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THE HYPERBOLIC CARNOT THEOREM IN THE POINCARÉ DISC MODEL OF HYPERBOLIC GEOMETRY

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Abstract. In this paper we present (i) the reverse triangle inequality, (ii) the reverse Möbius triangle inequality, and (iii) the hyperbolic Carnot theorem in the Poincaré disc model of hyperbolic geometry.

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1. Introduction

Hyperbolic geometry was created in the first half of the nineteenth century in the midst of attempts to understand Euclid's axiomatic basis for geometry. Hyperbolic geometry is also known as a type of non-Euclidean geometry, and it is similar to Euclidean geometry in many respects. It has concepts of distance and angle, and there are many theorems common to both.

There are many principal hyperbolic geometry models, for instance Poincaré disc model, Einstein relativistic velocity model, Weierstrass model, etc. In this paper we choose the Poincaré disc model of hyperbolic geometry for our study of the hyperbolic Carnot theorem.

Let $\mathbb{D}$ denote the complex open unit disc in the complex z-plane, i.e.

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of $\mathbb{D}$ is

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

which defines the Möbiüs addition $\oplus$ in $\mathbb{D}$, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by a rotation. Here θ is a real number, $z_0 \in \mathbb{D}$, and $\overline{z_0}$ is the complex conjugate of z_0. Möbius addition $\oplus$ is analogous to the common vector addition

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$+$ in Euclidean plane geometry, but Möbius addition $\oplus$ is neither commutative nor associative.

Let $\mathrm{Aut}(\mathbb{D},\oplus)$ be the automorphism group of the grupoid $(\mathbb{D},\oplus)$. If we define

$$\mathrm{gyr} : \mathbb{D}\times\mathbb{D} \to Aut(\mathbb{D},\oplus)$$

by the equation

$$\mathrm{gyr}\,[a,b] = \frac{a\oplus b}{b\oplus a} = \frac{1+a\bar{b}}{1+\bar{a}b},$$

then the following group-like properties of $\oplus$ can be verified by straightforward algebraic calculations:

$a\oplus b = \mathrm{gyr}\,[a,b]\,(b\oplus a),$	gyrocommutative law
$a\oplus(b\oplus c) = (a\oplus b)\oplus \mathrm{gyr}\,[a,b]\,c,$	left gyroassociative law
$(a\oplus b)\oplus c = a\oplus(b\oplus \mathrm{gyr}\,[b,a]\,c),$	right gyroassociative law
$\mathrm{gyr}\,[a,b] = \mathrm{gyr}\,[a\oplus b, b],$	left loop property
$\mathrm{gyr}\,[a,b] = \mathrm{gyr}\,[a, b\oplus a],$	right loop property

Thus, the breakdown of commutativity and associativity in Möbius addition is repaired. Clearly, with these properties, $(\mathbb{D},\oplus)$ is a gyrogroup. We refer readers to [2] for the definition of gyrogroups.

Define the secondary binary operation $\boxplus$ in $\mathbb{D}$ by

$$a\boxplus b = a\oplus \mathrm{gyr}\,[a,\ominus b]\,b.$$

The primary and secondary operations of $\mathbb{D}$ are collectively called the dual operations of the gyrogroups.

Let a,b be the elements of a gyrogroup G. Then the unique solution of the equation

$$a\oplus x = b$$

for the unknown x is

$$x = \ominus a\oplus b$$

and the unique solution of the equation

$$x\oplus a = b$$

for the unknown x is

$$x = b\boxminus a. \tag{1}$$

For further details, see [2, 4].

1.1. Reverse Möbius Triangle Inequality

Definition 1. *The hyperbolic distance function in $\mathbb{D}$ is defined by the equation*

$$d(a,b) = |a\ominus b| = \left|\frac{a-b}{1-\bar{a}b}\right|.$$

Here, $a\ominus b = a\oplus(-b)$ for $a,b\in\mathbb{D}$.

