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REGULARITY OF MULTISUBMEASURES WITH RESPECT TO THE WIJSMAN TOPOLOGY

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Abstract. In this article we introduce different types of regularity for multisubmeasures with respect to the Wijsman topology and establish several relationships with the types of regularity that we have studied in [3] and [5] with respect to the Hausdorff, respectively, the Vietoris topology.

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1. Basic notions, terminology and notations

Recently, because of its applications in many problems of optimization, convex analysis, economy, etc., the study of hypertopologies has become of a great interest.

On the other hand, it is well known that regularity is an important property which connects measure theory and topology, aproximating general Borel sets by compact and/or open sets (see, for instance, [1, 7, 9]).

That is why, in several papers (see [3, 4, 5, 6]) we have studied different problems concerning regularity for multisubmeasures with respect to the Hausdorff and the Vietoris topology. In this work we define and study different types of regularity with respect to another important hypertopology, the Wijsman topology, also pointing out linking results with the types of regularity that we have studied in [3] and [5] with respect to the Hausdorff, respectively, the Vietoris topology.

Let T be a locally compact, Hausdorff space, $\mathcal{C}$ a ring of subsets of T, X a linear, metrisable space (for instance, by a metric d), with the origin 0, and τ_d the topology induced by d. Let also $\mathcal{B}_0$ (respectively, $\mathcal{B}_0'$) be the Baire δ-ring (respectively, σ-ring) generated by the compact sets, which are G_δ (that is, countable intersections of open sets) and $\mathcal{B}$ (respectively, $\mathcal{B}'$) the Borel δ-ring (respectively, σ-ring) generated by the compact sets of T.

We consider $\mathcal{P}_0(X)$, the family of all non-empty subsets of $X, \mathcal{P}_f(X)$, the family of nonvoid, closed subsets of X, and $\mathcal{P}_{bf}(X)$, the family of all nonvoid, bounded, closed subsets of X.

Let also "$\dot{+}$" be the Minkowski addition on $\mathcal{P}_0(X)$, defined by $M \dot{+} N = \overline{M+N}$, for every $M, N \in \mathcal{P}_0(X)$.

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Definition 1.1. *A multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be a multisubmeasure if:*

$$\begin{cases} a) & \mu(\emptyset) = \{0\}, \\ b) & \begin{array}{l} [2]\ \mu(A \cup B) \subseteq \mu(A) \dot{+} \mu(B), \text{ for every } A, B \in \mathcal{C} \text{ with } A \cap B = \emptyset \\ \text{(or, equivalently, for every } A, B \in \mathcal{C}) \text{ and} \end{array} \\ c) & \mu(A) \subseteq \mu(B), \text{for every } A, B \in \mathcal{C} \text{ with } A \subseteq B. \end{cases}$$

All over this paper we shall consider an arbitrary multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$.

2. Regularity with respect to the Wijsman topology

It is well known from [8] that the family

$$\mathcal{F} = \{A \in \mathcal{P}_f(X); d(x, A) < \varepsilon\}_{\substack{x \in X \\ \varepsilon > 0}} \cup \{A \in \mathcal{P}_f(X); d(x, A) > \varepsilon\}_{\substack{x \in X \\ \varepsilon > 0}}$$

is a subbase for the Wijsman topology $\widehat{\tau}_W$ on $\mathcal{P}_f(X)$, where by $d(x, A)$ we mean the distance from x to the set A.

In the sequel, we introduce different types of regularity for multisubmeasures with respect to the Wijsman topology. Let $A \in \mathcal{C}$ be an arbitrary set.

Definition 2.1. *A is said to be R_l^w-regular with respect to μ if for every $x \in X$ and every $\varepsilon > 0$ with $d(x, \mu(A)) < \varepsilon$ or $d(x, \mu(A)) > \varepsilon$, there exists a compact set $K = K(x, \varepsilon) \in \mathcal{C}, K \subset A$, so that $d(x, \mu(B)) < \varepsilon$ or $d(x, \mu(B)) > \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset A$.*

Because for every $B \in \mathcal{C}$, with $B \subset A$, we have $d(x, \mu(B)) \geq d(x, \mu(A))$, the situation "$d(x, \mu(A)) > \varepsilon \Rightarrow d(x, \mu(B)) < \varepsilon$" is impossible. Also, the situation "$d(x, \mu(A)) > \varepsilon \Rightarrow d(x, \mu(B)) > \varepsilon$" is always valid and the situation "$d(x, \mu(A)) < \varepsilon \Rightarrow d(x, \mu(B)) > \varepsilon$" is not always valid, for instance if we put $B = A$.

