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# GEODESIC LINES IN $\widetilde{SL_2(R)}$ AND $Sol$

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**Abstract.** Locally, there are eight types of "nice", i.e., homogeneous metrics in dimension three. Besides metrics of constant curvature and their products we have also nil, sol and $SL_2(R)$ geometries, i.e., metrics which are locally isometric to the Heisenberg group, the group $Sol$ or the group $SL_2(R)$ of isometries of the Bolyai-Lobachevsky plane. In [6] we derived explicitly formulas for geodesic lines in the Heisenberg group $Heis$. In the present note we use the same method and obtain formulas for geodesic lines in the $SL_2(R)$ and equations of geodesics in *sol*-geometry, which may be solved in quadratures (with the help of elliptic functions). We also compute Cristoffel symbols and the curvature tensor and consider left-invariant Lorentz metrics.

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## 0. Introduction

Locally, there are eight types of "nice", i.e., homogeneous metrics in dimension three: spaces $S^3, R^3, H^3$ of constant sectional curvature (1,0 and -1 correspondingly), products $S^2 \times R$, $H^2 \times R$ and so called nil, sol and $SL_2(R)$ geometries, i.e., metrics which are locally isometric to the Heisenberg group $Heis^3$, the group $Sol$ or the group $SL_2(R)$ (of isometries of the Bolyai-Lobachevsky plane) equipped with some left-invariant Riemannian metric, see [10]. In [6] we derived explicitly formulas for geodesic lines in the Heisenberg group $Heis^{2n+1}$ (compare with [3]). Here we use the same method and obtain formulas for geodesic lines in the $SL_2(R)$. This is done in the following sequence: after introducing global coordinates in the universal cover $\widetilde{SL_2(R)}$ of $SL_2(R)$ we find a basis consisting of (three) left-invariant vector fields $E_i, i = 1, 2, 3$ and define the left-invariant metric on $\widetilde{SL_2(R)}$ by declaring this basis orthonormal, see [7]. Next, with the help of the Koszul's formula we compute the table of mutual covariant derivatives $\nabla_{E_i} E_j$ of these left-invariant fields. The obtained table then gives the (first) system of equations for the coordinates in the moving frame $\{E_i, i = 1, 2, 3\}$ of the unit speed vector $\dot{c}(t)$ of an arbitrary geodesic line $c(t)$. We solve this system and then compute coordinates of $\dot{c}(t)$ in the basis of coordinate vectors of the initial global coordinate system on $\widetilde{SL_2(R)}$ to obtain the (second) system of equations for the coordinates of the geodesic line $c(t)$. Finding the solutions of this (second) system, we determine explicitly formulas for

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the coordinates of an arbitrary geodesic line issuing from the given point in the given direction. For the group *Sol* we follow the same steps and for some particular choice of global coordinates, left-invariant vector fields $\{E_i, i = 1, 2, 3\}$ and corresponding left-invariant Riemannian metric we write down the table of mutual covariant derivatives $\nabla_{E_i} E_j$ and the (first) system of equations for the coordinates in the moving frame $\{E_i, i = 1, 2, 3\}$ of the unit speed vector $\dot{c}(t)$ of an arbitrary geodesic line $c(t)$. In this case this system can be solved in quadratures only (with the help of elliptic function); and hence we do not proceed till the last step and do not consider the (second) system to determine the coordinate functions of geodesic lines. We also compute some (interesting) components of the curvature tensor of *Sol*. In the first appendix we present similar tables and (first) systems of equations for the Lorentz left-invariant metrics on $Heis^3$, *Sol* and $SL_2(R)$.

Finely, note that our treatment of $Heis^3$, *Sol* or $SL_2(R)$ is purely differential geometric (say, in the line of [7]). This approach would be interesting to compare with the consideration of the same spaces from the "uniformization" (of three-dimensional manifolds) point of view. For instance, with the unified description of these space given by Molnár: in [8] the same spaces are presented as projective models $(X, G)$, where $X$ is a simply connected subset of the real projective sphere $RP^3$ or a sphere $S^3$, and $G$ is the complete group of isometries of $X$ acting transitively on it. Note that the complement of $X$ in $RP^3$, or in $S^3$ correspondingly, plays the role of the ideal boundary of $X$ with the given left-invariant Riemannian metric. Note further that the induced metric on this ideal boundary is not necessarily Riemannian, but reflects the existence of additional structures on $X$. Say, in the case of the $2n+1$-dimensional Heisenberg group its ideal boundary is a sphere $S^{2n-1}$ with a natural CR-structure and corresponding Carnot-Caratheodory (non-Riemannian) metric, i.e., it is a one-point compactification of the Heisenberg group $H^{2n-1}$ of the next dimension in a row, see [6]. Such interplay of different points of view on homogeneous spaces may bring their better understanding.

