Novi Sad J. Math.
Vol. 38, No. 2, 2008, 57-64

# A MEAN VALUE THEOREM FOR INTERNAL FUNCTIONS AND AN ESTIMATION FOR THE DIFFERENTIAL MEAN POINT

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**Abstract.** We present several results about the mean value theorem (MVT) with nonstandard analysis techniques. Using only the intermediate value theorem, we present a nonstandard proof of the MVT. In the next section we extend the MVT for internal SU-differentiable functions. In the end we discuss the location of the differential mean point $c$ in the segment $[x, y]$, as $y \to x$.

*AMS Mathematics Subject Classification (2000)*: 26E35, 26B05, 46T20.

*Key words and phrases:* Nonstandard analysis, Mean value theorem, Estimation for the differential mean point

## 1. Introduction

In this section we present some necessary terminology and some important facts needed for the present work. For further details, see [5], [7] or [8].

Let $E$ and $F$ be two standard normed spaces, ${}^*E$ and ${}^*F$ their nonstandard extensions and $U$ an open subset of $E$.

**Definition 1.** *Given $x$ and $y$ two vectors of ${}^*E$, we say that*

1. *$x$ is infinitesimal ($x \approx 0$) if for all positive real numbers $r$, $|x| < r$;*
2. *$x$ is finite ($x \in fin({}^*E)$) if, for some positive real number $r$, $|x| < r$;*
3. *$x$ is infinite ($x \approx \infty$) if it is not finite;*
4. *$x$ and $y$ are infinitely close ($x \approx y$) if $x - y \approx 0$;*
5. *$x$ is nearstandard ($x \in ns({}^*E)$) if there exists a standard $z \in E$ with $x \approx z$, and we write $z = st(x)$;*
6. *The monad of $x$ is the set $\mu(x) := \{z \in {}^*E |\ z \approx x\}$.*

If $x - y$ is not infinitesimal, we write $x \not\approx y$. The set of the positive hyperintegers will be denoted by ${}^*\mathbb{N}_\infty$.

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**Definition 2.** *Let $f : {}^*U \to {}^*F$ be an internal function. We say that $f$ is S-continuous at $a \in {}^*U$ if*

$$x \approx a \Rightarrow f(x) \approx f(a).$$

*If this is true for all $a \in U$, $f$ is called S-continuous. We say that $f$ is SU-continuous if it still holds for all $a \in {}^*U$.*

**Theorem 1.** *A function $f : U \to F$ is continuous (resp. uniformly continuous) if and only if it is S-continuous (resp. SU-continuous).*

Given an internal linear operator $L \in {}^*L(E, F)$, we say that $L$ is finite if $L(fin({}^*E)) \subseteq fin({}^*F)$

**Definition 3.** *Let $f : {}^*U \to {}^*F$ be an internal function. Given $a \in U$, we say that $f$ is S-differentiable at $a$ if*

1. *$f(ns({}^*U)) \subseteq ns({}^*F)$.*
2. *there exists a finite linear operator $Df_a \in {}^*L(E, F)$ such that, for each $x \approx a$ there exists some $\eta \approx 0$ satisfying*

$$f(x) - f(a) = Df_a(x - a) + |x - a|\eta$$

*If it is S-differentiable at all $a \in U$, $f$ is called S-differentiable. Furthermore, if the previous condition holds for all $a \in ns({}^*U)$, we say that $f$ is SU-differentiable.*

**Theorem 2.** *A function $f : U \to F$ is differentiable (resp. of class $C^1$) if and only if it is S-differentiable (resp. SU-differentiable).*

In opposite to standard functions, the derivative map for internal functions is not unique. In fact, let $\epsilon$ be a positive infinitesimal and let $f(x) = \epsilon$, $x \in {}^*\mathbb{R}$. Then $f$ is SU-differentiable and $f'(x) = \delta$, where $\delta$ is **any** infinitesimal number. In fact, given $x \approx a \in ns({}^*\mathbb{R})$, we have

$$\frac{f(x) - f(a)}{x - a} = 0 \approx f'(a).$$

Finally we present a nonstandard version of Taylor's theorem. We will denote by $SL^h(E, F)$ the symmetric $h$-linear operators from $E \times \ldots \times E = E^h$ into $F$.