That is why, in the following, by R_l^w**-regularity,** we shall mean only the situation

$$\text{"}d(x, \mu(A)) < \varepsilon \Rightarrow d(x, \mu(B)) < \varepsilon\text{"}.$$

We note that every compact set $K \in \mathcal{C}$ is R_l^w-regular.

Definition 2.2. *A is said to be R_r^w-regular with respect to μ if for every $x \in X$ and every $\varepsilon > 0$ with $d(x, \mu(A)) < \varepsilon$ or $d(x, \mu(A)) > \varepsilon$, there exists an open set $D = D(x, \varepsilon) \in \mathcal{C}, A \subset D$, such that $d(x, \mu(B)) < \varepsilon$ or $d(x, \mu(B)) > \varepsilon$, for every $B \in \mathcal{C}$, with $A \subset B \subset D$.*

Since for every $B \in \mathcal{C}$, with $A \subset B$, $d(x, \mu(B)) \leq d(x, \mu(A))$, we observe that the situation "$d(x, \mu(A)) < \varepsilon \Rightarrow d(x, \mu(B)) > \varepsilon$" is impossible. Also, the situation "$d(x, \mu(A)) < \varepsilon \Rightarrow d(x, \mu(B)) < \varepsilon$" always takes place, and the situation "$d(x,\mu(A)) > \varepsilon \Rightarrow d(x,\mu(B)) < \varepsilon$" is impossible if, for instance, we consider $B = A$.

That is why, by R_r^w**-regularity,** we shall understand only the situation

$$"d(x,\mu(A)) > \varepsilon \Rightarrow d(x,\mu(B)) > \varepsilon".$$

Obviously, every open set $D \in \mathcal{C}$ is R_r^w-regular.

By the same reasons of natural particularizations as before, we also introduce the following two notions:

Definition 2.3. *A is said to be $R_<^w$-regular with respect to μ if for every $x \in X$ and every $\varepsilon > 0$ with $d(x,\mu(A)) < \varepsilon$, there are a compact set $K = K(x,\varepsilon) \in \mathcal{C}$ and an open set $D = D(x,\varepsilon) \in \mathcal{C}, K \subset A \subset D$ so that $d(x,\mu(B)) < \varepsilon$, for every $B \in \mathcal{C}, K \subset B \subset D$.*

Definition 2.4. *A is said to be $R_>^w$-regular with respect to μ if for every $x \in X$ and every $\varepsilon > 0$ with $d(x,\mu(A)) > \varepsilon$, there exist a compact set $K = K(x,\varepsilon) \in \mathcal{C}$ and an open set $D = D(x,\varepsilon) \in \mathcal{C}$ such that $K \subset A \subset D$ and $d(x,\mu(B)) > \varepsilon$, for every $B \in \mathcal{C}, K \subset B \subset D$.*

Theorem 2.5. *If $\mathcal{C} = \mathcal{B}_0$ or $\mathcal{B}$, the following statements are equivalent:*

i) A is R_l^w-regular;

ii) A is $R_<^w$-regular.

Proof. The implication "ii) $\Rightarrow$ i)" follows immediately.

We prove now that "i) $\Rightarrow$ ii)". Let $x \in X$ and $\varepsilon > 0$ be so that $d(x,\mu(A)) < \varepsilon$. Since A is R_l^w-regular, there exists a compact set $K = K(x,\varepsilon) \in \mathcal{C}, K \subset A$ such that $d(x,\mu(B)) < \varepsilon$, for every $B \in \mathcal{C}, K \subset B \subset A$.

On the other hand, for $A \in \mathcal{C}$, there exists an open set $D \in \mathcal{C}$, with $A \subset D$. Let $B \in \mathcal{C}, K \subset B \subset D$. Because $K \subset A \cap B \subset A$, we get that $d(x,\mu(A \cap B)) < \varepsilon$. But $d(x,\mu(B)) \le d(x,\mu(A \cap B))$, hence $d(x,\mu(B)) < \varepsilon$, that is, A is $R_<^w$-regular. □

We note that the proof remains valid if we suppose, more generally, that $\mathcal{C}$ is a ring of subsets of T satisfying the habitual condition in the study of regularity:

(1) for every $A \in \mathcal{C}$, there is an open set $D \in \mathcal{C}$ so that $A \subset D$. □

Theorem 2.6. *If $\mathcal{C} = \mathcal{B}_0$ or $\mathcal{B}$, the following statements are equivalent:*

i) A is R_r^w-regular;

ii) A is $R_>^w$-regular.

Proof. The implication "ii) $\Rightarrow$ i)" is immediate.

For the implication "i) $\Rightarrow$ ii)", let $x \in X$ and $\varepsilon > 0$ be so that $d(x, \mu(A)) > \varepsilon$. By the hypothesis, there exists an open set $D = D(x, \varepsilon) \in \mathcal{C}$,with $A \subset D$ so that $d(x, \mu(B)) > \varepsilon$, for every $B \in \mathcal{C}$, with $A \subset B \subset D$.

