Novi Sad J. Math.
Vol. 38, No. 2, 2008, 25-31

ANTIEIGENVECTORS OF THE GENERALIZED EIGENVALUE PROBLEM AND AN OPERATOR INEQUALITY COMPLEMENTARY TO SCHWARZ'S INEQUALITY

Kallol Paul[1]

Abstract. We study the antieigenvectors of the generalized eigenvalue problem $Af = \lambda Bf$ by using the concept of stationary vectors and then obtain an operator inequality complementary to Schwarz's inequality in Hilbert space.

AMS Mathematics Subject Classification (2000): 47A63,47A75

Key words and phrases: Antieigenvectors

1. Introduction

Let A and B be two bounded linear operators on a complex Hilbert space Gustafson [6] and H. Krein [10] have studied the concept of antieigenvalue for the eigenvalue problem $Af = \lambda f$ which is denoted as $\mu_1(A)$ and is defined as follows :

$$\mu_1(A) = \min \ \{Re \ \frac{(Af, f)}{\|f\|\|Af\|} \ : \ f \in H, \ f \neq 0 \ \}.$$

Gustafson calls $\mu_1(A)$ the first antieigenvalue of A and f the corresponding antieigenvector. Davis [3] and Mirman [11] have also studied $\mu_1(A)$. In [2] we studied the structure of the antieigenvectors of a strictly accretive operator and in [9] we calculated the bounds for total antieigenvalue of a normal operator. Extending the idea of Krein [10] and Gustafson [6] we here define the antieigenvalue for the generalized eigenvalue problem $Af = \lambda Bf$ assuming

$$\mu_1(A, B) = \min \ \{ \ Re \ \frac{(Af, Bf)}{\|Bf\|\|Af\|} \ : \ f \in H, \ Af \neq 0, \ Bf \neq 0 \ \},$$

that $inf \ \{ \ Re(Af, Bf) \ / \ (\|Af\| \ \|Bf\|) \ \}$ is attained at a vector f if the space is infinite dimensional. We call $\mu_1(A, B)$ the generalized antieigenvalue and f the generalized antieigenvector.

To study the generalized antieigenvectors we use the concept of stationary vector studied by Das in [1], the definition of which is given below:

[1]Reader in Mathematics, Department of Mathematics, Jadavpur University, Kolkata 700032, INDIA, e-mail: kalloldada@yahoo.co.in, kpaul@math.jdvu.ac.in

Definition 1. Stationary vector.
Let $\phi(f)$ be a functional of a unit vector $f \in H$. Then $\phi(f)$ is said to have a stationary value at f if the function $w_g(t)$ of a real variable t, defined as

$$w_g(t) = \phi(\frac{f+tg}{\|f+tg\|})$$

has a stationary value at $t = 0$ for any arbitrary but fixed vector $g \in H$. The vector f is then called a stationary vector.

From now onwards, (A, B) will denote the generalized eigenvalue problem $Af = \lambda Bf$. So (A^*, B^*) denotes the generalized eigenvalue problem $A^*f = \lambda B^*f$.

2. Structure of generalized antieigenvectors

We write

$$\Phi(f) = Re\ \frac{(Af, Bf)}{\|Bf\|\|Af\|}\ ;\ f \in H,\ Af \neq 0,\ Bf \neq 0.$$

and find the necessary and sufficient condition for a unit vector f to be a stationary vector of $\Phi(f)$.
For this we define

$$w_g(t) \ = \ \frac{\left(\ \frac{(A^*B+B^*A)}{2}\ (f+tg), f+tg\ \right)^2}{\|A(f+tg)\|^{\ 2}\ \ \|B(f+tg)\|^{\ 2}}$$

where g is an arbitrary but fixed vector of H.
If f is a stationary vector then we have $w_g'(0) = 0$ and so we get

$$\|Af\|^2\|Bf\|^2 \cdot 2(\tfrac{A^*B+B^*A}{2}f, f) \cdot \Big\{(\tfrac{A^*B+B^*A}{2}f, g)$$
$$+(\tfrac{A^*B+B^*A}{2}g, f)\Big\} - \Big(\tfrac{A^*B+B^*A}{2}f, f\Big)^2 \cdot$$
$$\Big\{\|Af\|^2((Bf, Bg) + (Bg, Bf)) + \|Bf\|^2((Af, Ag) + (Ag, Af))\Big\} = 0.$$