# I
# Geodesics in $\widetilde{SL_2}$

First we obtain formulas for geodesic lines in $\widetilde{SL_2}$.

## 1. The ball model of the Bolyai-Lobachevsky plane and global coordinates in $\widetilde{SL_2(R)}$

In a standard way we identify the Bolyai-Lobachevsky hyperbolic plane $H^2$ with the unit disk $B^2 = \{z \in C | \, \|z\| < 1\}$ in the complex plane with the metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2},$$

where $z = x + iy$. In this coordinate system isometries can be described as transformations of $B^2$ of the form

$$z \to w(z) = -e^{i\phi} \frac{z - \alpha}{\bar{\alpha} z - 1} \quad \text{for} \quad \alpha \in B^2, 0 \le \phi \le 2\pi$$

(which is a composition of a parallel translation taking the point 0 to $\alpha$ and rotation on angle $\phi$). Then $(\alpha, \phi)$ for $\alpha \in B^2$ and $\phi \in S^1 = \{z \in C | \|z\| = 1\}$ are coordinates in the group $SL_2(R)$ of all isometries of $H^2$. Respectively, in the universal cover $\widetilde{SL_2(R)}$ of $SL_2(R)$ the system of global coordinates is $(\alpha, \phi)$, where now $\phi \in R$. Below we denote by $I(\alpha, \phi)$ the isometry with coordinates $(\alpha, \phi)$ and also $\alpha = x + iy$, so that $(x, y, \phi)$ are our global coordinates in $\widetilde{SL_2(R)}$.

## 2. The multiplication law in $\widetilde{SL_2(R)}$

In the coordinate system which we have introduced above the multiplication law giving coordinates of a composition of two isometries with the coordinates $(\alpha, \phi)$ and $(\beta, \psi)$ can be defined in the following way. Because

$$I(\alpha, \phi) \circ I(\beta, \psi)(z) = -e^{\phi + \psi} \frac{\gamma}{\bar{\gamma}} \frac{z - \delta}{\bar{\delta} z - 1}$$

for $\gamma = \gamma(\alpha, \beta) = 1 + \alpha \bar{\beta} e^{-i\psi}$ and $\delta = \delta(\alpha, \beta) = (\beta + \alpha e^{-i\psi})/\gamma$, we deduce that

$$I(\alpha, \phi) \circ I(\beta, \psi) = I(\delta, \phi + \psi + 2Arg(\gamma))$$

which we simply denote as

$$(\alpha, \phi) \circ (\beta, \psi) = (\delta(\alpha, \beta), \phi + \psi + 2Arg(\gamma(\alpha, \beta))). \tag{2.1}$$

## 3. Left-invariant vector fields, the Lie algebra and metric tensor of $SL_2(R)$

Take three coordinate vectors $X = \partial/\partial x$, $Y = \partial/\partial y$ and $T = \partial/\partial\phi$ of the coordinate system introduced above at the point $0 = (0, 0, 0)$, and applying the differential of the multiplication (2.1) by $(\alpha, \phi)$ from the left we obtain three left invariant vector fields $X(\alpha, \phi)$, $Y(\alpha, \phi)$ and $T(\alpha, \phi)$ correspondingly. As direct calculations show these vector fields in our coordinates $(x, y, \phi)$ are

$$\begin{cases} X(x, y, \phi) = (1 - x^2 + y^2, -2xy, 2y), \\ Y(x, y, \phi) = (-2xy, 1 + x^2 - y^2, -2x) \text{ and} \\ T(x, y, \phi) = (y, -x, 1). \end{cases} \tag{3.1}$$

Also, direct computations show that

$$[X, Y] = -4T \qquad [T, X] = Y \qquad [Y, T] = X,$$

or taking from now on $E_1 = X/2$, $E_1 = Y/2$ and $E_3 = T$ we have for the left-invariant vector fields $E_i, i = 1, 2, 3$

$$[E_1, E_2] = \lambda_3 E_3 \qquad [E_3, E_1] = \lambda_2 E_2 \qquad [E_2, E_3] = \lambda_1 E_1 \tag{3.2}$$

where

$$\lambda_1 = \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = -1.$$

**Definition 3.1.** Denote by $g_{sl_2}$ the left-invariant metric on $\widetilde{SL_2(R)}$ such that the vector fields $E_i, i = 1, 2, 3$ are orthonormal ones. The corresponding scalar product we denote as usual by ( , ).

