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CONNECTEDNESS IN IDEAL TOPOLOGICAL SPACES

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Abstract. In this paper we study the notion of connectedness in ideal topological spaces.

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1. Introduction

The notion of ideal topological spaces was studied by Kuratowski [7] and Vaidyanathaswamy [12]. Applications to various fields were further investigated by Janković and Hamlett [5]; Dontchev et al. [2]; Mukherjee et al. [8]; Arenas et al. [1]; Navaneethakrishnan et al. [10]; Nasef and Mahmoud [9], etc. The purpose of this paper is to introduce and study the notion of connectedness in ideal topological spaces. We study the notions of $\star$-connected ideal topological spaces, $\star$-separated sets, $\star_s$-connected sets and $\star$-connected sets in ideal topological spaces.

2. Preliminaries

Throughout the present paper, (X, τ) or (Y, σ) will denote a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively. A topological space X is said to be hyperconnected [11] if every pair of nonempty open sets of X has nonempty intersection.

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in I$ and $B \subset A$ implies $B \in I$,

(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a local function [7] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau :$

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$x \in U\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the $\star$-topology, finer than τ, is defined by $Cl^*(A) = A \cup A^*(I, \tau)$ [5]. When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* or $\tau^*(I)$ for $\tau^*(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or simply an ideal space. For any ideal space (X, τ, I), the collection $\{V \backslash J : V \in \tau \text{ and } J \in I\}$ is a basis for τ^*. A subset A of an ideal space (X, τ, I) is said to be $\star$-dense [2] if $Cl^*(A) = X$. An ideal space (X, τ, I) is said to be [3] $\star$-hyperconnected if A is $\star$-dense for every open subset $A \neq \varnothing$ of X.

Recall that if (X, τ, I) is an ideal topological space and A is a subset of X, then (A, τ_A, I_A), where τ_A is the relative topology on A and $I_A = \{A \cap J : J \in I\}$ is an ideal topological space.

Lemma 1. *([6]) Let (X, τ, I) be an ideal topological space and $B \subset A \subset X$. Then $B^*(\tau_A, I_A) = B^*(\tau, I) \cap A$.*

Lemma 2. *([4]) Let (X, τ, I) be an ideal topological space and $B \subset A \subset X$. Then $Cl^*_A(B) = Cl^*(B) \cap A$.*

3. Connectedness in ideal spaces

Definition 3. *An ideal space (X, τ, I) is called $\star$-connected [3] if X cannot be written as the disjoint union of a nonempty open set and a nonempty $\star$-open set.*

Definition 4. *A subset A of an ideal space (X, τ, I) is called $\star$-connected if (A, τ_A, I_A) is $\star$-connected.*

Remark 5. *([3]) The following implications hold for an ideal space (X, τ, I). These implications are not reversible.*

$$\begin{array}{ccc} (X,\tau,I) \text{ is } \star\text{-hyperconnected} & \Rightarrow & (X,\tau) \text{ is hyperconnected} \\ \Downarrow & & \Downarrow \\ (X,\tau,I) \text{ is } \star\text{-connected} & \Rightarrow & (X,\tau) \text{ is connected} \end{array}$$

Lemma 6. *([2]) Let (X, τ, I) be an ideal space. For each $U \in \tau^*$, $\tau^*_U = (\tau_U)^*$.*

Lemma 7. *Let (X, τ, I) be a topological space, $A \subset Y \subset X$ and $Y \in \tau$. The following are equivalent:*
(1) A is $\star$-open in Y,
(2) A is $\star$-open in X.

Proof. (1) $\Rightarrow$ (2) : Let A be $\star$-open in Y. Since $Y \in \tau \subset \tau^*$, by Lemma 6, A is $\star$-open in X.

(2) $\Rightarrow$ (1) : Let A be $\star$-open in X. By Lemma 6, $A = A \cap Y$ is $\star$-open in Y. □

Definition 8. *Nonempty subsets A, B of an ideal space (X, τ, I) are called $\star$-separated if $Cl^*(A) \cap B = A \cap Cl(B) = \varnothing$.*

Theorem 9. *Let (X, τ, I) be an ideal space. If A and B are $\star$-separated sets of X and $A \cup B \in \tau$, then A and B are open and $\star$-open, respectively.*

Proof. Since A and B are $\star$-separated in X, then $B = (A \cup B) \cap (X \backslash Cl^*(A))$. Since $A \cup B \in \tau$ and $Cl^*(A)$ is $\star$-closed in X, then B is $\star$-open. By similar way, we obtain that A is open. □

Theorem 10. *Let (X, τ, I) be an ideal space and A, $B \subset Y \subset X$. The following are equivalent:*

(1) A, B are $\star$-separated in Y,

(2) A, B are $\star$-separated in X.

Proof. It follows from Lemma 2 that $Cl^*_Y(A) \cap B = \varnothing = A \cap Cl_Y(B)$ if and only if $Cl^*(A) \cap B = \varnothing = A \cap Cl(B)$. □

Theorem 11. *Every continuous image of a $\star$-connected ideal space is connected.*

Proof. It is know that connectedness is preserved by continuous surjections. Since every $\star$-connected space is connected, the proof is obvious. □

Definition 12. *A subset A of an ideal space (X, τ, I) is called $\star_s$-connected if A is not the union of two $\star$-separated sets in (X, τ, I).*

Theorem 13. *Let Y be an open subset of an ideal space (X, τ, I). The following are equivalent:*

(1) Y is $\star_s$-connected in (X, τ, I),

(2) Y is $\star$-connected in (X, τ, I).