As g is arbitrary we get

$$\|Af\|^2\ \|Bf\|^2\ 2(\tfrac{A^*B+B^*A}{2})\ f-$$
$$(\tfrac{A^*B+B^*A}{2}\ f, f)\ \{\ \|Af\|^2B^*Bf + \|Bf\|^2A^*Af\ \} = 0.$$
$$\Rightarrow \|Af\|^2\ \|Bf\|^2\ (\ A^*B + B^*A\ )\ f-$$
$$(\tfrac{A^*B+B^*A}{2}\ f, f)\ \{\ \|Af\|^2B^*Bf + \|Bf\|^2A^*Af\ \} = 0.$$

This is the necessary and sufficient condition for $\Phi(f)$ to be stationary at a vector f.
We then prove the following theorem :

Theorem 1. *Suppose $A^*B = B^*A$ and f be a generalized antieigenvector of (A, B). Then Bf can be expressed as a linear combination of two generalized eigenvectors of (A^*, B^*).*
If further B is invertible then f can be expressed as the linear combination of two generalized eigenvectors of (A, B).

Proof. As f is a generalized antieigenvector, in particular, a stationary vector of $\Phi(f)$, we have the necessary and sufficient condition for f to be a stationary vector of $\Phi(f)$

$$\|Af\|^2 \ \|Bf\|^2 \ (\ A^*B + B^*A\ )\ f - \left(\tfrac{A^*B+B^*A}{2}\ f, f\right) \{\ \|Af\|^2 B^*Bf + \|Bf\|^2 A^*Af\ \} = 0.$$

As $A^*B = B^*A$ we get

$$\|Af\|^2 \ \|Bf\|^2 \ 2\ A^*B\ f - (\ A^*B\ f, f)\ \{\ \|Af\|^2 B^*Bf + \|Bf\|^2 A^*Af\ \} = 0.$$

$$\textit{Let } Af = \lambda Bf + h \textit{ where } (Bf, h) = 0, \text{ then } \|Af\|^2 - \frac{|\ (Af, Bf)\ |^{\ 2}}{(Bf, Bf)} = \|h\|^2.$$

Now

$$\begin{aligned} A^*Af \ - \ \frac{\|Af\|^2}{(A^*Bf, f)}\ A^*Bf \ &= \ \frac{\|Af\|^2}{(A^*Bf, f)}\ A^*Bf \ - \frac{\|Af\|^2}{\|Bf\|^2} B^*Bf \\ \Rightarrow \ A^*Af - \frac{\|Af\|}{\|Bf\|\Phi(f)} A^*Bf \pm \frac{\|h\|}{\|Bf\|\Phi(f)} A^*Bf &= \\ \pm \frac{\|h\|}{\|Bf\|\Phi(f)} A^*Bf + \frac{\|Af\|}{\|Bf\|\Phi(f)} A^*Bf - \frac{\|Af\|^2}{\|Bf\|^2} B^*Bf. & \\ \Rightarrow \ A^*\ [\ Af - \frac{\|Af\|}{\|Bf\|\Phi(f)} Bf \pm \frac{\|h\|}{\|Bf\|\Phi(f)} Bf\ ] &= \\ \frac{\|Af\| \pm \|h\|}{\Phi(f)\|Bf\|} B^*\ [\ Af - \frac{\|Af\|}{\|Bf\|\Phi(f)} Bf \pm \frac{\|h\|}{\|Bf\|\Phi(f)} Bf\ ]. & \end{aligned}$$

Let

$$g_1 = Af \ - \ \frac{\|Af\| - \|h\|}{\|Bf\|\ \Phi(f)} Bf\ , \quad \lambda_1 = \frac{\|Af\| + \|h\|}{\|Bf\|\ \Phi(f)}$$

and

$$g_2 = Af \ - \ \frac{\|Af\| + \|h\|}{\|Bf\|\ \Phi(f)}\ Bf\ , \quad \lambda_2 = \frac{\|Af\| - \|h\|}{\|Bf\|\ \Phi(f)}\ .$$

