A_n$ for all nonnegative integer n. □

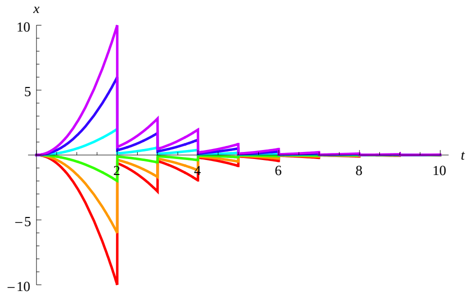

Figure 1: Constant delay case with property (H3)

Example 3.1. Figure 1 shows the graphs of solutions for the initial value problem $x(t) = 1/4\ x(t-1) + 1/8\ x(t-2)$, $x^{\varphi}(t) = jt^2$ for $0 \le t < 2$, where $j = -2.5, -1.5, -0.5, 0.5, 1.5, 2.5$.

Example 3.2. Figure 2 shows the graphs of solutions for the initial value problem $x(t) = 1/4\ x(t-1) + 1/8\ x(t/2)$, $x^{\varphi}(t) = jt^2 \quad \text{for} \quad 1 \le t < 2$, where $j = -2.5, -1.5, -0.5, 0.5, 1.5, 2.5$.

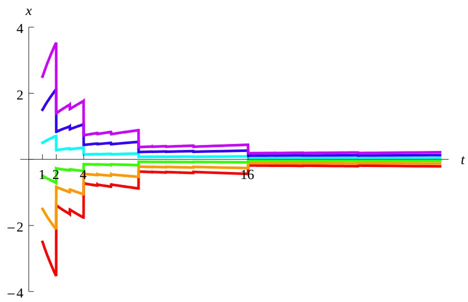

Figure 2: Pantograph delay case with property (H3)

Example 3.3. Figure 3 shows the graphs of solutions for the initial value problem $x(t) = 1/4\ x(t-1) + 1/8\ x\left(\sqrt{t}\right)$, $x^{\varphi}(t) = jt^2 \quad \text{for} \quad 1 \le t < 2$, where $j = -2.5, -1.5, -0.5, 0.5, 1.5, 2.5$.

Corollary 3.1. *Suppose that conditions (H_1), (H_2) and (H_3) hold. Then every solution $x = x^{\varphi}$ of the initial value problem* (1), (2) *is bounded and*

$$|x(t)| \le \lim_{n\to\infty} A_n \ .$$

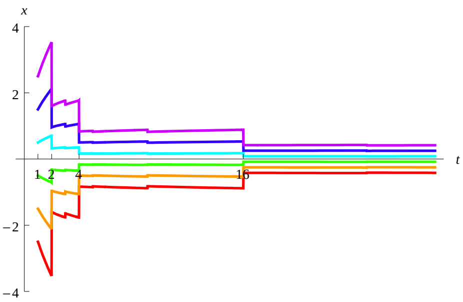

Figure 3: Power delay case with property (H3)

Remark 3.1. If the coefficient function b in Equation (1) is nonnegative for $t \geq t_0$, then the solution $x = x^{\varphi}$ of the initial value problem (1), (2) is nonnegative for $t \geq t_0$. If the coefficient function b in Equation (1) is nonpositive for $t \geq t_0$, then the solution $x = x^{\varphi}$ of the initial value problem (1), (2) is nonpositive for $t \geq t_0$.

Theorem 3.2. *Suppose that conditions (H_1), (H_2) and (H_4) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem* (1), (2), *and let*

$$B_n := \inf_{t_{n-1} \leq t < t_n} x(t), \quad C_n := \sup_{t_{n-1} \leq t < t_n} x(t), \quad n = 0, 1, 2, ...$$

Then for all nonnegative integer n is $B_{n+1} \geq B_n$ for $\varphi(t) \geq 0$, $t \in [t_{-1}, t_0)$ and $C_{n+1} \leq C_n$ for $\varphi(t) \leq 0$, $t \in [t_{-1}, t_0)$.

Proof. Let $\varphi(t) \geq 0$, $t \in [t_{-1}, t_0)$. For all natural numbers n and for the arbitrary number $\epsilon > 0$ there is a time $t \in [t_n, t_{n+1})$ such that $x(t) \leq B_{n+1} + \epsilon$. Then we obtain

$$\begin{aligned} B_{n+1} + \epsilon &\geq x(t) \\ &= x(t - k_n(t) - 1) \prod_{\ell = t - k_n(t)}^{t} a(\ell) + \sum_{\tau = t - k_n(t)}^{t} b(\tau) x(p(\tau)) \prod_{\ell = \tau + 1}^{t} a(\ell) \\ &\geq B_n \left(\prod_{\ell = t - k_n(t)}^{t} a(\ell) + \sum_{\tau = t - k_n(t)}^{t} (1 - a(\tau)) \prod_{\ell = \tau + 1}^{t} a(\ell) \right) \\ &= B_n \left(\prod_{\ell = t - k_n(t)}^{t} a(\ell) + \sum_{\tau = t - k_n(t)}^{t} \Delta_\tau \left(\prod_{\ell = \tau}^{t} a(\ell) \right) \right) = B_n. \end{aligned}$$

If $\epsilon \to 0$ we will obtain that $B_{n+1} \geq B_n$ for each nonnegative integer n.
In similar way for the case $\varphi(t) \leq 0$, $t \in [t_{-1}, t_0)$ we can obtain that $C_{n+1} \leq C_n$ for each nonnegative integer n. $\square$

Example 3.4. Figure 4 shows the graphs of solutions for the initial value problem $x(t) = 1/4\ x(t-1) + 3/2\ x(t-2)$, $x^{\varphi}(t) = \varphi(t)$ for $0 \le t < 2$, where $\varphi = jt^2$, $j = -2.5, -1.5, -0.5, 0.5, 1.5, 2.5$.