On the other hand, there is a compact set $K \in \mathcal{C}$ with $K \subset A$. Hence, $K \subset A \subset D$ and if $B \in \mathcal{C}$, with $K \subset B \subset D$, then $A \subset A \cup B \subset D$, which implies $d(x, \mu(A \cup B)) > \varepsilon$. Since $d(x, \mu(B)) \geq d(x, \mu(A \cup B))$, we get that $d(x, \mu(B)) > \varepsilon$, which yields the conclusion. □

Obviously, the proof remains valid if we suppose that $\mathcal{C}$ is a ring of subsets of T having the following natural property, which appears in the study of regularity:

(2) for every $A \in \mathcal{C}$, there exists a compact set $K \in \mathcal{C}$ with $K \subset A$.

In the sequel, we shall compare the above defined types of regularity with respect to the Wijsman topology $\widehat{\tau}_W$, with the types of regularity introduced and studied in [5] with respect to the Vietoris topology $\widehat{\tau}_V$. It is known from [8], that, even if X is simply a metric space, we have $\widehat{\tau}_W \subset \widehat{\tau}_V$. That is why, with respect to $\widehat{\tau}_W$, we have fewer types of regularity than with respect to $\widehat{\tau}_V$ (see [5]).

In the following, we prove that, although $\widehat{\tau}_V \neq \widehat{\tau}_W$, $R^w_<$-regularity (with respect to $\widehat{\tau}_W$) is equivalent to R^--regularity (with respect to $\widehat{\tau}_V$).

Theorem 2.7. *A is $R^w_<$-regular (with respect to $\widehat{\tau}_W$) if and only if A is R^--regular (with respect to $\widehat{\tau}_V$).*

Proof. The only if part.

Let $\varepsilon > 0$ and $x \in X$, with $d(x, \mu(A)) < \varepsilon$. There is $y_x \in \mu(A)$ with $d(x, y_x) < \varepsilon$, that is, $y_x \in S(x, \varepsilon)$, where $S(x, \varepsilon) = \{y \in X,\ d(x, y) < \varepsilon\}$. Consequently, $\mu(A) \cap S(x, \varepsilon) \neq \emptyset$.

By applying the hypothesis for $V_1 = V_2 = \ldots = V_n = S(x, \varepsilon) \in \tau_d$ (for which $\mu(A) \cap V_j \neq \emptyset$ for every $j = \overline{1, n}$) there are a compact set $K = K(x, \varepsilon) \in \mathcal{C}$ and an open set $D = D(x, \varepsilon) \in \mathcal{C}, K \subset A \subset D$ such that $\mu(B) \cap S(x, \varepsilon) \neq \emptyset$, for every $B \in \mathcal{C}, K \subset B \subset D$.

Let $B \in \mathcal{C}, K \subset B \subset D$ be abitrarily, but fixed.

Because $\mu(B) \cap S(x, \varepsilon) \neq \emptyset$, there exists $z_0 \in \mu(B)$, with $d(x, z_0) < \varepsilon$, hence

$$d(x, \mu(B)) = \inf_{z \in \mu(B)} d(x, z) \leq d(x, z_0) < \varepsilon.$$

Consequently, A is $R^w_<$-regular.

The if part.

Let $V_1, V_2, \ldots, V_n \in \tau_d$ be so that $\mu(A) \cap V_j \neq \emptyset$, for every $j = \overline{1, n}$.

Since $\mu(A) \cap V_1 \neq \emptyset$, there are $x_1 \in \mu(A)$ and $\varepsilon_1 > 0$ such that $S(x_1, \varepsilon_1) \subset V_1$.

Because $d(x_1, \mu(A)) = 0 < \varepsilon_1$, for ε_1 there are a compact set $K_1 \in \mathcal{C}$ and an open set $D_1 \in \mathcal{C}, K_1 \subset A \subset D_1$ so that $d(x_1, \mu(B)) < \varepsilon_1$, for every $B \in \mathcal{C}$, with $K_1 \subset B \subset D_1$.

Since $d(x_1, \mu(B)) < \varepsilon_1$, there is $y_1 \in \mu(B)$ with $d(x_1, y_1) < \varepsilon_1$, that is, $y_1 \in S(x_1, \varepsilon_1)$. Consequently, $\mu(B) \cap S(x_1, \varepsilon_1) \neq \emptyset$ and, because $S(x_1, \varepsilon_1) \subset V_1$, we get that $\mu(B) \cap V_1 \neq \emptyset$. Continuing this way, we find the compact sets $\{K_j\}_{j=\overline{1,n}}$ and the open sets $\{D_j\}_{j=\overline{1,n}}$ of $\mathcal{C}$ such that $K_j \subset A \subset D_j$ and $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$ and every $B \in \mathcal{C}$, with $K_j \subset B \subset D_j$.

