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ABOUT ALMOST SYMPLECTIC STRUCTURES ON THE TOTAL SPACE OF THE TANGENT BUNDLE [1]

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Abstract. In this paper, starting from the general theory of the pseudoriemannian conjugation, which was systematically elaborated for the first time in [3], and from the general theory of the almost symplectic ω -conjugation, systematically elaborated in [4], the authors obtain a partition of the relativistic models $\left\{ \overset{G}{L} = \{E, G, D\} \right\}$, and also of the hamiltonian models $\left\{ \overset{(\omega)}{L} = \{E, \omega, D\} \right\}$, based on a general criterion of conjugation.

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1. Introduction

Let us consider a C^∞ differentiable, p-dimensional, paracompact and connected $(E_p = (E, [A]), \Re^p)$ manifold.

Example 1.1. Let us consider a vector bundle $\xi = (E, \pi, M)$, where the base M is a C^∞, n-dimensional, paracompact differentiable manifold, with the m-dimensional fiber type $\Re^m$. Then we will obtain on E the structure of C^∞, paracompact, differentiable manifold E_p, $(p = 2n)$.

Example 1.2. Let us consider the tangent bundle $\xi = (E = TM, \pi, M)$. Under the above conditions, with respect to M_n, we will obtain on $E = TM$ a structure of C^∞ differentiable manifold, E_p, $(p = 2n)$.

We chose these two examples because the following theory will be applied in these cases.

Under the given conditions:

a) There are riemannian structures on E_p (globally).

b) There are linear connections, $\{D\}$, on E_p (globally), with or without torsion.

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For the modelling in relativistic mechanics in the relativity theory it is necessary to consider pseudo-riemannian metrics $\{G\}$.

The existence of the almost symplectic structures, ω, or symplectic structures ($d\omega = 0$) is necessary for the modelling of the hamiltonian mechanical systems.

But, as is well-known, the study can be simplified if it can be done by splitting in two directions with a certain property. In this way the idea arises of a classification of all the models $\left\{ \overset{G}{L} = \{E, G, D\} \right\}$, respectively of all the models $\left\{ \overset{(\omega)}{L} = \{E, \omega, D\} \right\}$. This idea is developed in this paper.

2. G-conjugations

Definition 2.1 ([3])**.** Let us consider E_p a C^∞-differentiable, p-dimensional, paracompact, connected manifold, endowed with a pseudoriemannian metric G and $\overset{(1)}{D}, \overset{(2)}{D}$ two linear connections on E. Let us take $L_1 = \left(E, G; \overset{(1)}{D} \right); L_2 = \left(E, G; \overset{(2)}{D} \right)$ two models associated to the structure (E, G) and $\overset{(1)}{D}_{(1)}, \overset{(2)}{D}_{(1)}$ two one-dimensional distributions, $\overset{(1)}{D}_{(1)} : u \to \overset{(1)}{D}_{u(1)} \subset T_u E$, $\overset{(2)}{D}_{(1)} : u \to T_u E$ such that $\overset{(1)}{D}_{(1)} \perp \overset{(2)}{D}_{(1)}$:

$$G\left(\underset{(1)}{V}, \underset{(2)}{V} \right) = 0; (\forall)\ \underset{(1)}{V} \in \overset{(1)}{D}_{u(1)},\ \underset{(2)}{V} \in \overset{(2)}{D}_{u(1)} \tag{1}$$

If (1) is preserved at the parallel transport of the distribution $\overset{(1)}{D}_{(1)}$ with respect to $\overset{(1)}{D}$ and at the parallel transport of $\overset{(2)}{D}_{(1)}$ with respect to $\overset{(2)}{D}$, then we will say that the two models are G-conjugated or that the two linear connections $\overset{(1)}{D}, \overset{(2)}{D}$ are G-conjugated. We will write $L_1 \overset{G}{\sim} L_2$ or $\overset{(1)}{D} \overset{G}{\sim} \overset{(2)}{D}$.

