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WELL-SUITED MULTIPROCESSOR TOPOLOGIES WITH SMALL NUMBER OF PROCESSORS[1]

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Abstract. Homogeneous multiprocessor systems are usually modelled by undirected graphs. Vertices of these graphs represent the processors, while edges denote the connection links between adjacent processors. Let G be a graph with diameter D, maximum vertex degree Δ, the largest eigenvalue λ_1 and m distinct eigenvalues. The products $m\Delta$ and $(D+1)\lambda_1$ are called the tightness of G of the first and second type, respectively. In the recent literature it was suggested that graphs with a small tightness of the first type are good models for the multiprocessor interconnection networks. We extended analysis to four types of tightness and found all graphs with tightness values at most eight.

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1. Introduction

Multiprocessor interconnection networks [7] are usually modelled by undirected, connected graphs [9, 10]. Vertices of these graphs represent the processors, while edges denote the connection links between adjacent processors. Weights on vertices can be introduced to ensure modelling of heterogeneous multiprocessor systems (vertex weight represents, for example, the speed of a corresponding processor). One can also introduce weights on edges in order to model communication links with different speed and/or capacity. In this paper we deal with homogeneous multiprocessor systems, and therefore we do not consider weighted graphs.

To avoid the main drawback of multiprocessor systems (communication overhead) interconnection networks have to satisfy two contradictory properties: to minimize the "number of wires" and to maximize the data exchange rate. This means that the paths connecting each two processors have to be as short as possible while the average number of connections per processor has to be as small as possible.

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Recently, the link between the design of multiprocessor topologies and the theory of graph spectra [4] has been recognized [6]. The main conclusion of [6] regarding the multiprocessor design and, particularly the load balancing within given multiprocessor systems was the following: if the product $m\Delta$ of the number m of distinct eigenvalues of a graph adjacency matrix and the maximum vertex degree Δ is small for a given graph G, the corresponding multiprocessor topology was expected to have good communication properties and has been called *well-suited*. It has been pointed out that there exists an efficient algorithm which provides optimal load balancing within $m - 1$ computational steps. This algorithm is summarized in [5]. The graphs with large $m\Delta$ were called *ill-suited* in [6] and are not considered suitable for design multiprocessor networks.

The clustering of multiprocessor topologies into well-suited and ill-suited classes as proposed in [6] raised very natural question: are all of the widely used multiprocessor architectures well-suited according to this classification? Let us point out the classes for some of the widely used multiprocessor architectures:

well-suited	ill-suited
hypercubes;	stars;
butterflies;	meshes;
some trees;	processor arrays;
(complete) bipartite graphs;	circuits.

According to the authors' opinion, not all of these architectures should be considered ill-suited. For example, 2-dimensional processor arrays (also known as mesh architectures) have nice properties as multiprocessor topologies. Stars are always used to model the well known master-slave multiprocessor architecture which has been used in the bulk of the parallel applications [8]. A survey of frequently used interconnection networks is given in [5].

In order to extend the application of the theory of graph spectra to the design of multiprocessor topologies, in [2] we have considered also some other related types of graph invariants under common name *tightness*, and investigated their suitability for describing the corresponding interconnection networks.

In this paper we provide detailed algorithms on how to determine graphs with the value of tightness less than or equal to a given number. We determine all graphs for which tightness value does not exceed 8. In [3], we pass to the next step and identify all graphs for which tightness value (all four types) does not exceed 9. These graphs happen to be of small order (not exceeding 10 vertices) and there are 69 such graphs. In this way we made a catalogue of models for small well-suited (according to each tightness) multiprocessor networks. Additional result provided in [3] was the catalogue of extremal graphs with up to $n = 10$ vertices. For each n, we have identify graphs with minimum value of each tightness, and obtain usually star graphs and circuits.

2. Preliminaries

In [2], we introduced the following definitions. (To avoid trivial technical discussions, here and in the whole paper we shall assume that the graphs have at least two vertices. Moreover, we deal with connected graphs only.)

Definition 1. *The* type one mixed tightness t_1 *of a graph* G *is defined as the product of the number of distinct eigenvalues* m *and the maximum vertex degree* Δ *of* G*, i.e.* $t_1(G) = m\Delta$.

Definition 2. Structural tightness $stt(G)$ *is the product* $(D+1)\Delta$*, where* D *is diameter and* Δ *is the maximum vertex degree of a graph* G.

Definition 3. Spectral tightness $spt(G)$ *is a product of the number of distinct eigenvalues* m *and the largest eigenvalue* λ_1 *of a graph* G.

Definition 4. Type two mixed tightness $t_2(G)$ *is defined as a function of the diameter* D *of* G *and the largest eigenvalue* λ_1*, i.e.* $t_2(G) = (D+1)\lambda_1$.