In [3], Ungar has proved the beautiful inequality (*Möbius Triangle Inequality*), i.e

$$d(a,c) \le d(a,b) \oplus d(b,c)$$

for all $a, b, c \in \mathbb{D}$.

Naturally, one may wonder whether the reverses of the triangle inequality and the Möbius triangle inequality exist? Now we give the affirmative answers as follows:

Theorem 2. (Reverse Triangle Inequality) *For all $a, b \in \mathbb{D}$ we have*

$$||a| \ominus |b|| \le |a \ominus b|\,.$$

Proof. Similar to Ungar's proof (see [3, pp. 762]), we start the proof by defining a function. $f_a = \left(1 - \left|a^2\right|\right)^{-\frac{1}{2}}$ is a monotically increasing function of $|a|$. It is easy to see that $f_{|a|} = f_a$ and the identity

$$f_{a\ominus b} = f_a f_b \left|1 - \overline{a}b\right|$$

holds. Thus, we obtain

$$f_{||a|\ominus|b||} = f_{|a|\ominus|b|} = f_{|a|} f_{|b|} \left|1 - |ab|\right|.$$

Using the elementary inequality in the complex plane

$$||z_1| - |z_2|| \le |z_1 - z_2| \text{ for all } z_1, z_2 \in \mathbb{C},$$

we have

$$f_{|a\ominus b|} = f_{a\ominus b} = f_{|a|} f_{|b|} \left|1 - \overline{a}b\right| \ge f_{|a|} f_{|b|} \left|1 - |ab|\right| = f_{|a|\ominus|b|} = f_{||a|\ominus|b||}$$

and therefore we obtain $|a \ominus b| \ge ||a| \ominus |b||$ for all $a, b \in \mathbb{D}$. □

Theorem 3. (Reverse Möbius Triangle Inequality) *For all $x, y, z \in \mathbb{D}$ we have*

$$|\ominus d(x,y) \oplus d(x,z)| \le d(y,z).$$

Proof. In [3, pp. 762], Ungar proved the Möbius triangle inequality, i.e.,

$$d(x,y) \le d(x,z) \oplus d(y,z) \text{ for all } x, y, z \in \mathbb{D}.$$

From (1), we have

$$d(x,y) \boxminus d(y,z) \le d(x,z),$$

i.e.

$$d(x,y) \ominus d(y,z) \le d(x,z)$$

and this implies

$$\ominus d(y,z) \le \ominus d(x,y) \oplus d(x,z).$$

Moreover, Möbius triangle inequality $d(x,z) \le d(x,y) \oplus d(y,z)$ implies that

$$\ominus d(x,y) \oplus d(x,z) \le d(y,z)$$

holds and we obtain

$$\ominus d(y,z) \le \ominus d(x,y) \oplus d(x,z) \le d(y,z),$$

i.e.,

$$|\ominus d(x,y) \oplus d(x,z)| \le d(y,z).$$

Therefore, the reverse Möbius triangle inequality holds. □

2. The hyperbolic Carnot theorem in the Poincaré disc model of hyperbolic geometry

In Euclidean Geometry, Carnot's theorem is a direct application of the theorem of Pythagoras and the theorem states that for a triangle ΔABC and the points A', B', C', where be located on the sides BC, AC and AB respectively, then the perpendiculars to the sides of the triangle at the points A', B' and C' are concurrent if and only, if

$$AC'^2 - BC'^2 + BA'^2 - CA'^2 + CB'^2 - AB'^2 = 0.$$

For the proof of the theorem see [1].

In this section, we prove Carnot's theorem in the Poincaré disc model of hyperbolic geometry.