Remark 1. If G is a pseudo-riemannian metric then the distributions can be selfconjugated, i.e. the pseudo-riemannian indicator $\varepsilon_v = 0$ for $V \in \overset{(1)}{D}_{(1)}$ or $V \in \overset{(2)}{D}_{(1)}$ (the distributions can be null or isotropic, which is different from the case of riemannian G).

Definition 2.2. Two models L_1, L_2 will be also called G-compatible if they are G-conjugated (comparison criteria).

We obtain two special situations:

Theorem 2.1. *Two models L_1, L_2 cannot be G-compatible if $\overset{(1)}{D}, \overset{(2)}{D}$ are metric G-compatible ($\overset{(1)}{D}G = 0, \overset{(2)}{D}G = 0$).*

Theorem 2.2. *No model $L = (E, G, D)$ with symmetric linear connection ($T = 0$) can be compatible with the model $L = (E, G, \nabla)$, where ∇ is the Levy-Civita connection.*

It results in:

Theorem 2.3. *Any model from the theory of the general relativity $L = (E, G, D)$, with D symmetric, is not compatible with the model of Einstein ((E, G, ∇)).*

Theorem 2.4. *There are relativistic models which are G-compatible with the Einstein model.*

Because of Theorem 2.1, Theorem 2.2 we can give a classification of the relativistic models. It easily results in:

Theorem 2.5. *The relation " $\overset{G}{\sim}$ " is not, in the general case, an equivalence one. It is symmetric but is not reflexive and transitive one.*

Theorem 2.6. *If $L_1 = \left(E_p, G; \overset{(1)}{D}\right) \overset{G}{\sim} L_2 = \left(E, G; \overset{(2)}{D}\right)$ and $\overset{(1)}{D}G = 0$ then $\overset{(1)}{D}_X G = \rho(X)G;\ \rho \neq 0.$*

Theorem 2.7. *Let us consider $C_{\underset{(G)}{\overline{D}}}$ the set of the models which are G-compatible with the model $\overline{L} = \left(E, G, \overline{D}\right)$, where $\overline{D}G = 0$. Then:*

1) on the set $C_{\underset{(G)}{\overline{D}}}$ the relation "$\overset{G}{\sim}$" is an equivalence one.

2) Two classes $C_{\underset{(G)}{\overset{(1)}{\overline{D}}}}, C_{\underset{(G)}{\overset{(1)}{\overline{D}}}}$ are disjointed. If $\overset{(1)}{D} \in C_{\underset{(G)}{\overset{(1)}{\overline{D}}}}, \overset{(2)}{D} \in C_{\underset{(G)}{\overset{(2)}{\overline{D}}}}$ then the models L_1, L_2 are not G-compatible.

From these results we get:

Theorem 2.8 (The partition theorem)**.** *The set of the relativistic models*

$$\{L = (E, G, \{D\})\} = \underset{(G)}{M}(D)$$

admit the partition: $\underset{(G)}{M}(D) = \underset{(G)}{M_1}(D) \cup \underset{(G)}{M_2}(D);\ \underset{(G)}{M_1}(D) \cap \underset{(G)}{M_2}(D) = \emptyset$, where $\underset{(G)}{M_1}(D) = \underset{\{\overline{D}\}}{\cup} C_{\underset{(G)}{\overline{D}}}; C_{\underset{(G)}{\overset{(1)}{\overline{D}}}} \cap C_{\underset{(G)}{\overset{(2)}{\overline{D}}}} = \emptyset\ \forall \overset{(1)}{\overline{D}}, \overset{(2)}{\overline{D}} \in \left\{\overline{D} | \overline{D}G = 0\right\}$, $\underset{(G)}{M_2}(D) = \underset{(G)}{M}(D) - \underset{(G)}{M_1}(D)$.

This partition completes the classification of the relativistic models given in [3].

Corollary 2.1. *The Einstein model with non-symmetric field, which was elaborated by Einstein in the last part of his life and was presented in an extended version, is included in* $C_{\underset{(G)}{\nabla}}$.