Because of (3.1) the coordinate vectors are

$$\frac{\partial}{\partial x} = \frac{1}{1-(x^2+y^2)}(2E_1 - 2yE_3), \qquad \frac{\partial}{\partial y} = \frac{1}{1-(x^2+y^2)}(2E_2 + 2xE_3),$$

and

$$\frac{\partial}{\partial \phi} = E_3 - y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{1}{1-(x^2+y^2)}(-2yE_1 + 2xE_2 + (1 + x^2 + y^2)E_3),$$

and by our choice $\{E_1, E_2, E_3\}$ is an orthonormal basis we arrive at the following formula for the metric tensor of our left-invariant metric in coordinates $(x, y, \phi)$:

$$g_{sl_2} = \frac{1}{(1-(x^2+y^2))^2}\begin{pmatrix} y^2 & -4xy & -4y-2y(1+x^2+y^2) \\ -4xy & 4+4x^2 & 4x+2x(1+x^2+y^2) \\ -4y-2y(1+x^2+y^2) & 4x+2x(1+x^2+y^2) & (1+x^2+y^2)^2+4x^2+4y^2 \end{pmatrix}.$$

## 4. The curvature tensor of $\widetilde{SL_2(R)}$

Applying well-known formulas (see [7]) for the curvature tensor of the left-invariant metric $g_{sl_2}$ we obtain the following. The Ricci curvature of $g_{sl_2}$:

$$Ric(E_1) = -\frac{3}{2} \quad Ric(E_2) = -\frac{3}{2} \quad Ric(E_3) = \frac{1}{2}, \tag{4.1}$$

scalar curvature

$$\rho = -\frac{5}{2}, \tag{4.2}$$

and for sectional curvatures $K_{ij}$ in two-dimensional directions generated by $\{E_i, E_j\}$

$$K_{12} = -\frac{7}{4} \quad K_{13} = \frac{1}{4} \quad K_{23} = \frac{1}{4}. \tag{4.3}$$

Note, that there is a natural projection $\pi : \widetilde{SL_2(R)} \to B^2$ such that $\pi(x, y, \phi) = (x, y)$, which is a Riemannian submersion (i.e., is an isometry in horizontal directions normal to the fibres of the projection $\pi$). Indeed, by the well-known O'Neill's formula (see [9]) we see that the sectional curvature of the base $H^2$ of this submersion equals

$$K(H^2) = K_{12} + \frac{3}{4}\|[E_1, E_2]\|^2 = -\frac{7}{4} + \frac{3}{4} = -1. \tag{4.4}$$

## 5. Cristoffel symbols of the Levi-Civita connection

**Lemma 1.** *For the covariant derivatives of the Riemannian connection of the left-invariant metric $g_{sl_2}$ defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & -\frac{1}{2}E_3 & \frac{1}{2}E_2 \\ \frac{1}{2}E_3 & 0 & -\frac{1}{2}E_1 \\ \frac{3}{2}E_2 & -\frac{3}{2}E_1 & 0 \end{pmatrix}, \tag{5.1}$$

*where the $(i,j)$-element in the table above equals $\nabla_{E_i}E_j$ for our orthonormal basis $E_i, i = 1,2,3$.*

*Proof.* If in the Koszul's formula (see [4])

$$A(B,C)+B(C,A)-C(A,B) = 2(\nabla_A B, C)+(A,[B,C])+(B,[A,C])-(C,[A,B])$$

the vector fields $A, B, C$ are left-invariant, then all derivatives in the left part of the equality above are zeros and from the bracket relations (3.2) above for our left-invariant fields we easily see that $(\nabla_A B, C)$ is not zero if only the set $\{A, B, C\}$ is of the type $\{E_i, E_j, E_k\}$ for different $i, j, k$. Thus direct computations lead to the table of covariant derivatives above. Lemma 5.1 is proved. □