**Theorem 3.** *Let $f : U \to F$ be a function. Then $f$ is of class $C^k$ if and only if there exist unique maps $D^h f_{(.)} : U \to SL^h(E, F)$, $h \in \{1, \ldots, k\}$ such that, given any $a \in ns({}^*U)$ and $x \approx a$, there is an infinitesimal $\eta \in {}^*F$ satisfying*

$$f(x) = \sum_{h=0}^{k} \frac{1}{h!} D^h f_a (x - a)^{(h)} + |x - a|^k \eta.$$

## 2. A Nonstandard Proof of the Mean Value Theorem

As usual, we use the symbol $[x, y]$, where $x$ and $y$ are two vectors in $E$, to denote the elements of the closed line segment joining $x$ with $y$. First we prove a nonstandard analogous of the intermediate value theorem for internal functions.

**Theorem 4.** ***Intermediate Value Theorem*** *Let $a, b \in {}^*\mathbb{R}$ and let $f : [a, b] \to {}^*\mathbb{R}$ be an internal SU-continuous function such that $f(a) < f(b)$. Then, for all $K \in {}^*\mathbb{R}$ with $f(a) < K < f(b)$, there exists $c \in [a, b]$ with $f(c) \approx K$.*

*Proof.* Fix $N \in {}^*\mathbb{N}_\infty$ with $(b-a)/N \approx 0$ and define

$$A := \left\{ j \in \{0, 1, \ldots, N\} \mid f\left(a + j\frac{b-a}{N}\right) < K \right\}.$$

The set $A$ is nonempty since $0 \in A$. Let $l$ be the maximum of $A$. Then $l \neq N$ and therefore we have

$$K \leq f\left(a + (l+1)\frac{b-a}{N}\right) \approx f\left(a + l\frac{b-a}{N}\right) < K$$

then

$$f\left(a + l\frac{b-a}{N}\right) \approx K.$$

$\square$

**Theorem 5.** *Let $U$ be an open convex subset of $E$ and $f : U \to \mathbb{R}$ a $C^1$ function. Then, for all $x, y \in U$, there exists $c \in [x, y]$ with*

$$f(x) - f(y) = Df_c(x - y).$$

*Proof.* Fix an infinite $N \in {}^*\mathbb{N}_\infty$ and define $\delta := (y - x)/N \approx 0$. Then, for some infinitesimal numbers $\eta_1, \ldots, \eta_N$, the following holds:

$$\begin{aligned} f(x) - f(y) &= \sum_{n=1}^{N} [f(x + (n-1)\delta) - f(x + n\delta)] \\ &= \frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N} + \frac{\sum_{n=1}^{N} \eta_n}{N}|x - y|. \end{aligned}$$

Once

$$\left| \frac{\sum_{n=1}^{N} \eta_n}{N} \right| \leq \max\{|\eta_1|, ..., |\eta_N|\} \approx 0,$$

we obtain

$$f(x) - f(y) = st\left(\frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N}\right). \tag{2.1}$$

Let $t_m, t_M \in \{0, ..., N-1\}$ be the hyper-integers satisfying

$$Df_{x+t_m\delta}(x-y) = \min_{t\in\{0,\dots,N-1\}} Df_{x+t\delta}(x-y)$$

and

$$Df_{x+t_M\delta}(x-y) = \max_{t\in\{0,\dots,N-1\}} Df_{x+t\delta}(x-y).$$

Then the following is verified:

$$Df_{x+t_m\delta}(x-y) \leq \frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N} \leq Df_{x+t_M\delta}(x-y).$$

As $Df_{(\cdot)}(x-y)$ is a S-continuous function, the map

$$t \mapsto Df_{x+t\delta}(x-y),\ t \in [0, N-1]$$

is an internal SU-continuous function and so, by the Intermediate Value Theorem, there exists $k \in [x + t_m\delta, x + t_M\delta] \subseteq {}^*[x, y]$ with

$$Df_k(x-y) \approx \frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N}.$$

Therefore, taking standard parts on the last equation and by (2.1), we obtain

$$f(x) - f(y) = Df_c(x-y)$$

where $c = st(k)$. □

In the last theorem we proved that, for $N \approx \infty$ and $\delta = (y-x)/N$,

$$f(x) - f(y) = st\left(\frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N}\right).$$

So, if $c \in [x, y]$ satisfies the condition

$$f(x) - f(y) = Df_c(x-y),$$

then

$$Df_c(x-y) = st\left(\frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N}\right).$$

In particular, if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a $C^1$ function, we get

$$f'(c)(x-y) = st\left(\frac{\sum_{n=1}^{N} f'(x+(n-1)\delta)(x-y)}{N}\right)$$

hence

$$f'(c) = st\left(\frac{f'(x) + f'(x+\delta) + f'(x+2\delta) + \ldots + f'(x+(N-1)\delta)}{N}\right),$$

*i.e.*, $c$ is the point in $[x, y]$ for which the derivative of $f$ at $c$ is the limit of the arithmetic *mean* of the derivatives of $f$ at $x+(n-1)\delta$, $n = 1, \ldots, N$, as $N \to \infty$.

