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SPACES WITH NON-SYMMETRIC AFFINE CONNECTION[1]

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Abstract. The beginning of the study of non-symmetric affine connection spaces is especially in relation with the works of A. Einstein on United Field Theory (UFT). The paper is a short survey of the development of the theory of these spaces.

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1. Notion, covariant derivatives

Although the notion of non-symmetric affine connection is used in several works before A. Einstein, for example in [6] (Eisenhart, 1927), [10] (Hayden, 1932), the use of non-symmetric connection became especially actual after appearance the works of Einstein, related to creating the Unified Field Theory (UFT).

Einstein was not satisfied with his General Theory of Relativity (GTR, 1916), and from 1923 to the end of his life (1955), he worked on various variants of UFT. This theory had the aim to unite the gravitation theory, to which is related GTR, and the theory of electromagnetism.

Remark that at UFT the symmetric part $g_{\underline{ij}}$ of the non-symmetric basic tensor g_{ij} is related to gravitation, and antisymmetric $g_{\underset{\vee}{ij}}$ to the electromagnetism. The same is valid for $\Gamma^i_{\underline{jk}}$ and $\Gamma^i_{\underset{\vee}{jk}}$. While at the Riemannian space (the space of GTR) the connection coefficients are expressed by virtue of g_{ij}, in Einstein's works on UFT (1945-1955) (e.g. [1]-[5]) the connection between these magnitudes are determined by the equations

$$(1)\qquad g_{\underset{+-}{ij};m} \equiv g_{ij,m} - \Gamma^p_{im} g_{pj} - \Gamma^p_{mj} g_{ip} = 0, \quad (g_{ij,m} = \frac{\partial g_{ij}}{\partial x^m}).$$

Beginning with 1951 L. P. Eisenhart ([7], [8]) has dealt in several works with the problems of spaces with non-symmetric basic tensor and non-symmetric connection. At [21], [22], [24], are also used non-symmetric connections.

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Because of nonsymmetry of the connection coefficients L^i_{jk}, it is possible to define 4 kinds of covariant derivative in the space L_N. For example:

$$t^i_{j\underset{1,2,3,4}{|}m} = t^i_{j,m} + L^i_{\underset{mp,\,pm,\,pm,\,mp}{pm}} t^p_j - L^p_{\underset{mj,\,mj,\,jm,\,jm}{jm}} t^i_p. \tag{2}$$

2. Ricci type identities, curvature tensors and pseudotensors in L_N

In the place of the difference $t^i_{j;mn} - t^i_{j;nm}$, which gives the Ricci identity in Riemannian space, now we have 10 cases based on the 1^{st} and 2^{nd} kind of covariant derivative:

$$t^{i_1\ldots i_r}_{j_1\ldots j_s\underset{\lambda}{|}m\underset{\mu}{|}n} - t^{i_1\ldots i_r}_{j_1\ldots j_s\underset{\nu}{|}m\underset{\omega}{|}n}, \tag{3}$$

where

$(\lambda,\mu;\nu,\omega) = \{(1,1;1,1),(2,2;2,2),(1,2;1,2),(2,1;2,1),(1,1;2,2),(1,1;1,2),$
$(1,1;2,1),(2,2;1,2),(2,2;2,1),(1,2;2,1).$

In [11] Ricci identities are derived in the space L_N for a tensor of the type (r,s). For example:

$$t^i_{j\underset{1}{|}mn} - t^i_{j\underset{1}{|}nm} = \underset{1}{R}{}^i{}_{pmn} t^p_j - \underset{1}{R}{}^p{}_{jmn} t^i_p - 2L^p_{\underset{\vee}{mn}} t^i_{j\underset{1}{|}p}, \tag{4}$$

$$t^i_{j\underset{2}{|}mn} - t^i_{j\underset{2}{|}nm} = \underset{2}{R}{}^i{}_{pmn} t^p_j - \underset{2}{R}{}^p{}_{jmn} t^i_p + 2L^p_{\underset{\vee}{mn}} t^i_{j\underset{2}{|}p}, \tag{5}$$

$$\begin{aligned} t^i_{j\underset{1}{|}m\underset{2}{|}n} - t^i_{j\underset{1}{|}n\underset{2}{|}m} &= \underset{1}{A}{}^i{}_{pmn} t^p_j - \underset{2}{A}{}^p{}_{jmn} t^i_p + 4(L^i_{\underset{\vee}{pm}} t^p_{j,n} - L^p_{\underset{\vee}{jm}} t^i_{p,n})_{\underset{\vee}{mn}} \\ &+2(L^i_{mp}L^s_{jn} - L^i_{pm}L^s_{nj})_{\underset{\vee}{mn}} t^p_s + 2L^p_{\underset{\vee}{mn}} t^i_{j\underset{1}{|}p} \end{aligned} \tag{6}$$