Application: In the case of the vector bundle $p = n + m$ is sufficient to consider a nonlinear connection $\overset{G}{N}\,(G\,(hX, vY)) = 0\ \forall X, Y \in X(E))$ and the theory will become more beautiful if we will consider the models on the set of linear d-connections because these preserve the orthogonality of the horizontal distribution H and the vertical one V ([2]). Therefore, the condition of G-conjugation for $\overset{(1)}{D}_{(1)}, \overset{(2)}{D}_{(1)}$ will be considered only for the cases when $\overset{(1)}{D}_{(1)}, \overset{(2)}{D}_{(1)}$ are both horizontal or $\overset{(1)}{D}_{(1)}, \overset{(2)}{D}_{(1)}$ are both vertical.

3. Almost symplectic conjugations

Definition 3.1. Let us consider $L_1 = \left(E, \omega; \overset{1}{D}\right), L_2 = \left(E, \omega; \overset{2}{D}\right)$ two models associated to the structures (E, ω), where ω is a general almost symplectic structure on E. Let us consider two arbitrary one-dimensional distributions, $\overset{(1)}{D}_{(1)} : u \to \overset{(1)}{D}_{u(1)} \subset T_uE$; $\overset{(2)}{D}_{(1)} : u \to \overset{(2)}{D}_{u(1)} \subset T_uE$ such that, in $u \in E$, we have:

$$\omega\left(\underset{(1)}{V}, \underset{(2)}{V}\right) \quad \forall \underset{(1)}{V} \in \overset{(1)}{D}_{u(1)}; \underset{(2)}{V} \in \overset{(2)}{D}_{u(1)} \tag{2}$$

If the condition (2) is preserved at the parallel transport of the distributions $\overset{(1)}{D}_{(1)}, \overset{(2)}{D}_{(1)}$ with respect to the connections $\overset{(1)}{D}, \overset{(2)}{D}$, then we will say that the two models are ω -conjugated $\left(L_1 \overset{\omega}{\sim} L_2\right)$ or $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$.

Definition 3.2. Let us consider (E_p, ω), where p is an even number and ω is a general almost symplectic structure on E. If D is a linear connection on E, then we will say that D is ω -compatible if we have:

$$D_X\omega = 0; \quad \forall X \in X(E) \tag{3}$$

Theorem 3.1. *There are linear connections* $\left\{\overline{D}\right\}$*, on E, which are* ω *-compatible.*

Proof. Let us consider D, an arbitrary linear connection on E, and $\{\overline{D}\}$, a linear connection on E, defined by:

$$\omega\left(Y, \overline{D}_X Z\right) = \omega\left(Y, D_X Z\right) + \frac{1}{2}\left(D_X \omega\right)(YZ) \ X, Y, Z \in X(E). \tag{4}$$

It results in: $\left(\overline{D}_X \omega\right)(Y, Z) = 0 \ \forall X, Y, Z \in X(E)$ so $\overline{D}$ is ω -compatible $\left(\overline{D}_X \omega = 0\right)$.

Definition 3.3. The linear connection $\overline{D}$, defined by (3), will be called the ω -compatibilisation of the connection D on E.

In the given conditions there are linear connections, D, on E, with torsion $(T \neq 0)$.

$$T(XZ) = D_X Z - D_Z X - [XZ] \neq 0 \tag{5}$$

but also symmetric linear connections (torsion free i.e. $T = 0$).

For the almost symplectic structures (E, ω) such connection, ∇, which must be ω -compatibile $(\nabla \omega = 0)$ and torsion free $\left(\overset{\nabla}{T} = 0\right)$, does not exist if ω is a general almost symplectic one. More precisely we have:

Theorem 3.2. *If ω is a general one, that means that it is not integrable, then:*

a) There are linear connections, $\overline{D}$ on E, which are ω -compatible $\left(\overline{D}\omega = 0\right)$.

b) There are linear connections, on E, without torsion.

c) Linear connections, $\overline{D}$, do not exist on E, which are ω -compatible and without torsion.