## 6. The equation of geodesics in $\widetilde{SL_2(R)}$

First we find equations of geodesics issuing from **0**=(0,0,0). Let $c(t)$ be such a geodesics with a natural parameter $t$, and let its vector of velocity be given by

$$\dot{c}(t) = u'(t)E_1(c(t)) + v'(t)E_2(c(t)) + w'(t)E_3(c(t)). \tag{6.1}$$

Then the equation of a geodesic $\nabla_{\dot{c}(t)}\dot{c}(t) \equiv 0$ and our table of covariant derivatives (5.1) give:

$$(u"(t) - 2v'(t)w'(t))E_1(c(t)) + (v"(t) + 2u'(t)w'(t))E_2(c(t)) + w"(t)E_3(c(t)) = 0,$$

which easily gives

$$w'(t) \equiv w'(0) \tag{6.2}$$

and

$$u'(t) = u'(0)\cos(2w'(0)t) + v'(0)\sin(2w'(0)t) \quad \text{and}$$
$$(6.3) \qquad v'(t) = v'(0)\cos(2w'(0)t) - u'(0)\sin(2w'(0)t).$$

To find equations for the coordinates $(x(t), y(t), \phi(t))$ of the geodesic line $c(t)$ in our coordinate system $(x, y, \phi)$ recall that due to (3.1) we have

$$E_1 = (1 - x^2 + y^2)\frac{\partial}{\partial x} + (-2xy)\frac{\partial}{\partial y} + (2y)\frac{\partial}{\partial \phi},$$

$$E_2 = (-2xy)\frac{\partial}{\partial x} + (1 + x^2 - y^2)\frac{\partial}{\partial y} + (-2x)\frac{\partial}{\partial \phi},$$

and

$$E_3 = y\frac{\partial}{\partial x} + (-x)\frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}.$$

Therefore, (6.2)-(6.3) substituted in (6.1) gives

$$\dot{c}(t) = (u'(t)(1 - x^2 + y^2) + v'(t)(-2xy) + w'(0)y)\frac{\partial}{\partial x} +$$

(6.4)

$$(u'(t)(-2xy) + v'(t)(1 + x^2 - y^2) - w'(0)x)\frac{\partial}{\partial y} + (u'(t)(2y) + v'(t)(-2x) + w'(0))\frac{\partial}{\partial \phi},$$

so that from $\dot{c}(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y} + \phi'(t)\frac{\partial}{\partial \phi}$ and that obviously $x'(0) = u'(0)$, $y'(0) = v'(0)$ and $\phi'(0) = w'(0)$ we arrive at the following equations

$$(6.5) \quad \begin{cases} \dot{x}(t) &= (1 - x^2(t) + y^2(t))(u'\cos(2w't) + v'\sin(2w't)) \\ & \quad +(-2x(t)y(t))(v'\cos(2w't) - u'\sin(2w't)) + w'y(t) \\ \dot{y}(t) &= (-2x(t)y(t))(u'\cos(2w't) + v'\sin(2w't)) \\ & \quad +(1 + x^2(t) - y^2(t))(v'\cos(2w't) - u'\sin(2w't)) - w'x(t) \\ \dot{\phi}(t) &= (2y(t))(u'\cos(2w't) \\ & \quad +v'\sin(2w't)) + (-2x(t))(v'\cos(2w't) - u'\sin(2w't)) + w' \end{cases}$$

where we drop the argument 0 in $u'(0)$, $v'(0)$ and $w'(0)$ so that $(u', v'w')$ equals the unit vector $\dot{c}(0)$. First two equations are independent of the third one, hence we first concentrate on the equation for $\alpha(t) = (x(t) + iy(t))$. If we denote $u'\cos(2w't) + v'\sin(2w't) = \sqrt{1 - (w')^2}\cos(\tau)$, then $v'\cos(2w't) - u'\sin(2w't) = (\sqrt{1 - (w')^2})\sin(\tau)$ for $\tau = 2w't$, and due to $x^2 - y^2 = Re(\alpha^2)$ and $2xy = Im(\alpha^2)$ we find

$$(\dot{x}(t) + i\dot{y}(t)) = \dot{\alpha}(t) =$$

$$\sqrt{1 - (w')^2}((1 - Re(\alpha^2(t)))\cos(\tau) + (iIm(\alpha^2(t)))(i\sin(\tau) + (-iIm(\alpha^2(t)))\cos(\tau)$$

$$+i(1 + Re(\alpha^2(t)))\sin(\tau)) - (iw')(iIm(\alpha(t))) - (iw')Re(\alpha(t)) =$$

$$\sqrt{1-(w')^2}((1-\alpha^2(t)))\cos(\tau)+(1+\alpha^2(t))(i\sin(\tau)))-iw'\alpha(t)=$$