Theorem 4. *Let ΔABC be a hyperbolic triangle in the Poincaré disc, whose vertices are the points A, B and C of the disc and whose sides (directed counterclockwise) are $a = -B \oplus C$, $b = -C \oplus A$ and $c = -A \oplus B$. Let the points A', B' and C' be located on the sides a, b and c of the hyperbolic triangle ΔABC respectively. If the perpendiculars to the sides of the hyperbolic triangle at the points A', B' and C' are concurrent, then the following holds:*

$$\begin{aligned} &|-A \oplus C'|^2 \ominus |-B \oplus C'|^2 \oplus |-B \oplus A'|^2 \\ &\ominus |-C \oplus A'|^2 \oplus |-C \oplus B'|^2 \ominus |-A \oplus B'|^2 = 0. \end{aligned} \tag{2}$$

Proof. We assume that three perpendiculars meet at a point of ΔABC and let denote this point by P. The geodesic segments $-A \oplus P$, $-B \oplus P$, $-C \oplus P$, $-A' \oplus P$, $-B' \oplus P$, $-C' \oplus P$ split the hyperbolic triangle into six right-angled hyperbolic triangles. Notice that three pairs of them share a hypotenuse, whilst three other pairs share a leg with a vertex at P. Now we apply the Hyperbolic Pythgorean theorem to these six right-angled hyperbolic triangles one by one, and we easily obtain:

$$\begin{aligned}
|-P \oplus A|^2 &= |-A \oplus C'|^2 \oplus |-C' \oplus P|^2, \\
|-B \oplus P|^2 &= |-P \oplus C'|^2 \oplus |-C' \oplus B|^2, \\
|-P \oplus B|^2 &= |-B \oplus A'|^2 \oplus |-A' \oplus P|^2, \\
|-C \oplus P|^2 &= |-P \oplus A'|^2 \oplus |-A' \oplus C|^2, \\
|-P \oplus C|^2 &= |-C \oplus B'|^2 \oplus |-B' \oplus P|^2, \\
|-A \oplus P|^2 &= |-P \oplus B'|^2 \oplus |-B' \oplus A|^2.
\end{aligned}$$

Using the equalities

$$\begin{aligned}
|-P \oplus A|^2 &= |-A \oplus P|^2, \\
|-B \oplus P|^2 &= |-P \oplus B|^2, \\
|-C \oplus P|^2 &= |-P \oplus C|^2,
\end{aligned}$$

we have

$$\begin{aligned}
\alpha &= |-A \oplus C'|^2 \oplus |-C' \oplus P|^2 = |-P \oplus B'|^2 \oplus |-B' \oplus A|^2 = \alpha', \\
\beta &= |-B \oplus A'|^2 \oplus |-A' \oplus P|^2 = |-P \oplus C'|^2 \oplus |-C' \oplus B|^2 = \beta', \\
\gamma &= |-C \oplus B'|^2 \oplus |-B' \oplus P|^2 = |-P \oplus A'|^2 \oplus |-A' \oplus C|^2 = \gamma'.
\end{aligned}$$

This implies

$$(\alpha \oplus \beta) \oplus \gamma = (\alpha' \oplus \beta') \oplus \gamma'.$$

Since $((-1,1), \oplus)$ is a commutative group, we immediately obtain

$$|-A \oplus C'|^2 \oplus |-B' \oplus A|^2 \oplus |-C \oplus B'|^2 = |-B' \oplus A|^2 \oplus |-C' \oplus B|^2 \oplus |-A' \oplus C|^2,$$

i.e.

$$\begin{aligned}
&|-A \oplus C'|^2 \ominus |-B \oplus C'|^2 \oplus |-B \oplus A'|^2 \\
&\ominus |-C \oplus A'|^2 \oplus |-C \oplus B'|^2 \ominus |-A \oplus B'|^2 = 0.
\end{aligned}$$