Proof. Generally speaking, we have for any linear connection D on E:

$$(d\omega)(XYZ) = \sum_{(XYZ)} \left\{ \left(D_X \omega\right)(YZ) + \omega\left(T(XY), Z\right) \right\} \ \forall X, Y, Z \tag{6}$$

where $d\omega$ is the external differential.

If $D_X \omega = 0$, $T = 0$ then $d\omega = 0$ so ω is integrable (the structure ω is symplectic). Therefore, if ω is a general one $(d\omega \neq 0)$, $\overline{D}$ does not exist on E, with the properties:

$$\overline{D}_X \omega = 0 \tag{7}$$

$$\overline{T} = 0 \tag{8}$$

Theorem 3.3. *If ω is a symplectic one then the connections $\overline{D} = \nabla$ exists with the properties* (7) *and* (8).

A proof can be found in [1] or, for $E = TM$ (tangent bundle) in [4], with some completions related to the horizontal distribution, HTM, which is supplementary to the vertical one, VTM.

From the above considerations, in the general case $d\omega \neq 0$, the partition of the almost symplectic conjugated models $\{(E, \omega \{D\})\}$ will be studied in a different way from the case of $\{(E, G, D)\}$.

To compare two almost symplectic models $L_1 = \left(E, \omega; \overset{1}{D}\right), L_2 = \left(E, \omega; \overset{2}{D}\right)$, a more general criterion is necessary. This criterion is given by the preservation condition of the ω -conjugation of the distributions $\overset{(1)}{D}_{u(1)}, \overset{(2)}{D}_{u(1)}$, in $u \in E$, at the parallel transport with respect to the linear connections $\overset{(1)}{D}, \overset{(2)}{D}$, that means it is given by the condition $L_1 \overset{\omega}{\sim} L_2$. The physical meaning of this criterion is obvious in the particular case of the models associated to the hamiltonian mechanical systems (on the phase space $E = T^*M$), such as the physical meaning of the G-conjugation criterion is obvious in Lagrangean modelling, on $E = TM$ (on the speed space). These special cases show that a classification criterion is necessary.

Definition 3.4. Let us consider two almost symplectic ω -conjugated models, $L_1 \overset{\omega}{\sim} L_2$. We will say that L_1, L_2 are ω -compatible.

In [4], starting from the mixed, covariant derivative, associated to $\overset{(1)}{D}, \overset{(2)}{D}$:

$$\left(\overset{(12)}{D}_X \omega\right)(YZ) = \left(\overset{(1)}{D}_X \omega\right)(YZ) - \omega\left(Y, \overset{(21)}{\tau}(XZ)\right) \tag{9}$$

and similarly for $\left(\overset{(21)}{D}_X \omega\right)(YZ)$ where:

$$\overset{(21)}{\tau}(XZ) = \overset{(2)}{D}_X Z - \overset{(1)}{D}_X Z = -\overset{(12)}{\tau}(XZ) \tag{10}$$

is elaborated a systematic theory of ω -conjugation, $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$. In the general case it is obtained:

Theorem 3.4. *([4]). We have* $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$ *if and only if :*

$$\left(\overset{(12)}{D}_X \omega\right)(YZ) = \alpha(X)\omega(YZ), \alpha \in \Lambda_1(E) \tag{11}$$

Also it results in:

Theorem 3.5. *Let us consider two models L_1, L_2. If $\overset{(1)}{D}, \overset{(2)}{D}$ are ω -compatible $\left(\overset{(1)}{D}\omega = 0; \overset{(2)}{D}\omega = 0\right)$ then L_1, L_2 cannot be ω -conjugated, i.e.they are not ω -compatible.*

Theorem 3.6. *Let us consider L_1, L_2 two models. If $\overset{(1)}{D}$ is ω -compatible or $\overset{(2)}{D}$ is ω -compatible and $\overset{(1)}{T} = \overset{(2)}{T}$ then L_1, L_2 are not ω -compatible.*

Theorem 3.7. *Let us consider the ω -compatible models L_1, L_2 . If $\overset{(1)}{D}$ is ω -compatible then we will have:*

$$\overset{(2)}{D}_X\omega = 2\alpha(X)\omega;\ X \in X(M), \alpha \neq 0 \tag{12}$$