Let be the compact set $K = \overset{n}{\underset{j=1}{\cup}} K_j$ and the open set $D = \overset{n}{\underset{j=1}{\cap}} D_j$. Then $K, D \in \mathcal{C}$, $K \subset A \subset D$ and if $B \in \mathcal{C}$, with $K \subset B \subset D$, then $K_j \subset B \subset D_j$, hence $\mu(B) \cap V_j \neq \emptyset$, for every $j = \overline{1,n}$. This means A is R^--regular (with respect to $\widehat{\tau}_V$).

Theorem 2.8. *If A is R^+-regular (with respect to $\widehat{\tau}_V$), then A is $R^w_>$-regular (with respect to $\widehat{\tau}_W$).*

Proof. Let $x \in X$ and $\varepsilon > 0$ be such that $d(x, \mu(A)) > \varepsilon$. There exists $\varepsilon_1 > 0$ with $d(x, \mu(A)) > \varepsilon_1 > \varepsilon$. Therefore,

$$\mu(A) \subset \{y \in X; d(x, y) > \varepsilon_1\} := U,$$

because for every $y \in \mu(A)$,

$$d(x, y) \geq d(x, \mu(A)) = \inf_{y \in \mu(A)} d(x, y) > \varepsilon_1.$$

Obviously, $U \in \tau_d$.

Since A is R^+-regular, there are a compact set $K = K(x, \varepsilon) \in \mathcal{C}$ and an open set $D = D(x, \varepsilon) \in \mathcal{C}, K \subset A \subset D$ so that $\mu(B) \subset U$, for every $B \in \mathcal{C}, K \subset B \subset D$. Consequently, for every $y \in \mu(B)$, we have $d(x, y) > \varepsilon_1$, which implies $d(x, \mu(B)) = \inf_{y \in \mu(B)} d(x, y) \geq \varepsilon_1 > \varepsilon$ and the proof is thus finished. □

We shall compare in the following the types of regularity defined with respect to the Wijsman topology $\widehat{\tau}_W$, with those that we have introduced and studied in [3] with respect to the Hausdorff topology $\widehat{\tau}_H$, induced by the Hausdorff pseudometric h. In order to do this, from now on, we shall suppose that, moreover, X is a normed space.

We recall that $h(M, N) = \max\{e(M, N), e(N, M)\}$, where $e(M, N) = \sup_{x \in M} d(x, N)$.

It is well known from [8] that $\widehat{\tau}_W \subset \widehat{\tau}_H$, the equality taking place if X is a compact metric space. But this situation is impossible since X is a normed space. Although, we obtain the following characterization:

Theorem 2.9. *Let $\mu : \mathcal{C} \to \mathcal{P}_k(X)$, where $\mathcal{C} = \mathcal{B}_0$ or $\mathcal{B}$. Then A is $R^w_<$-regular (with respect to $\widehat{\tau}_W$) if and only if it is R_l-regular (with respect to $\widehat{\tau}_H$).*

Proof. It follows immediately from [3], [5] and Theorem 2.7.

Theorem 2.10. *Let $\mu : \mathcal{C} \to \mathcal{P}_k(X)$, where $\mathcal{C} = \mathcal{B}_0$ or $\mathcal{B}$. If A is R_r-regular (with respect to $\widehat{\tau}_H$) then A is $R^w_>$-regular (with respect to $\widehat{\tau}_W$).*

Proof. By [3], [5] and Theorem 2.8, we immediately get the conclusion.

We can also indicate a direct proof, which is valid if $\mu : \mathcal{C} \to \mathcal{P}_f(X)$. Indeed, according to Theorem 2.6, let us prove that, equivalently, A is R^w_r-regular. Let $\varepsilon > 0$ and $x_0 \in X$, with $d(x_0, \mu(A)) > \varepsilon$. Then $\varepsilon' = d(x_0, \mu(A)) - \varepsilon > 0$ and, since A is R_r-regular, we obtain the existence for ε' of an open set $D = D(\varepsilon') = D(x_0, \varepsilon) \in \mathcal{C}, A \subset D$ so that $e(\mu(B), \mu(A)) < \varepsilon'$, for every $B \in \mathcal{C}$, with $A \subset B \subset D$.

Because

$$d(x_0, \mu(B)) \geq d(x_0, \mu(A)) - e(\mu(B), \mu(A)) > d(x_0, \mu(A)) - \varepsilon' = \varepsilon,$$

we get that A is $R^w_>$-regular (with respect to $\widehat{\tau}_W$), as claimed.

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