Then $A^*g_1 = \lambda_1 B^*g_1$ and $A^*g_2 = \lambda_2 B^*g_2$ so that g_1 and g_2 are two eigenvectors of the equation $A^*f = \lambda B^*f$ with eigenvalues λ_1 and λ_2 respectively.
Then

$$Bf = \frac{\|Bf\|\ \Phi(f)}{2\ \|h\|}\ \ (g_1 - g_2)\ .$$

If B is invertible then for any $g \in H$ we have

$$(A^* - \lambda B^*)g = 0 \ \Leftrightarrow \ (A - \lambda B)B^{-1}g = 0\ .$$

So

$$A(B^{-1}g_1) = \lambda_1 B(B^{-1}g_1), \quad A(B^{-1}g_2) = \lambda_2 B(B^{-1}g_2)$$

and

$$f \;=\; \frac{\|Bf\|\Phi(f)}{2\|h\|}(B^{-1}g_1 - B^{-1}g_2)\ .$$

This completes the proof of the theorem. □

3. An inequality complementary to Schwarz's inequality

Here we develop an inequality complementary to Schwarz's inequality in Hilbert space. With Schwarz's inequality we always have

$$\forall f \in H\ (Af, Af)(Bf, Bf) \geq |\ (Af, Bf)\ |^2.$$

We reverse the sign of inequality and then improve it under some restrictions on A and B. Assuming A and B to be positive and permutable Greub and Rheinboldt [5] proved that if $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_2 I \leq B \leq M_2 I$ then for all $f \in H$

$$\text{(I)} \qquad (Af, Af)(Bf, Bf) \leq \frac{(M_1M_2 + m_1m_2)^2}{4M_1M_2m_1m_2}(Af, Bf)^2$$

With the same assumtions Diaz J.B. and Metcalf F.T. [4] improved on the inequality to prove that for all $f \in H$,

$$\text{(II)} \qquad m_1M_1(Bf, Bf) + m_2M_2(Af, Af) \leq (M_1M_2 + m_1m_2)(Af, Bf).$$

Greub and Rheinboldt [5] also proved the generalized Kantorovich inequality which states that if C is a positive operator with $0 < mI \leq C \leq MI$ then for all $f \in H$

$$\text{(III)} \qquad (Cf, f)(C^{-1}f, f) \leq \frac{(M + m)^2}{4mM}(f, f)^2$$

and they also proved that inequalities (I) and (III) are equivalent.
Instead of assuming A and B to be positive and permutable we only assume here that A^*B is positive and prove that for all $f \in H$

$$\text{(IV)} \qquad (Af, Af)(Bf, Bf) \leq \frac{(M + m)^2}{4mM}(Af, Bf)^2$$

where m and M are the least and greatest generalized eigenvalues of (A^*, B^*). We then show that inequalities (III) and (IV) are equivalent. We first prove the following theorem :

Theorem 2. *Suppose m and M are the least and greatest generalized eigenvalues of* (A^*, B^*).
Then

$$\forall f \in H\ 4mM\ (Af, Af)(Bf, Bf) \leq (M + m)^2(Af, Bf)^2.$$

Proof. If f is a generalized antieigenvector then we have by previous theorem $A^*g_1 = \lambda_1 B^* g_1$ and $A^* g_2 = \lambda_2 B^* g_2$ where

$$g_1 = Af \; - \; \frac{\|Af\| - \|h\|}{\|Bf\| \; \Phi(f)} Bf \; , \quad \lambda_1 = \frac{\|Af\| + \|h\|}{\|Bf\| \; \Phi(f)}$$

and

$$g_2 = Af \; - \; \frac{\|Af\| + \|h\|}{\|Bf\| \; \Phi(f)} Bf \; , \quad \lambda_2 = \frac{\|Af\| - \|h\|}{\|Bf\| \; \Phi(f)}.$$

So

$$\lambda_1 + \lambda_2 = \frac{2\|Af\|}{\Phi(f)\|Bf\|} \text{ and } \sqrt{\lambda_1 \lambda_2} = \frac{(Af, Bf)}{\Phi(f)\|Bf\|^2}.$$

Also

$$\frac{2\,\sqrt{\lambda_1\lambda_2}}{\lambda_1 + \lambda_2} = \frac{(Af, Bf)}{\|Af\|\|Bf\|} = \Phi(f).$$