Analogously, we have

**Theorem 6.** *Let $U$ be an open convex subset of $E$ and $f : U \to F$ a $C^1$ function. Then, for all $x, y \in U$, there exists $c \in [x, y]$ with*

$$|f(x) - f(y)| \leq |Df_c(x - y)|.$$

*Proof.* Since

$$|f(x) - f(y)| \leq st\left(\frac{\sum_{n=1}^{N} |Df_{x+(n-1)\delta}(x-y)|}{N}\right)$$

and taking $t_m, t_M$ with

$$|Df_{x+t_m\delta}(x - y)| = \min_{t\in\{0,\dots,N-1\}} |Df_{x+t\delta}(x - y)|$$

and

$$|Df_{x+t_M\delta}(x - y)| = \max_{t\in\{0,\dots,N-1\}} |Df_{x+t\delta}(x - y)|,$$

we obtain the desired result. □

## 3. A Mean Value Theorem for Internal Functions

We now present the mean value theorem for internal functions. Since the derivative function of an internal function is not unique, in the formula we must add an error. The proof will be omitted for it is similar to the proof of the same theorem for standard functions.

**Theorem 7.** *Let $U$ be an open convex subset of $E$. If $f : {}^*U \to {}^*\mathbb{R}$ is an internal SU-differentiable function then*

$$\forall x, y \in ns({}^*U)\, \exists c \in [x, y] \qquad f(x) - f(y) = Df_c(x - y) + |x - y|\eta,$$

*for some $\eta \approx 0$.*

*More generally, if $f : {}^*U \to {}^*F$ is an internal SU-differentiable function, then*

$$\forall x, y \in ns({}^*U)\, \exists c \in [x, y] \qquad |f(x) - f(y)| \leq |Df_c(x - y)| + |x - y|\eta,$$

*with $\eta \approx 0$.*

## 4. An Estimation for the Differential Mean Point

Fix $x \in U$ and assume that $y \in {}^*E$ is infinitely close to $x$. Where in the interval $[x, y]$ might $c$ be located? We will begin by proving that, under some conditions, $c$ approaches the midpoint of the segment $[x, y]$.

Let $f : U \subseteq E \to \mathbb{R}$ be a $C^2$ function, where $U$ is an open convex set, and fix $x \in U$. Then, for all $y \in U$, by the mean value theorem, we can ensure the existence of $c \in [x, y]$ with $f(x) - f(y) = Df_c(x - y)$. Therefore, if $y \in {}^*U$ with $y \approx x$, there still exists such $c \in [x, y]$. We give a generalization of a result due to Jacobson, presented in [6]:

**Theorem 8.** *Under the previous assumptions and, if*

$$D^2f_x\left(\frac{x-y}{|x-y|}\right)^{(2)} \not\approx 0,$$

*then*

$$\frac{|x-c|}{|x-y|} \approx \frac{1}{2}.$$

*Proof.* Since $f$ is twice continuously differentiable, we have:

- $f(x)-f(y) = Df_x(x-y)+1/2D^2f_x(x-y)^{(2)}+|x-y|^2\eta$, for some $\eta \approx 0$;
- $Df_c(x-y) = Df_x(x-y)+D^2f_x(x-y,c-x)+|x-c|\cdot\theta(x-y)$, where $\theta \in Lin(^*E, ^*\mathbb{R})$ is an operator such that $\theta(fin(^*E)) \subseteq inf(^*\mathbb{R})$ (see ([8]);

and also the equality

- $f(x)-f(y) = Df_c(x-y)$.