In [2], it has been noted that the four tightness values are partially ordered by the relation $\leq$ in the following way

$$t_2(G) \leq spt(G) \leq t_1(G)$$

and

$$t_2(G) \leq stt(G) \leq t_1(G).$$

Therefore, replacing t_1 with stt in the criterion for a good interconnection network (i.e. replacing the number of distinct eigenvalues m with the quantity $D+1$ where D is the diameter) captures graphs with small values for diameter and maximum vertex degree as good models for multiprocessor topologies.

On the other hand, introducing spt instead of t_1 (i.e. replacing maximal vertex degree Δ with the largest eigenvalue λ_1, a kind of average vertex degree [2]) causes the acceptance of star graphs as good models.

If we finally pass to t_2, we get additional graphs that are characterized by a suitable combination of small values for the diameter D and for the largest eigenvalue λ_1.

Let $\mathcal{G}_c$ be the set of connected graphs with at least two vertices. Let us introduce the following notation:

$$T_1^a = \{G : G \in \mathcal{G}_c,\ t_1(G) \leq a\}, \quad T_{stt}^a = \{G : G \in \mathcal{G}_c,\ stt(G) \leq a\},$$

$$T_{spt}^a = \{G : G \in \mathcal{G}_c,\ spt(G) \leq a\} \ , \qquad T_2^a = \{G : G \in \mathcal{G}_c,\ t_2(G) \leq a\}.$$

It is obvious that $T_1^a \subseteq T_{stt}^a \subseteq T_2^a$ and $T_1^a \subseteq T_{spt}^a \subseteq T_2^a$ because of the partial order between tightness values. Having in mind the inclusions between these sets we can represent them in the form

$$T_1^a = A,$$

$$T_{stt}^a = A \cup B, \quad T_{spt}^a = A \cup C,$$

$$T_2^a = A \cup B \cup C \cup D,$$

where A, B, C, D are sets of graphs illustrating the influence of each particular tightness definition. Moreover, according to Theorem 1 from [2], each of these sets is finite.

3. Graphs with Small Type 1 Mixed Tightness

Let us consider the case T_1^6. More precisely, we are looking for graphs $G \in \mathcal{G}_c$ such that $t_1(G) = m\Delta \leq 6 = a$. Since both values ($m$ and Δ) are integers, we can distinguish the following cases:

a° $m = 1$. This is trivial case satisfied only for K_1 which is excluded from considerations.

b° $m = 2$, $\Delta \leq 3$. Two distinct eigenvalues appear only in complete graphs, K_n, $n = 2, 3, \ldots$. On the other hand, $\Delta \leq 3$ defines the upper bound on the number of vertices to $n = 4$ since each vertex has to be connected to all the others. Consequently, this case involves K_2, K_3 and K_4.

c° $m = 3$, $\Delta \leq 2$. Graphs with $m = 3$ are well studied in the literature (cf. [4], p. 108). For example, *stars* S_n, containing central vertex connected to all the others by $n - 1$ edges belong to this class. Having the conditions $m = 3$, $\Delta \leq 2$ in mind, we can consider only graphs where each vertex has at most 2 neighbors. Therefore, the only star that we can take into account is $S_3 = P_3$. Graphs with $\Delta \leq 2$ are also circuits C_n, $n = 2, 3, \ldots$. They have $m = \lfloor \frac{n}{2} \rfloor + 1$ distinct eigenvalues. Among all circuits, $m = 3$ have C_4 and C_5. Consequently, graphs satisfying these conditions are two circuits and 3-vertex path, namely, C_4, C_5 and P_3.

d° $m = 4, 5, 6$, $\Delta \leq 1$. This case involves only disconnected graphs which are excluded from consideration.

Now, it is easy to see that $T_1^6 = \{K_2, K_3, P_3, K_4, C_4, C_5\}$.

If we analyze T_1^8, the following cases are of interest:

a° $m = 2$, $\Delta \leq 4$. Here we have complete graphs K_2, K_3, K_4, K_5 with the same reasoning as in the case b° for T_1^6.

b° $m = 3$, $\Delta \leq 2$. This case is the same as c° for T_1^6, and we get again the graphs C_4, C_5 and P_3.

c° $m = 4$, $\Delta \leq 2$. Several classes of regular graphs with four distinct eigenvalues are described in [11], but the whole set is not determined yet. In this case we are limited to the circuits and paths by the condition $\Delta \leq 2$, and therefore we have only C_6, C_7 and P_4.

Finally, we conclude with the following theorem.