$$t^i_{j\underset{1}{|}m\underset{2}{|}=n} - t^i_{j\underset{2}{|}n\underset{1}{|}m} = \underset{3}{R}{}^i{}_{pmn} t^p_j - \underset{3}{R}{}^p{}_{jmn} t^i_p, \tag{7}$$

where

$$\underset{1}{R}{}^i{}_{pmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm}, \tag{8}$$

$$\underset{2}{R}{}^i{}_{pmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp}, \tag{9}$$

$$(10)\qquad \underset{3}{R}{}^{i}{}_{pmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}(L^i_{pj} - L^i_{jp}),$$

$$(11)\qquad \underset{1}{A}{}^{i}{}_{jmn} = L^i_{jmn} - L^i_{jn,m} + L^p_{jm}L^i_{np} - L^p_{jn}L^i_{mp}$$

$$(12)\qquad \underset{2}{A}{}^{i}{}_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{mj}L^i_{pn} - L^p_{nj}L^i_{pm}$$

....

$$(13)\qquad \underset{15}{A}{}^{i}{}_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm},$$

The magnitudes $\underset{t}{R}{}^{i}{}_{pmn}$, $t = 1, 2, 3$ are tensors and we call them curvature tensors of the first, second and third kind respectively and $\underset{t}{A}{}^{i}{}_{jmn}$ $(t = 1, 2, \ldots, 15)$ are not tensors and we call them curvature pseudotensors of the first,..., the fifteenth kind respectively.

If in forming the Ricci-type identities, we use the third and the fourth kind of covariant derivative, we get ten new identities analogous to $(4) - (7)$. In these identities appear the same quantities $\underset{1}{R}, \underset{2}{R}, \underset{3}{R}; \underset{1}{A}, \underset{2}{A}, \ldots, \underset{15}{A}$, but in a different distribution. Only in the last case there appears a new curvature tensor $\underset{4}{R}$:

$$(14)\qquad a^i_{j\underset{3}{|}m\underset{4}{|}n} - a^i_{j\underset{4}{|}n\underset{3}{|}m} = \underset{4}{R}{}^{i}{}_{pmn}a^p_j + \underset{3}{R}{}^{p}{}_{jnm}a^i_p,$$

where

$$(15)\qquad \underset{4}{R}{}^{i}{}_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}(L^i_{pj} - L^i_{jp}).$$

In [12] the author proved that by Ricci-type identities, in which appear curvature pseudotensors, we can obtain identities in which appear 8 new curvature tensors $\underset{1}{\tilde{R}}, \ldots, \underset{8}{\tilde{R}}$, and which we call derived curvature tensors.

All above identities and all curvature tensors and pseudotensors of the space L_N are the generalizations of the Ricci-identity, respectively curvature tensor of the space with symmetric connection and of the Riemannian space.

In [13] is shown that from all tensors $\underset{1}{R}, \ldots, \underset{4}{R}; \underset{1}{\tilde{R}}, \ldots, \underset{8}{\tilde{R}}$ we have five independent ones, while the others are linear combinations of these five tensors and the curvature tensor R, formed by the symmetric connection $L^i_{\underline{jk}}$ (symmetric part of L^i_{jk}).

3. Geometric interpretations of the curvature tensors and pseudotensors in L_N

In L_N one can define two kinds of parallel displacement of vectors. For a vector field $v^i(t)$, defined along a curve $C : x^i = x^i(t)$, we say that it is a field of the first kind parallel vectors, respectively a second kind, if for the differential we have

$$\underset{1}{d}v^i = -L^i_{pm} v^p dx^m, \quad \underset{2}{d}v^i = -L^i_{mp} v^p dx^m. \tag{16a, b}$$

According to [9], one can give the next geometric interpretation of two kinds parallel displacement and the torsion in L_N. Consider at the tangent space a surface element, determined with help of two infinitesimal vectors, whose origin is the point $P(x^i)$ and let the ends of these vectors be $Q(x^i + dx^i), R(x^i + \delta x^i)$. Making the same kind of parallel displacement, e.g. the first, of the vector dx^i along δx^i and δx^i along dx^i, we get different points S, T for the ends and for their coordinates:

$$\underset{T}{x^i} - \underset{S}{x^i} = \underset{1}{d}(\delta x^i) - \underset{1}{\delta}(dx^i) = (L^i_{pm} - L^i_{mp}) dx^p \delta x^m = 2L^i_{\underset{\vee}{pm}} dx^p \delta x^m. \tag{17}$$

Consequently, using the same kind of parallel displacement, we obtain $S \neq T$, if the connection is non-symmetric, i.e. if $L^i_{\underset{\vee}{pm}} \neq 0$. But, calculating $\delta(dx^i)$ with respect to $(16a)$, and $d(\delta x^i)$ with respect to $(16b)$, we get

$$d(\delta x^i) = \underset{2}{d}(\delta x^i) = -L^i_{mp} \delta x^p dx^m, \delta(dx^i) = \underset{1}{\delta}(dx^i) = -L^i_{pm} \delta x^m dx^p,$$

from where

$$\underset{T}{x^i} - \underset{S}{x^i} = \underset{2}{d}(\delta x^i) - \underset{2}{\delta}(dx^i) = 0,$$

that is the points S and T coincide and we obtain an infinitesimal parallelogram $PQSR$.