Proof. (1) $\Rightarrow$ (2) : Suppose that Y is not $\star$-connected. There exist nonempty disjoint open and $\star$-open sets A, B in Y such that $Y = A \cup B$. Since Y is open in X, by Lemma 7, A and B are open and $\star$-open in X, respectively. Since A and B are disjoint, then $Cl^*(A) \cap B = \varnothing = A \cap Cl(B)$. This implies that

A, B are $\star$-separated sets in X. Thus, Y is not $\star_s$-connected in X. This is a contradiction.

(2) $\Rightarrow$ (1) : Suppose that Y is not $\star_s$-connected in X. There exist $\star$-separated sets A, B such that $Y = A \cup B$. By Theorem 9, A and B are open and $\star$-open in X, respectively. By Lemma 7, A and B are open and $\star$-open in Y, respectively. Since A and B are $\star$-separated in X, then A and B are nonempty disjoint. Thus, Y is not $\star$-connected. This is a contradiction. □

Theorem 14. *Let (X, τ, I) be an ideal space. If A is a $\star_s$-connected set of X and H, G are $\star$-separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Proof. Let $A \subset H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, then $(A \cap G) \cap Cl^*(A \cap H) \subset G \cap Cl^*(H) = \varnothing$. By similar way, we have $(A \cap H) \cap Cl(A \cap G) = \varnothing$.

Suppose that $A \cap H$ and $A \cap G$ are nonempty. Then A is not $\star_s$-connected. This is a contradiction.

Thus, either $A \cap H = \varnothing$ or $A \cap G = \varnothing$. This implies that $A \subset H$ or $A \subset G$. □

Theorem 15. *If A is a $\star_s$-connected set of an ideal space (X, τ, I) and $A \subset B \subset Cl^*(A)$, then B is $\star_s$-connected.*

Proof. Suppose that B is not $\star_s$-connected. There exist $\star$-separated sets H and G such that $B = H \cup G$. This implies that H and G are nonempty and $G \cap Cl^*(H) = \varnothing = H \cap Cl(G)$. By Theorem 15, we have either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $Cl^*(A) \subset Cl^*(H)$ and $G \cap Cl^*(A) = \varnothing$. This implies that $G \subset B \subset Cl^*(A)$ and $G = Cl^*(A) \cap G = \varnothing$. Thus, G is an empty set. Since G is nonempty, this is a contradiction. Suppose that $A \subset G$. By similar way, it follows that H is empty. This is a contradiction. Hence, B is $\star_s$-connected. □

Corollary 16. *If A is a $\star_s$-connected set in an ideal space (X, τ, I), then $Cl^*(A)$ is $\star_s$-connected.*

Theorem 17. *If $\{M_i : i \in I\}$ is a nonempty family of $\star_s$-connected sets of an ideal space (X, τ, I) with $\underset{i \in I}{\cap} M_i \neq \varnothing$, then $\underset{i \in I}{\cup} M_i$ is $\star_s$-connected.*

Proof. Suppose that $\underset{i \in I}{\cup} M_i$ is not $\star_s$-connected. Then we have $\underset{i \in I}{\cup} M_i = H \cup G$, where H and G are $\star$-separated sets in X. Since $\underset{i \in I}{\cap} M_i \neq \varnothing$, we have a point x in $\underset{i \in I}{\cap} M_i$. Since $x \in \underset{i \in I}{\cup} M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then M_i and H intersect for each $i \in I$. By Theorem 14, $M_i \subset H$ or $M_i \subset G$. Since H and G are disjoint, $M_i \subset H$ for all $i \in I$ and hence

$\bigcup\limits_{i\in I} M_i \subset H$. This implies that G is empty. This is a contradiction. Suppose that $x \in G$. By similar way, we have that H is empty. This is a contradiction. Thus, $\bigcup\limits_{i\in I} M_i$ is $\star_s$-connected. □

Theorem 18. *Suppose that $\{M_n : n \in N\}$ is an infinite sequence of $\star$-connected open sets of an ideal space (X, τ, I) and $M_n \cap M_{n+1} \neq \varnothing$ for each $n \in N$. Then $\bigcup\limits_{n\in N} M_n$ is $\star$-connected.*

Proof. By induction and Theorems 13 and 17, the set $N_n = \bigcup\limits_{k\leq n} M_k$ is a $\star$-connected open set for each $n \in N$. Also, N_n have a nonempty intersection. Thus, by Theorems 13 and 17, $\bigcup\limits_{n\in N} M_n$ is $\star$-connected. □

Definition 19. *Let X be an ideal space and $x \in X$. The union of all $\star_s$-connected subsets of X containing x is called the $\star$-component of X containing x.*

Theorem 20. *Each $\star$-component of an ideal space (X, τ, I) is a maximal $\star_s$-connected set of X.*

Theorem 21. *The set of all distinct $\star$-components of an ideal space (X, τ, I) forms a partition of X.*

Proof. Let A and B be two distinct $\star$-components of X. Suppose that A and B intersect. Then, by Theorem 17, $A \cup B$ is $\star_s$-connected in X. Since $A \subset A \cup B$, then A is not maximal. Thus, A and B are disjoint. □

Theorem 22. *Each $\star$-component of an ideal space (X, τ, I) is $\star$-closed in X.*

Proof. Let A be a $\star$-component of X. By Corollary 16, $Cl^*(A)$ is $\star_s$-connected and $A = Cl^*(A)$. Thus, A is $\star$-closed in X. □

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