From the above theorems it results:

Theorem 3.8. *Let us consider $\underset{(\omega)}{M}(D)$ the set of the ω -compatibile models. On the set $\underset{(\omega)}{M}(D)$, the relation "$\overset{\omega}{\sim}$ " is not an equivalence one. It is symmetrical: $L_1 \overset{\omega}{\sim} L_2 \Leftrightarrow L_2 \overset{\omega}{\sim} L_1$, but, in the general case, it is not reflexive or transitive.*

Let us take $\underset{(\omega)}{C_{\overline{D}}} = \left\{(E, \omega, D)\,|D\ \overset{\omega}{\sim} \overline{D}; \overline{D}\omega = 0\right\}$. It results in:

Theorem 3.9. *The restriction of the relation "$\overset{\omega}{\sim}$" to $\underset{(\omega)}{C_{\overline{D}}}$ is an equivalence relation, i.e. $\underset{(\omega)}{C_{\overline{D}}}$ is an equivalence class.*

Let us consider $\underset{(\omega)}{M_1}(D) = \underset{\overline{D}}{\cup}\underset{(\omega)}{C_{\overline{D}}}$; $\overline{D}\omega = 0$. Now we will take $\underset{(\omega)}{M_2}(D) = \underset{(\omega)}{M}(D) - \underset{(\omega)}{M_1}(D)$. It results in:

Theorem 3.10. *(the partition theorem).We have:*

$$\underset{(\omega)}{M}(D) = \underset{(\omega)}{M_1}(D) \cup \underset{(\omega)}{M_2}(D) \tag{13}$$

where $\underset{(\omega)}{M_1}(D) \cap \underset{(\omega)}{M_2}(D) = \emptyset$, $\underset{(\omega)}{M_1}(D) = \underset{\overline{D}}{\cup}\underset{(\omega)}{C_{\overline{D}}}; \underset{(\omega)}{C_{\overset{(1)}{\overline{D}}}} \cap \underset{(\omega)}{C_{\overset{(2)}{\overline{D}}}} = \emptyset \left(\overline{D}\omega = 0; \overset{(1)}{\overline{D}}\omega = 0; \overset{(2)}{\overline{D}}\omega = 0\right)$

This theorem emphasizes how to analyze the ω -compatibilisation of the almost symplectic conjugated models.

Corollary 3.1. *If ω is a symplectic structure then there are classes,* $\left\{ \underset{(\omega)}{C_{\nabla}} \right\}$ *with* $\nabla\omega = 0; \overset{\nabla}{T} = 0$.

Corollary 3.2. *If ω is symplectic then $D \in \underset{(\omega)}{C_{\nabla}}$ cannot be torsion free.*

Corollary 3.3. *If ω is symplectic then $D \in \underset{(\omega)}{C_{\nabla}}$ is a linear semisymmetric connection in Schouten-Thomas way.*

Example 3.1. If $D \overset{\omega}{\sim} \overset{(0)}{D}$, with $\overset{(0)}{D}$ an arbitrary fixed linear connection, then from (2.6) the following transformations result $\omega\left(Y, D_X Z\right) = \omega\left(Y, \overset{(0)}{D}_X Z\right) + \left(\overset{(0)}{D}_X \omega\right)(YZ) - \alpha(X)\omega(YZ)$ with $\alpha \in \Lambda_1(E)$, arbitrary chosen.

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References

[1] Gheorghiev, V., Oproiu, V., Differentiable manifolds. (in Romanian) Vol.I-II, Ed.Acad.Române, 1976-1979

[2] Miron R., Anastasiei M., Vector bundles. Lagrange Spaces. Applications to

[3] Stavre P., Vector bundles, Ed. EUC,vol.I- 2004, II- 2005, III- 2006

[4] Stavre, P., An introduction to the theory of the conjugated geometrical structures. Applications to the relativity, Ed. Matrix, Bucarest, 2007

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