To obtain a geometric interpretation of curvature tensor, F. Graif considers the first kind parallel displacement of a vector v^i along the whole considered contour $PQSR$ and for the increment obtains

$$\underset{1}{\triangle} v^i = \underset{1}{R}{}^i{}_{jmn} v^j dx^m \delta x^n. \tag{18}$$

Using the second kind parallel displacement, we obtain

$$\underset{2}{\triangle} v^i = \underset{2}{R}{}^i{}_{jmn} v^j dx^m \delta x^n. \tag{19}$$

M. Prvanović ([23], 1977) uses for two opposite sides of mentioned parallelogram the first kind displacement, and for the rest of sides over sides - the second kind displacement. In this manner one obtains

$$\underset{3}{\triangle} v^i = -\underset{3}{R}{}^i{}_{jnm} v^j dx^m \delta x^n, \tag{20}$$

and changing the kind of the displacement along all the sides obtains

$$\underset{4}{\triangle} v^i = \underset{4}{R}{}^i{}_{jmn} v^j dx^m \delta x^n. \tag{21}$$

On the basis of the above one may attempt to examine all the possibilities that appear by changing the kind of displacement of vectors along the contour $PQSR$. We have in all $2^4 = 16$ cases (4 sides, 2 kinds of displacement), which are presented at the table [15]:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
PQ:	1	2	2	1	1	1	1	1	1	1	2	2	2	2	2	2
QS:	1	2	1	2	1	1	1	2	2	2	1	1	1	2	2	2
RS:	1	2	2	1	1	2	2	1	2	2	1	1	2	1	1	2
PR:	1	2	1	2	2	1	2	1	1	2	1	2	2	1	2	1

Consider, for example, the ninth case in the cited table. In this case one obtains

$$\underset{9}{\triangle} v^i = 2L^i_{\underset{\vee}{jm}} v^j (dx^m - \delta x^m) + \underset{2}{A}{}^i{}_{jmn} v^j dx^m \delta x^n, \tag{22}$$

where $\underset{2}{A}{}^i{}_{jmn}$ is the curvature pseudotensor (12) of the space L_N. In [15] in this manner are obtained geometric interpretations of all cited curvature tensors and pseudotensors in L_N.

4. Properties of the curvature tensor in L_N

4.1. Mixed curvature tensors

Staying only on independent curvature tensors $\underset{1}{R}, \dots, \underset{5}{R}$, at L_N we obtain [14], [19]:

$$\underset{\theta}{R}{}^i{}_{jmn} = -\underset{\theta}{R}{}^i{}_{jnm}, \theta = 1, 2, \quad \underset{jmn}{Cycl} \underset{\theta}{R}{}^i{}_{jmn} = 0, \theta = 4, 5, \tag{23}$$

The Bianchi identity which is valid in the symmetric connection space, in L_N is not valid, but we obtain 20 identities [16], [17], examining the expressions

$$\underset{mnv}{Cycl} \underset{\theta}{R}{}^i{}_{jmn \underset{\omega}{|} v} \quad \text{for} \quad \theta = 1, \dots, 5; \omega = 1, \dots, 4.$$

So, for example, we get

$$\underset{mnv}{Cycl} \underset{1}{R}{}^i{}_{jmn \underset{1}{|} v} = 2 \underset{mnv}{Cycl} L^p_{\underset{\vee}{mn}} \underset{1}{R}{}^i{}_{jpv}, \tag{24}$$

4.2. Covariant curvature tensors

In GR_N the covariant curvature tensors are defined by the equation

$$\underset{\theta}{R}{}_{ijmn} = g_{\underline{ip}} \underset{\theta}{R}{}^p{}_{jmn}, \theta = 1, \dots, 5$$

and from here one obtains that the following properties hold [14], [19]:

$$R_{\underset{\theta}{}ijmn} = -R_{\underset{\theta}{}jimn} = -R_{\underset{\theta}{}ijnm}, \quad \theta = 1,2, \quad R_{\underset{\theta}{}ijmn} = -R_{\underset{\theta}{}jimn}, \quad \theta = 3,4 \tag{25}$$

$$\begin{aligned} &R_{\underset{5}{}ijmn} = R_{\underset{5}{}mnij}, \quad \underset{\substack{imn\\jmn}}{Cycl} R_{\underset{4}{}ijmn} = 0, \quad \underset{\alpha\beta\gamma}{Cycl} R_{\underset{5}{}ijmn} = 0,\\ &\{\alpha,\beta,\gamma\} \subset \{i,j,m,n\}. \end{aligned} \tag{26}$$