$$\sqrt{1-(w')^2}(Exp^{i\tau}-\alpha^2(t)Exp^{-i\tau})-iw'\alpha(t). \tag{6.6}$$

Changing variable $t$ to $s=\sqrt{1-(w')^2}t$ last equality yields for

$$u(s)=\alpha(s/\sqrt{1-(w')^2})Exp^{i\frac{w's}{\sqrt{1-(w')^2}}}$$

the following equations

$$\dot{u}(s)=Exp^{isA}-u^2(s)Exp^{-isA}$$

where $A=6w'/\sqrt{1-(w')^2}$, which in turn gives Ricatti equation

$$\dot{r}(s)=1+(iA)r(s)-r^2(s)$$

for $r(s)=Exp^{iAs}u^{-1}(s)$. If $B$, $C$ are such that $1=B+C^2$ and $iAC=B$ then

$$\dot{r}(s)=(iA)(C+r(s))(C-r(s))$$

which for $\mu(s)=1/(C+r(s))$ leads to

$$\dot{\mu}(s)=2B\mu(s)-iA$$

and

$$\mu(s)=\frac{iA}{B}(1-Exp^{2Bs})$$

due to the initial condition $\mu(0)=0$, following from $\alpha(0)=0$. Returning back this implies

$$u(s)=\frac{iA}{B}Exp^{iAs}\frac{1-Exp^{2Bs}}{1+Exp^{2Bs}}$$

and finally, we arrive at the following formula for the first two coordinates of the geodesic line:

$$\alpha(t)=\frac{Exp^{\frac{2w'}{\sqrt{1-(w')^2}}s}}{\frac{1}{2}(\frac{6w'i}{\sqrt{1-(w')^2}}+(\sqrt{\frac{40(w')^2-4}{1-(w')^2}})ctg[(\sqrt{10(w')^2-1})t])}. \tag{6.7}$$

Substituting $x(t)=Re(\alpha(t))$ and $y(t)=Im(\alpha(t))$ into the equation (6.5) for the third coordinate, we define $\phi(t)$. It would be interesting to analyze further $\phi(t)$, but to the moment we will restricted ourself only to computer-generated pictures (see [5]), which show not so simple behavior of the geodesics in $\widetilde{SL_2(R)}$.

## II
## Geodesics in *Sol*

## 7. Left-invariant vector fields, the Lie algebra, the metric tensor of $Sol$ and covariant derivatives

Remind that the group $Sol$ can be described as the $R^3$ with the following multiplication:

$$(x, y, z,)(x', y', z') = (x + x'e^{-z}, y + y'e^{z}, z + z'), \tag{7.1}$$

see [10]. Thus the following are left-invariant vector fields
(7.2)
$X(x, y, z) = (e^{-z}, 0, 0)$, $\quad Y(x, y, z) = (0, e^{z}, 0) \quad$ and $Z(x, y, z) = (0, 0, 1)$.

From now on we denote the basis $\{X, Y, Z\}$ by $E_i, i = 1, 2, 3$, and by declaring it orthonormal we define the left-invariant metric $g_{sol}$ on $Sol$:

$$g_{sol} = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.3}$$

see again [10]. Also direct computations show that

$$[E_1, E_2] = 0 \qquad [E_1, E_3] = E_1 \qquad [E_2, E_3] = -E_2. \tag{7.4}$$

**Lemma 2.** *For the covariant derivatives of the Riemannian connection of the left-invariant metric $g_{sol}$ defined above the following is true:*

$$\nabla = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.5}$$

*where the $(i, j)$-element in the table above equals $\nabla_{E_i} E_j$ for our orthonormal basis $E_i, i = 1, 2, 3$.*

*Proof.* If in the Koszul's formula

$$A(B, C)+B(C, A)-C(A, B) = 2(\nabla_A B, C)+(A, [B, C])+(B, [A, C])-(C, [A, B])$$

the vector fields $A, B, C$ are left-invariant, then all derivatives in the left part of the equality above are zeros and from the bracket relations (7.4) we deduce (7.5) by direct computations. Lemma 7.1 is proved. □

## 8. The equation of geodesic lines in $Sol$

We find equations of geodesics issuing from **0**=(0,0,0). Let $c(t)$ be such a geodesics with a natural parameter $t$, and let its vector of velocity be given by

$$\dot{c}(t) = u'(t)E_1(c(t)) + v'(t)E_2(c(t)) + w'(t)E_3(c(t)). \tag{8.1}$$

Then the equation of a geodesic $\nabla_{\dot{c}(t)}\dot{c}(t) \equiv 0$ and our table of covariant derivatives (7.5) give:

$$\begin{aligned}\nabla_{\dot{c}(t)}\dot{c}(t) &= (u''(t) - u'(t)w'(t))E_1(c(t)) + (v''(t) + v'(t)w'(t))E_2(c(t)) \\ &\quad +(w''(t) - (u'(t))^2 + (v'(t))^2)E_3(c(t)) \\ &= 0\end{aligned}$$

which infer

$$\left\{\begin{array}{c} u''(t) - u'(t)w'(t) = 0 \\ v''(t) + v'(t)w'(t) = 0 \\ w''(t) + (-(u'(t))^2 + (v'(t))^2) = 0 \end{array}\right. \tag{8.2}$$