Therefore

$$D^2f_x(x-y,c-x)+|x-c|\cdot\theta(x-y) = \frac{1}{2}D^2f_x(x-y)^{(2)}+|x-y|^2\eta \Leftrightarrow$$

$$|x-y|\cdot|x-c|\left[D^2f_x\left(\frac{x-y}{|x-y|},\frac{c-x}{|x-c|}\right)+\theta\left(\frac{x-y}{|x-y|}\right)\right]$$
$$= |x-y|^2\left[\frac{1}{2}D^2f_x\left(\frac{x-y}{|x-y|}\right)^{(2)}+\eta\right]$$

which implies

$$\frac{|x-c|}{|x-y|} = \frac{1}{2}\frac{\left|D^2f_x\left(\frac{x-y}{|x-y|}\right)^{(2)}+2\eta\right|}{\left|-D^2f_x\left(\frac{x-y}{|x-y|}\right)^{(2)}+\theta\left(\frac{x-y}{|x-y|}\right)\right|} \approx \frac{1}{2}.$$

□

In *A Note on the Mean Value Theorem for Integrals*, Zhang Bao-Lin extends the result of Jacobson (see [4]). Next we generalize his work for arbitrary normed spaces.

**Theorem 9.** *Let $f: U \subseteq E \to \mathbb{R}$ be a $C^3$ function, where $U$ is an open convex set. If*

1. *$x \in U$, $y \in {}^*U$ with $y \approx x$;*
2. *$c \in [x,y]$ with $f(x)-f(y) = Df_c(x-y)$;*

*3.* $D^2 f_x \left(\frac{x-y}{|x-y|}\right)^{(2)} = 0$ *and* $D^3 f_x \left(\frac{x-y}{|x-y|}\right)^{(3)} \not\approx 0,$

*then*

$$\frac{|x-c|}{|x-y|} \approx \frac{1}{\sqrt{3}}.$$

*Proof.* Taking the Taylor's expansions:

- $f(x)-f(y) = Df_x(x-y)+1/2D^2 f_x(x-y)^{(2)}+1/6D^3 f_x(x-y)^{(3)}+|x-y|^3\eta;$
- $Df_c(x-y) = Df_x(x-y) + D^2 f_x(x-y, c-x) + 1/2D^3 f_x(x-y, c-x, c-x) + |x-c|^2 \cdot \theta(x-y);$

roughly as before

$$\left(\frac{|x-c|}{|x-y|}\right)^2 = \frac{\left|\frac{1}{2|x-y|} D^2 f_x \left(\frac{x-y}{|x-y|}\right)^{(2)} + \frac{1}{6} D^3 f_x \left(\frac{x-y}{|x-y|}\right)^{(3)} + \eta\right|}{\left|-\frac{1}{|x-c|} D^2 f_x \left(\frac{x-y}{|x-y|}\right)^{(2)} + \frac{1}{2} D^3 f_x \left(\frac{x-y}{|x-y|}\right)^{(3)} + \theta\left(\frac{x-y}{|x-y|}\right)\right|} \approx \frac{1}{3}.$$

□

Iterating this procedure we obtain the following result:

**Theorem 10.** *Let* $f : U \subseteq E \to \mathbb{R}$ *be a* $C^{k+1}$ *function, where* $U$ *is an open convex set. If*

*1.* $x \in U$, $y \in {}^*U$ *with* $y \approx x$*;*

*2.* $c \in [x, y]$ *with* $f(x) - f(y) = Df_c(x-y)$*;*

*3.* $D^j f_x \left(\frac{x-y}{|x-y|}\right)^{(j)} = 0$, *for* $j = 2, \ldots, k$ *and* $D^{k+1} f_x \left(\frac{x-y}{|x-y|}\right)^{(k+1)} \not\approx 0,$

*then*

$$\frac{|x-c|}{|x-y|} \approx \frac{1}{\sqrt[k]{k+1}}.$$

*Proof.* Just observe that the Taylor's expansions are now

$$f(x) - f(y) = Df_x(x-y) + 1/2D^2 f_x(x-y)^{(2)} + 1/3!D^3 f_x(x-y)^{(3)} + \ldots + 1/(k+1)!D^{k+1} f_x(x-y)^{(k+1)} + |x-y|^{k+1}\eta;$$

and

$$Df_c(x-y) = Df_x(x-y) + D^2 f_x(x-y, c-x) + 1/2D^3 f_x(x-y, c-x, c-x) + \ldots + 1/k!D^{k+1} f_x(x-y, c-x, c-x, \ldots, c-x) + |x-c|^k \cdot \theta(x-y).$$

□

**Acknowledgments.** *Work partially supported by Centre for Research on Optimization and Control (CEOC) from the "Fundação para a Ciência e a Tecnologia"(FCT), cofinanced by the European Community Fund FEDER/POCI 2010.*

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*Received by the editors March 5, 2008*