In the Einstein's condition $g_{\underset{+-}{ij};m} = 0$ we see that the index i is treated as in the first kind derivative, and j as in the second one. Proceeding in that sense, Einstein, in his theory for covariant curvature tensor, obtains a Bianchi-type identity [4]:

$$R_{\underset{-+-+}{iklm};n} + R_{\underset{-+++}{ikmn};l} + R_{\underset{-+--}{iknl};m} = 0. \tag{27}$$

5. Infinitesimal deformations and mappings in L_N

An application of more kinds of covariant derivative at L_N makes possible to express more concise some results. For example, let us consider the infinitesimal deformations defined by

$$\bar{x}^i = x^i + \varepsilon z^i(x), \quad x = (x^1,\dots,x^N), \quad i = 1,\dots,N, \tag{28}$$

where ε is an infinitesimal parameter and $z^i(x)$ a vector field. As is known, a deformed geometric object $\bar{\mathcal{A}}(x)$ (e.g. a tensor, a connection) of the object $\mathcal{A}(x)$ is

$$\bar{\mathcal{A}} = \mathcal{A} + \varepsilon \mathcal{L}_z \mathcal{A}, \tag{29}$$

where $\mathcal{L}_z\mathcal{A}$ is the Lie derivative of $\mathcal{A}$ in the direction of the field $z^i(x)$. Then, e.g. for a tensor t^{ij}_{kl} [26], [28], [29] we have:

$$\mathcal{L}_z t^{ij}_{kl} = t^{ij}_{kl,p} z^p - z^i_{,p} t^{pj}_{kl} - z^j_{,p} t^{ip}_{kl} + z^p_{,k} t^{ij}_{pl} + z^p_{,l} t^{ij}_{kp}, \tag{30}$$

Using the covariant derivatives of one kind, for example the first, instead of partial derivatives, we have

$$\begin{aligned} \mathcal{L}_z t^{ij}_{kl} &= t^{ij}_{kl\underset{1}{|}p} z^p - z^i_{\underset{1}{|}p} t^{pj}_{kl} - z^j_{\underset{1}{|}p} t^{ip}_{kl}\\ &+ z^p_{\underset{1}{|}k} t^{ij}_{pl} + z^p_{\underset{1}{|}l} t^{ij}_{kp} + 2(L^i_{\underset{\vee}{ps}} t^{sj}_{kl} + L^j_{\underset{\vee}{ps}} t^{is}_{kl} + L^s_{\underset{\vee}{kp}} t^{ij}_{sl} + L^s_{\underset{\vee}{lp}} t^{ij}_{ks}) z^p, \end{aligned} \tag{30'}$$

from where we see that the Lie derivative is a tensor. But, using more kinds of covariant derivative, we obtain in the considered case [27]:

$$\begin{aligned} \mathcal{L}_z t^{ij}_{kl} &= t^{ij}_{kl\underset{\lambda}{|}p} z^p - z^i_{\underset{\mu}{|}p} t^{pj}_{kl} - z^j_{\underset{\mu}{|}p} t^{ip}_{kl}\\ &+ z^p_{\underset{\nu}{|}k} t^{ij}_{pl} + z^p_{\underset{\nu}{|}l} t^{ij}_{kp}, \end{aligned} \tag{30''}$$

where $(\lambda,\mu,\nu) \in \{(1,2,2),(2,1,1),(3,4,3),(4,3,4)\}$ i.e. in $(30'')$ we have 4 manners of the presenting Lie derivative.

In his doctoral thesis [25] and other works M. Stanković examines geodesic and other mappings of the spaces L_N and GR_N and obtains certain invariant objects of these mappings. In this way one generalizes results, known for Riemannian spaces.

The equations

$$x^i = x^i(u^1,\dots,u^M), (i = 1,\dots,N, rank(x^i_\alpha) = M, x^i_\alpha = \partial x^i/\partial u^\alpha) \tag{31}$$

define a subspace of GR_N with induced basic tensor

$$\bar{g}_{\alpha\beta} = x^i_\alpha x^j_\beta g_{ij}, \tag{32}$$

which generally is non-symmetric too. The question posed by M. Prvanović is: *Can a given generalized Riemannian space(with nonsymmetric g_{ij} in (32)) possess any subspace, whose induced basic tensor is symmetric?* It has been proved in [18, 20] that the answer is affirmative and several examples were constructed.

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