From the first two equations it follows that the product $u'(t)v'(t)$ is constant, say

$$u'(t)v'(t) \equiv k \tag{8.3}$$

for some $|k| \leq 1/4$. Hence, from $1 = \|\dot{c}(t)\|^2 = (u'(t))^2 + (v'(t))^2 + (w'(t))^2$ we deduce that

$$w'(t) = \sqrt{1 - (u'(t))^2 - \frac{k^2}{(u'(t))^2}}$$

and, consequently, the first line of (8.2) implies that $u'(t)$ is the solution of the following equation:

$$(ln(f(t)))' = \sqrt{1 - f^2(t) - \frac{k^2}{f^2(t)}}, \tag{8.4}$$

for the unknown function $f$. This equation may be solved in quadratures. For instance, after re-writing (8.4) in the form

$$f'(t) = \sqrt{f^2(t) - f^4(t) - k^2}, \tag{8.5}$$

we see that $f(t) = f(0) + R^{-1}_{f(0),k}(t)$ where the $R^{-1}_{a,k}(t)$ is the function which is inverse to the one given by the integral:

$$R_{a,k}(s) = \int_a^s \frac{df}{\sqrt{f^2 - f^4 - k^2}}. \tag{8.6}$$

## 9. The curvature of *Sol*

Table (7.5) of mutual covariant derivatives of the left-invariant vector fields easily gives curvature tensor components. For instance,

1).

$$R_{XY}Y = \nabla_{E_1}\nabla_{E_2}E_2 - \nabla_{E_2}\nabla_{E_1}E_2 - \nabla_{[E_1,E_2]}E_2 = \nabla_{E_1}E_3 = E_1, \tag{9.1}$$

which implies that the sectional curvature $K(X,Y)$ in the two-dimensional direction $\{X,Y\}$ equals $(R_{XY}Y,X)=(E_1,E_1)=1$.

In the same way

$$R_{XZ}Z=\nabla_{E_1}\nabla_{E_3}E_3-\nabla_{E_3}\nabla_{E_1}E_3-\nabla_{[E_1,E_3]}E_3=-\nabla_{E_1}E_3=-E_1, \tag{9.2}$$

so that $K(X,Z)=-1$; and

$$R_{YZ}Z=\nabla_{E_2}\nabla_{E_3}E_3-\nabla_{E_3}\nabla_{E_2}E_3-\nabla_{[E_2,E_3]}E_3=-\nabla_{-E_2}E_3=-E_2, \tag{9.3}$$

so that $K(Y,Z)=-1$.

Due to the anonymous referees for the previous versions of these notes we find out that for the *Sol*-geometry the table of mutual covariant derivatives, the equation for coordinates of the tangent vector for the geodesic (similar to ours (7.5) and (8.2)) and components of the curvature tensor (9.1 – 9.3) may be extracted from the Grayson's Ph. D thesis [2] (available through the pay service UMI Dissertation Express). See also the recent paper [1] where the authors obtained the above results on the *Sol*-geometry in a different way.

## Appendix 1.
## On geodesics of left-invariant Lorentz metrics in $H^3$, *Sol* and $SL_2(R)$

Obviously, the same arguments work for Lorentz left-invariant metrics on $Heis$, *Sol* and $SL_2(R)$: having the same left-invariant vector fields $E_i, i=1,2,3$ as above in the Riemannian case, to define the Lorentz left-invariant metric $h$ (on $Heis$, *Sol* or $SL_2(R)$) we choose their mutual scalar products $h_{ij}=(E_i,E_j)$ constant and such that the matrix $h=(h_{ij})$ has prescribed signature. Note that in this Lorentz case the matrix of covariant derivatives changes compare to the Riemannian case. Correspondingly, the equations of geodesic lines will be different. Below we present the results for the basic cases when the Lorentz metric $h$ in our bases $E_i, i=1,2,3$ is diagonal. We would not bother to obtain exact formulas for geodesic lines in given coordinates as we deed for $\widetilde{SL_2(R)}$ above in the section 6. Instead, we only list 1) the tables of mutual covariant derivatives of the left-invariant vector fields and 2) equations for the coordinates of unit speed vectors of the geodesic lines.