Theorem 1.

$$T_1^8 = \{K_2, K_3, K_4, K_5, P_3, P_4, C_4, C_5, C_6, C_7\} = T_1^6 \cup \{P_4, K_5, C_6, C_7\}.$$

4. Graphs with Small Structural Tightness

This type of tightness also takes integer values, since $stt = (D + 1)\Delta$.

If we analyze the set T_{stt}^6, we can distinguish several cases (quite similar to those considered when looking for T_1^6):

a° $D = 0$, $\Delta \leq 6$. There are no non-trivial graphs satisfying these conditions.

b° $D = 1$, $\Delta \leq 3$. First condition implies that graphs must be complete, i.e. only K_n are involved while the second one in that case limits the number of vertices, i.e. $n \leq 4$. Therefore, we can include only the graphs K_2, K_3 and K_4.

c° $D = 2$, $\Delta \leq 2$. This case involves paths and circuits, namely P_3, C_4 and C_5.

d° $D = 3, 4, 5$, $\Delta \leq 1$ are contradictory conditions.

Therefore, we can conclude that $T_{stt}^6 = \{K_2, K_3, P_3, K_4, C_4, C_5\} = T_1^6$.

The analysis for T_{stt}^8 involves the following cases

a° $D = 1$, $\Delta \leq 4$. Here again only K_n are involved with the limit on the number of vertices $n \leq 5$. Therefore, we can include only the graphs K_2, K_3, K_4 and K_5.

b° $D = 2$, $\Delta \leq 2$. This case is the same as *c*° for T_{stt}^6 and we have P_3, C_4 and C_5.

c° $D = 3$, $\Delta \leq 2$. This case also involves paths and circuits, but with $D = 3$ and we get P_4, C_6 and C_7.

d° $D = 4, 5, 6, 7$, $\Delta \leq 1$ are contradictory conditions.

Therefore, we obtain the next theorem.

Theorem 2.

$$T_{stt}^8 = \{K_2, K_3, K_4, K_5, P_3, P_4, C_4, C_5, C_6, C_7\} = T_1^8.$$

5. Graphs with Small Spectral Tightness

As for the definition of *spt*, we have to analyze product of two positive numbers, one of them not always being integer. This may cause our analysis to be more difficult, but we can use the well known theoretical results from the theory of graph spectra.

Within this analysis an important role play the graphs with $\lambda_1 \leq 2$ (Smith graphs and their subgraphs). All relevant parameters for Smith graphs and their subgraphs have been calculated and summarized in Table 1 in [3]. As a useful tool for this study we explored newGRAPH programming package [1] to calculate values of m and λ_1 of obtained subgraphs. It is interesting to note that for C_n we always have $D = \lfloor \frac{n}{2} \rfloor$, $\Delta = 2$, $m = \lfloor \frac{n}{2} \rfloor + 1$, $\lambda_1 = 2$. Therefore, it holds $t_1(C_n) = stt(C_n) = spt(C_n) = t_2(C_n)$. Also for $S_5 = K_{1,4}$ we have $D = 2$, $\Delta = 4$, $m = 3$ and $\lambda_1 = 2$. Stars are only the special case in more general class of bipartite graphs. The main representative of this class is complete bipartite graph K_{n_1,n_2} having vertices divided into two sets and edges connecting each vertex from one set to all the vertices in the other set. For K_{n_1,n_2} we have $m = 3$, $\Delta = \max\{n_1, n_2\}$, $D = 2$, $\lambda_1 = \sqrt{n_1 n_2}$.

Let us start with $a = 6$, i.e. T^6_{spt}. The following cases can be recognized:

a° $m = 1$, $\lambda_1 \leq 6$ includes no non-trivial connected graphs because it has to be $D \leq 0$ according to the well known relation $m \geq 1 + D$ (see Theorem 3.13. from [4]).

b° $m = 2$, $\lambda_1 \leq 3$. Similarly to the previous case we have $D = 1$ and consequently, this case involves only complete graphs. Since for K_n, $\lambda_1 = n - 1$, here we have K_2, K_3, and K_4.

c° $m = 3$, $\lambda_1 \leq 2$. For this particular case, we have C_4 and C_5 as strongly regular graphs and $K_{1,2} = P_3$, $K_{1,3} = S_4$, and $K_{1,4} = S_5$, which are complete bipartite graphs [11].

d° $m \geq 4$. Here $\lambda_1 \leq 1.5$ and these cases do not give any new graphs (as can be verified in Table 1 from [3]).