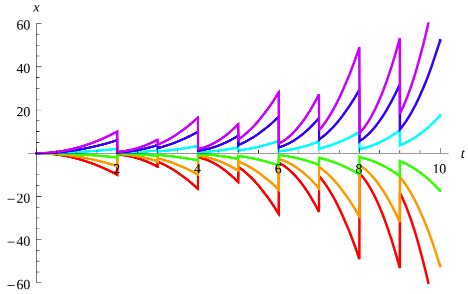

Figure 4: Constant delay case with property (H4)

Example 3.5. Figure 5 shows the graphs of solutions for the initial value problem $x(t) = 1/(t+1)^2 x(t-1) + 3/2\ x\,(t/2)$, $x^{\varphi}(t) = j\sqrt{t}$ for $1 \le t < 2$, where $j = -2.5, -1.5, -0.5, 0.5, 1.5, 2.5$.

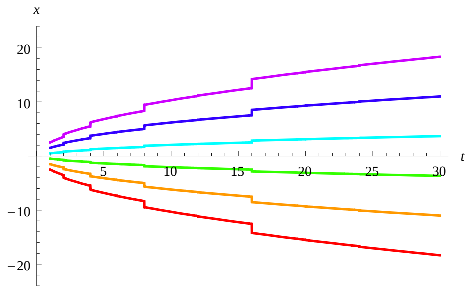

Figure 5: Pantograph delay case with property (H4)

Example 3.6. Figure 6 shows the graphs of solutions for the initial value problem $x(t) = 1/(t+1)^2 x(t-1) + 3/2\ x\left(\sqrt{t}\right)$, $x^{\varphi}(t) = j\sqrt{t}$ for $1 \le t < 2$, where $j = -2.5, -1.5, -0.5, 0.5, 1.5, 2.5$.

A special case of Equation (1), when $b(t) = 1 - a(t)$, is the equation

$$x(t) = a(t)x(t-1) + (1 - a(t))x(p(t)). \tag{4}$$

From the above theorems we get the following results.

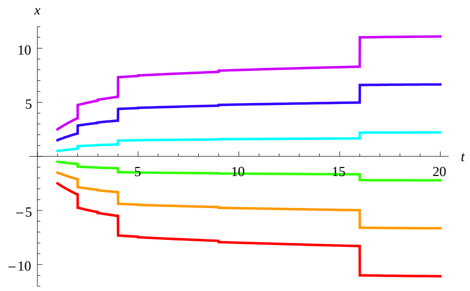

Figure 6: Power delay case with property (H4)

Corollary 3.2. *Suppose that (H_1) and (H_2) hold. Let $x = x^\varphi$ be the solution of the initial value problem* (4), (2), *and let $\varphi(t) \geq 0$ or $\varphi(t) \leq 0$ for $t \in [t_{-1}, t_0)$,*

$$B_n := \inf_{t_{n-1} \leq t < t_n} x(t), \quad A_n := \sup_{t_{n-1} \leq t < t_n} x(t), \quad n = 0, 1, 2, \dots$$

Then every solution $x = x^\varphi$ of the initial value problem (4), (2) *is bounded and*

$$\lim_{n \to \infty} B_n \leq x(t) \leq \lim_{n \to \infty} A_n \ .$$

Example 3.7. Figure 7 shows the graphs of solutions for the initial value problem $x(t) = 1/(t+1)^2 x(t-1) + \left(1 - 1/(t+1)^2\right) x\,(t-2)$, $x^\varphi(t) = j\sqrt{t}$, $0 \leq t < 2$, where $j = -2.5, -1.5, -0.5, 0.5, 1.5, 2.5$.

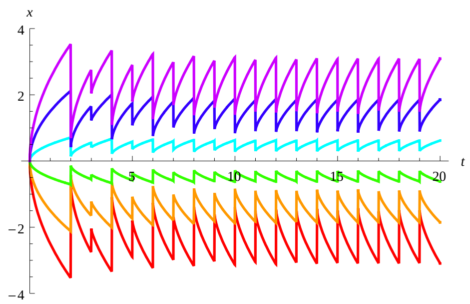

Figure 7: Constant delay case with property (H3) and (H4)

Remark 3.2. Under conditions of Corollary 3.2 we can conclude that for

$$\lim_{n \to \infty} B_n = \lim_{n \to \infty} A_n \ ,$$

the solution of the initial value problem (4), (2) is asymptotically constant.

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