□

Naturally, one may wonder whether the converse of the Carnot theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 5. *Let ΔABC be a hyperbolic triangle in the Poincaré disc, whose vertices are the points A, B and C of the disc and whose sides (directed counterclockwise) are $a = -B \oplus C$, $b = -C \oplus A$ and $c = -A \oplus B$. Let the points A', B' and C' be located on the sides a, b and c of hyperbolic triangle ΔABC respectively. If (2) holds and two of the three perpendiculars to the sides of the hyperbolic triangle at the points A', B' and C' are concurrent, then the three perpendiculars are concurrent.*

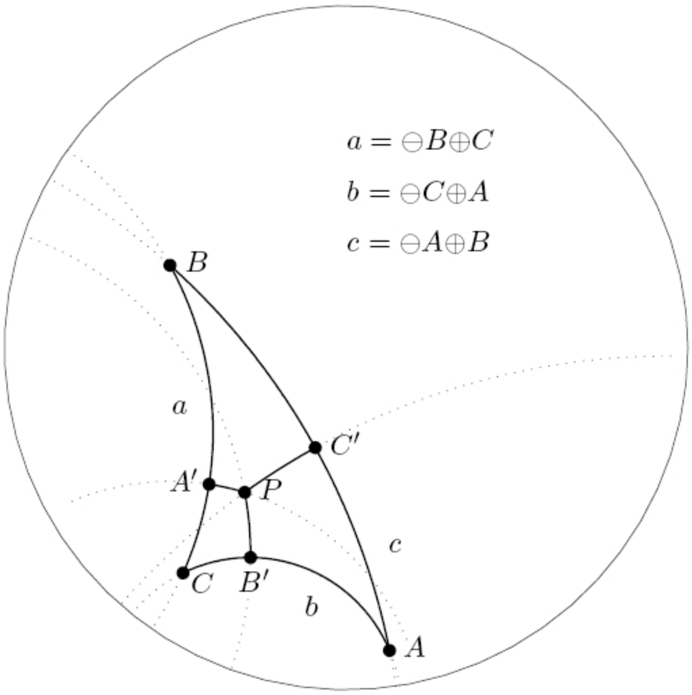

Figure 1: The hyperbolic Carnot theorem in the open unit disc. Here the geodesic lines are circular arcs that intersect the boundary of the disc orthogonally

Proof. Let $-A' \oplus P$ and $-B' \oplus P$ be perpendiculars to the sides a and b respectively and P be the intersection point of these perpendiculars. Draw a perpendicular $-K \oplus P$ from P to c. Using the already proven equality (2), we obtain

$$|-A \oplus K|^2 \ominus |-B \oplus K|^2 \oplus \left|-B \oplus A'\right|^2 \ominus \left|-C \oplus A'\right|^2 \oplus \left|-C \oplus B'\right|^2 \ominus \left|-A \oplus B'\right|^2 = 0,$$

then we get

$$|-A \oplus K|^2 \ominus |-B \oplus K|^2 = \left|-A \oplus C'\right|^2 \ominus \left|-B \oplus C'\right|^2.$$

This equation holds only for $K = C'$. Indeed, if we take $x := d(B, K)$ and $h := |-A \oplus B|$, then we get $d(A, K) = h \ominus x$. For $x, y \in (-1, 1)$ define

$$f(x) = (h \ominus x)^2 \ominus x^2.$$

Since the following equality holds

$$f(x) - f(y) = \frac{-2h\left(1 - h^2\right)(1 - xy)}{(1 - 2hx + x^2)(1 - 2hy + y^2)}(x - y),$$

we get $f(x)$ is an injective function and this implies $K = C^{\prime}$. $\square$

Remark 6. *Two perpendiculars to the sides of the hyperbolic triangle at any two points of the three points $A^{\prime}, B^{\prime}$ and $C^{\prime}$ need not be concurrent, as we can see from Figure 1. If we place $B^{\prime}$ close enough to A, and $A^{\prime}$ close enough to B, then the two perpendiculars do not intersect, and the point P does not exist in this case. Hence, the converse of the hyperbolic Carnot theorem is valid only partially.*

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