Let

$$u = \frac{\lambda_1}{\lambda_2} \; , \; \lambda_1 > \lambda_2.$$

Then

$$\begin{aligned} F(u) &= \frac{2\sqrt{\lambda_1\lambda_2}}{\lambda_1 + \lambda_2} \\ &= \frac{2}{\sqrt{\frac{\lambda_1}{\lambda_2}} + \sqrt{\frac{\lambda_2}{\lambda_1}}} \\ &= \frac{2}{\sqrt{u} + \frac{1}{\sqrt{u}}} \end{aligned}$$

is a decreasing function of u so that $F(u)$ attains its minimum at the maximum value of u. Hence if m and M are the least and the greatest generalized eigenvalues of (A^*, B^*) then

$$\begin{aligned} \min_{Af, Bf \neq 0} \frac{(Af, Bf)}{\|Af\|\|Bf\|} &= \frac{2\sqrt{mM}}{m + M} \\ \Rightarrow \frac{(Af, Bf)^2}{\|Af\|^2\|Bf\|^2} &\geq \frac{4mM}{(m+M)^2} \; , \end{aligned}$$

where $f \in H$ is such that $Af \neq 0, \; Bf \neq 0$.
Thus we get

$$\forall f \in H \; 4mM \; (Af, Af)(Bf, Bf) \leq (M + m)^2 (Af, Bf)^2.$$

This completes the proof. □

Now we show that inequalities (III) and (IV) are equivalent.
Inequality (III) clearly follows from (IV) by taking $A = C^{\frac{1}{2}}$ and $B = C^{\frac{-1}{2}}$. For

the other part we have $A^*B > 0$ and so B is invertible. Let $C = AB^{-1}$. Then $C^* = (B^{-1})^* A^* = (B^{-1})^* (A^*B)B^{-1}$ so that C > 0.
As m and M are the least and greatest eigenvalues of $(B^{-1})^* A^* = C^* = C$ so we get by inequality (III)

$$(Cf, f)(C^{-1}f, f) \leq \frac{(M+m)^2}{4mM}(f, f)^2 \ \forall f \in H \ .$$

Substituting $g = C^{\frac{1}{2}}h$ we get

$$\forall h \in H \ (CC^{\frac{1}{2}}h, C^{\frac{1}{2}}h)(C^{-1}C^{\frac{1}{2}}h, C^{\frac{1}{2}}h) \leq \frac{(M+m)^2}{4mM}(C^{\frac{1}{2}}h, C^{\frac{1}{2}}h)^2.$$

So we get

$$\forall h \in H \ (Ch, Ch)(h, h) \leq \frac{(M+m)^2}{4mM}(Ch, h)^2.$$

Again substituting $h = Bg$ we get

$$\forall g \in H \ (Ag, Ag)(Bg, Bg) \leq \frac{(M+m)^2}{4mM}(Ag, Bg)^2.$$

Thus the inequalities (III) and (IV) are equivalent.
Also the inequality (I) can be deduced easily from inequality (IV), for if A, B are selfadjoint with $AB = BA$, $0 < m_1 I \leq A \leq M_1 I$, $0 < m_2 I \leq B \leq M_2 I$ then $\frac{m_1}{M_2}$ and $\frac{M_1}{m_2}$ are the least and greatest real eigenvalues of $Af = \lambda Bf$ so that

$$\min_{Af, Bf \neq 0} \frac{(Af, Bf)}{\|Af\|\|Bf\|} = \frac{2\sqrt{m_1 m_2 M_1 M_2}}{m_1 m_2 + M_1 M_2} \ .$$

Thus we get

$$\forall f \in H \ (Af, Af)(Bf, Bf) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 M_1 M_2 m_1 m_2}(Af, Bf)^2.$$

The inequality (II) by Diaz J.B. and Metcalf F.T. stated earlier is better than our inequality but with more restrictions on operators A and B.
We finally give an easy example of two operators A and B for which inequality (IV) holds but inequality (I) is not applicable.

Example 1. *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then $A \neq A^*$ *and* $B \neq B^*$*. Also* $AB \neq BA$*. But* $A^*B > 0$ *so that inequality (IV) holds to give*

$$\forall f \in H \ (Af, Af)(Bf, Bf) \leq 2(Af, Bf)^2.$$

Clearly, inequality (I) is not applicable.

From this example we can conclude that inequality (IV) is applicable to a larger class of operators than inequality (I).

I thank a referee for pointing out the additional references [7, 8, 12], the first for history and background on antieigenvalue theory, the second and third also treating the generalized eigenvalue problem, each with somewhat different perspective.

Acknowledgement

The author thanks Professor K.C. Das and Professor T.K. Mukherjee for their invaluable suggestions while preparing this paper. A part of the research work was done by the author during PhD Dissertation.

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Received by the editors August 14, 2007