## 10. The Lorentz left-invariant metrics $h_{heis}$ on $Heis$

As in [6], we identify the Heisenberg group $Heis^3$ with the Euclidean space $R^3$ with the multiplication

$$(x,y,t)(x',y',t')=(x+x',y+y',t+t'+(xy'-x'y)). \tag{10.1}$$

Correspondingly, we have left-invariant vector fields

$$X(x,y,t)=(1,0,-y), Y(x,y,t)=(0,1,x) \text{ and } T(x,y,t)=(0,0,1), \tag{10.2}$$

which we denote $E_i, i = 1,2,3$. We consider left-invariant Lorentz metrics $h$ such that the matrix of mutual products $h_{ij} = h(E_i, E_j)$ is diagonal:

$$h_{ij} = diag(\pm 1, \pm 1, \pm 1).$$

The general case may be easily reduced to this particular case.

Next we use the Koszul's formula for left-invariant vector fields $A, B, C$ which reads:

$$-2(\nabla_A B, C) = (A, [B,C]) + (B, [A,C]) - (C, [A,B]), \tag{10.3}$$

to find matrix of mutual covariant derivatives $\nabla_{E_i} E_j$ for different $h$.

10.A. Let $h_{ij} = diag(-1,1,1)$. Then (10.3) (after some lengthly but straightforward calculations) gives

$$\nabla = \nabla^{(-1,1,1)} = \begin{pmatrix} 0 & E_3 & -E_2 \\ -E_3 & 0 & -E_1 \\ -E_2 & -E_1 & 0 \end{pmatrix}. \tag{10.4}$$

Accordingly for a geodesic line $c(t)$ with the unit speed vector $\dot{c}(t) = u'(t)E_1 + v'(t)E_2 + w'(t)E_3$ the geodesic equation $\nabla_{\dot{c}(t)}\dot{c}(t)$ implies

$$\begin{cases} u\text{''}(t) - 2v'(t)w'(t) = 0 \\ v\text{''}(t) - 2u'(t)w'(t) = 0 \\ w\text{''}(t) = 0 \end{cases} \tag{10.5}$$

The solution of this system is

$$\begin{cases} u(t) = rcosh(\gamma t + \phi) \\ v(t) = rsinh(\gamma t + \phi) \\ w'(t) \equiv \gamma \end{cases} \tag{10.6}$$

where $|\gamma| \leq 1$ and $r = \sqrt{1-\gamma^2}$. (Note the difference from the Riemannian case.)

10.B. Now let $h_{ij} = diag(1,-1,1)$. Then again (10.3) gives

$$\nabla = \nabla^{(1,-1,1)} = \begin{pmatrix} 0 & E_3 & E_2 \\ -E_3 & 0 & E_1 \\ E_2 & E_1 & 0 \end{pmatrix}. \tag{10.7}$$

Accordingly, for a geodesic line $c(t)$ with the unit speed vector $\dot{c}(t) = u'(t)E_1 + v'(t)E_2 + w'(t)E_3$ the geodesic equation $\nabla_{\dot{c}(t)}\dot{c}(t)$ implies

$$\begin{cases} u\text{''}(t) + 2v'(t)w'(t) = 0 \\ v\text{''}(t) + 2u'(t)w'(t) = 0 \\ w\text{''}(t) = 0 \end{cases} \tag{10.8}$$

The solution of this system is

$$(10.9)\qquad \begin{cases} u(t) = rcosh(\gamma t + \phi) \\ v(t) = -rsinh(\gamma t + \phi) \\ w'(t) \equiv \gamma \end{cases}$$

where $|\gamma| \leq 1$ and $r = \sqrt{1-\gamma^2}$. (Note again the difference from the Riemannian case.)

10.C. For the last case we consider - when $h_{ij} = diag(1, 1, -1)$; it holds

$$(10.10)\qquad \nabla = \nabla^{(1,1,-1)} = \begin{pmatrix} 0 & E_3 & E_2 \\ -E_3 & 0 & -E_1 \\ E_2 & -E_1 & 0 \end{pmatrix}$$

Accordingly, for a geodesic line $c(t)$ with the unit speed vector $\dot{c}(t) = u'(t)E_1 + v'(t)E_2 + w'(t)E_3$ the geodesic equation $\nabla_{\dot{c}(t)}\dot{c}(t)$ implies

$$(10.11)\qquad \begin{cases} u"(t) - 2v'(t)w'(t) = 0 \\ v"(t) + 2u'(t)w'(t) = 0 \\ w"(t) = 0 \end{cases}$$

The solution of this system is

$$(10.12)\qquad \begin{cases} u(t) = rcos(\gamma t + \phi) \\ v(t) = -rsin(\gamma t + \phi) \\ w'(t) \equiv \gamma \end{cases}$$

where $|\gamma| \leq 1$ and $r = \sqrt{1-\gamma^2}$.