Hence we have $T^6_{spt} = \{K_2, K_3, K_4, C_4, C_5, P_3, S_4, S_5\}$

Considering T^8_{spt} the following cases are of interest:

a° $m = 2$, $\lambda_1 \leq 4$. These conditions give us K_2, K_3, K_4, K_5.

b° $m = 3$, $\lambda_1 \leq 2.667$. C_4, C_5, $K_{1,2} = P_3$, $K_{1,3} = S_4$, $K_{1,4} = S_5$, $K_{1,5} = S_6$, $K_{1,6} = S_7$, $K_{1,7} = S_8$, $K_{2,3}$ appear in this case.

c° $m = 4$, $\lambda_1 \leq 2$. We have C_6, C_7, for $\lambda_1 = 2$ and P_4 for $\lambda_1 < 2$ according to Table 1 from [3].

d° $m \geq 5$. Now $\lambda_1 \leq 1.6$ and there are no graphs satisfying these conditions.

Hence, the following theorem holds.

Theorem 3.

$$T_{spt}^{8} = T_{spt}^{6} \cup \{C_6, C_7, K_5, S_6, S_7, S_8, K_{2,3}\} = T_1^8 \cup \{S_4, S_5, S_6, S_7, S_8, K_{2,3}\}.$$

6. Graphs with Small Type 2 Mixed Tightness

Considering $t_2 = (D+1)\lambda_1$ we also perform case analysis in a similar way.

For T_2^6 we distinguish the following cases:

a° $D = 1$, $\lambda_1 \leq 3$. This case involves only complete graphs and since for K_n, $\lambda_1 = n - 1$, here we have K_2, K_3, and K_4.

b° $D = 2$, $\lambda_1 \leq 2$. Again we have to consider Smith graphs and their subgraphs. Smith graphs with $D = 2$ are S_5, C_4, and C_5. By deleting their vertices we obtain also S_4 and $S_3 = P_3$.

c° $D = 3, 4, 5$. This implies $\lambda_1 \leq 1.5$ and checking in Table 1 from [3] we can see that no graphs satisfying these conditions exist.

Therefore, we conclude that $T_2^6 = \{K_2, K_3, K_4, S_3, S_4, S_5, C_4, C_5\}$.

The analysis of T_2^8 involves the following cases:

a° $D = 1$, $\lambda_1 \leq 4$. We obtain K_2, K_3, K_4, K_5.

b° $D = 2$, $\lambda_1 \leq 2.667$. Denote the set of graphs satisfying these conditions by Q_1. The restriction on λ_1 combined with the relation $\lambda_1 \geq \sqrt{\Delta}$ (cf. [4], p. 112) implies $\Delta \leq \lambda_1^2 < 8$, hence $\Delta \leq 7$. Therefore, from the inequality $n \leq 1 + \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \cdots + \Delta(\Delta-1)^{D-1}$ we have $n \leq 1 + 7 + 7 \cdot 6 = 50$. The set Q_1 is completed by Lemma 1.

c° $D = 3$, $\lambda_1 \leq 2$. For $\lambda_1 = 2$ we get C_6, C_7 and W_6, while $\lambda_1 < 2$ leads to the graph Z_5 (see Fig. 1).

d° $D \geq 4$, $\lambda_1 \leq 1.6$. There are no graphs satisfying these conditions.

Figure 1 contains some examples of graphs with $t_2 \leq 8$.

Lemma 1. *The set Q_1 consists of 17 graphs listed below.*

$n = 3:$ P_3;
$n = 4:$ G_5, G_6, C_4, S_4;
$n = 5:$ $G_{20}, G_{21}, G_{23}, K_{2,3}, C_5, S_5$;
$n = 6:$ $CP(93), CP(94), S_6$;
$n = 7:$ $S_7, N(7, 337)$;
$n = 8;$ S_8.

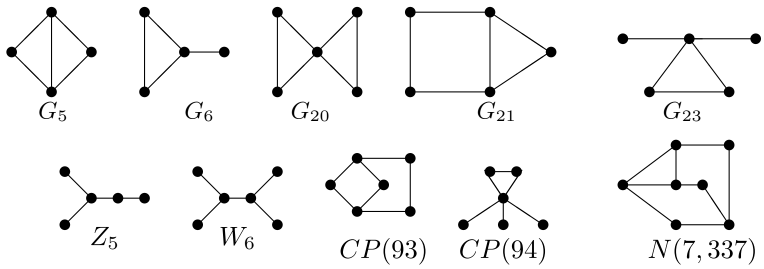

Figure 1: Some graphs from the set T_2^8

A proof of this Lemma is given in [3].

Summarizing these results we get the following theorem.

Theorem 4.

$$\begin{aligned} T_2^8 &= T_2^6 \cup Q_1 \cup \{K_5, C_6, C_7, Z_5, W_6\} \\ &= T_1^8 \cup \{S_4, S_5, S_6, S_7, S_8, G_5, G_6, G_{20}, G_{21}, G_{23}, K_{2,3}, \\ &\quad CP(93), CP(94), Z_5, W_6, N(7, 337)\}. \end{aligned}$$

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