## 11. The Lorentz left-invariant metrics on *Sol*

Similar as above considerations we can show that the left-invariant Lorentz metric on *Sol* such that

11.A. $h = diag(-1, 1, 1)$ we have:

$$(11.1)\qquad \nabla^{(-1,1,1)} = \begin{pmatrix} E_3 & 0 & E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix},$$

The system of equations for geodesic lines reads:

$$(11.2)\qquad \begin{cases} u"(t) + u'(t)w'(t) = 0 \\ v"(t) - v'(t)w'(t) = 0 \\ w"(t) + (u'(t))^2 + (v'(t))^2 = 0 \end{cases}$$

11.B. For $h = diag(1, -1, 1)$ we have:

$$(11.3)\qquad \nabla^{(1,-1,1)} = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & -E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix},$$

giving the following system of equations for geodesic lines:

$$(11.4)\qquad \begin{cases} u"(t) + u'(t)w'(t) = 0 \\ v"(t) - v'(t)w'(t) = 0 \\ w"(t) - (u'(t))^2 + (v'(t))^2 = 0 \qquad \text{or} \qquad w"(t) = 1 - (w'(t))^2 \end{cases}$$

11.C Lastly, for $h = diag(1, 1, -1)$ it holds

$$(11.5)\qquad \nabla^{(1,1,-1)} = \begin{pmatrix} E_3 & 0 & E_1 \\ 0 & -E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix},$$

which in turn implies

$$(11.6)\qquad \begin{cases} u"(t) + u'(t)w'(t) = 0 \\ v"(t) - v'(t)w'(t) = 0 \\ w"(t) + (u'(t))^2 - (v'(t))^2 = 0 \end{cases}$$

## 12. The Lorentz left-invariant metrics on $\widetilde{SL_2(R)}$

Repeating once again in the same way, it is not difficult to obtain.

12.A. When $h = diag(-1, 1, 1)$ we have:

$$(12.1)\qquad \nabla^{(-1,1,1)} = \begin{pmatrix} 0 & \frac{1}{2}E_3 & -\frac{1}{2}E_2 \\ \frac{3}{2}E_3 & 0 & \frac{3}{2}E_1 \\ \frac{1}{2}E_2 & \frac{1}{2}E_1 & 0 \end{pmatrix},$$

The system of equations for geodesic lines reads:

$$(12.2)\qquad \begin{cases} u"(t) + 2v'(t)w'(t) = 0 \\ v"(t) = 0 \\ w"(t) + 2u'(t)v'(t) = 0 \end{cases}$$

12.B. For $h = diag(1, -1, 1)$ we have:

$$(12.3)\qquad \nabla^{(1,-1,1)} = \begin{pmatrix} 0 & -\frac{3}{2}E_3 & -\frac{3}{2}E_2 \\ -\frac{1}{2}E_3 & 0 & \frac{1}{2}E_1 \\ -\frac{1}{2}E_2 & -\frac{1}{2}E_1 & 0 \end{pmatrix},$$

giving the following system of equations for geodesic lines:

$$(12.4)\qquad \begin{cases} u"(t) = 0 \\ v"(t) - 2u'(t)w'(t) = 0 \\ w"(t) - 2u'(t)v'(t) = 0 \end{cases}$$

12.C- Lastly, for $h = diag(1,1,-1)$ it holds

$$\nabla^{(1,-1,1)} = \begin{pmatrix} 0 & -\frac{1}{2}E_3 & -\frac{1}{2}E_2 \\ \frac{1}{2}E_3 & 0 & \frac{3}{2}E_1 \\ \frac{1}{2}E_2 & \frac{1}{2}E_1 & 0 \end{pmatrix}, \tag{12.5}$$

which in turn implies

$$\begin{cases} u"(t) + v'(t)w'(t) = 0 \\ v"(t) = 0 \\ w"(t) = 0 \end{cases} \tag{12.6}$$

It seems, that this last case deserves more attention, which will probably have in the next paper.

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