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EXTENSION OF NONADDITIVE MEASURES ON LOCALLY COMPLETE σ-CONTINUOUS LATTICES

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Abstract. We introduce the concept of M_σ-approachability for a semi-continuous function (i.e. a nonadditive measure) on a σ-complete sublattice of a locally complete σ-continuous lattice L and using it we extend a nonadditive measure from a sublattice of L to a σ-complete sublattice of L.

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1. Introduction

In measure theory, a basic procedure is that of extending the notion of a measure on a given class of sets to a larger class of sets. Kelley, Nayak and Srinivasan [12] proved that a nonnegative real-valued function μ defined on a lattice L of sets is a premeasure (meaning that it extends to a countably additive measure on a δ-ring containing L) provided μ is tight and continuous at $\emptyset$. The extension of this theorem to the class of real-valued (not necessarily nonnegative real-valued) function is dealt in [19]. In 1981, Morales [18] established a quite general extension theorem for a uniform semigroup valued tight set function λ on a lattice L of subsets of a set X, the domain of extension being the σ-ring generated by the lattice L. He also discussed the extension of λ on the σ-algebra of a locally L-measurable sets.

Riečan [22] proved an extension theorem for a positive real-valued modular functions defined on a suborthomodular lattice of a σ-continuous, σ-complete orthomodular lattice. An extension theorem has been proved for measures on MV-algebra in fuzzy measure theory ([4, 5, 23]; see also [10, 13, 25]). In [2], Avallone and Simone proved an extension theorem for nonnegative real-valued modular functions defined on suborthomodular lattices of a σ-continuous, σ-complete orthomodular lattice, using topological approach. They used the theory of lattice uniformities (i.e. a uniformity which makes the lattice operations $\vee$ and $\wedge$ uniformly continuous). They further extended the theory in context of lattice ordered effect algebras in [3].

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A variety of structural characteristics of nonadditive set functions are introduced and discussed by Dobrakov [7, 8], Drewnowski [9], Wang [24], Wang and Klir [26], Pap [20] and Denneberg [6], and the relevant theories are developed by them separately. Nonadditive measures appear today in many branches of pure mathematics with many important applications ([21, 24], see also [14, 15, 16]).

The aim of the present paper is to study an extension problem for nonadditive measures defined on a sublattice of a locally complete σ-continuous lattice L. Some basic definitions are collected in Section 2. In Section 3, we introduce notions of absolute continuity and M_σ-approachability and we extend a semicontinuous function (i.e. a nonadditive measure, or simply a measure) μ on a sublattice M of L to a unique M_σ-approachable measure $\widehat{\mu}$ on a σ-complete sublattice N containing M under suitable conditions. This goal has been achieved in three steps: firstly we extend any *lsc*-measure on M to an *lsc*-measure on M_σ. To obtain extension of a measure μ on M to a measure $\widetilde{\mu}$ on M_σ, we need a sufficient condition involving the notion of absolute continuity; $\widetilde{\mu}$ preserves absolute continuity. Finally, in the third step we prove that μ can uniquely be extended to $\widehat{\mu}$ on N containing M_σ, using M_σ-approachability. Extensions obtained at each step are uniquely determined. Some basic results on modular functions are also obtained and we prove that if μ is submodular, then $\widetilde{\mu}$ is submodular and consequently $\widehat{\mu}$ is submodular.

2. Preliminaries and Basic Results

Let $(P, \leq)$ be a poset. An element $x \in P$ is called an *upper bound* of $A \subseteq P$ if $a \leq x$ for every $a \in A$; x is called a *lower bound* of A, if $x \leq a$ for every $a \in A$. An element $x \in L$ is called the *join* (or the *least upper bound*, or the *sup*) of $A \subseteq L$, denoted by $\bigvee A$, if

(i) x is an upper bound of A,

(ii) if y is an upper bound of A, then $x \leq y$.

If A is finite, we call $\bigvee A$ (if it exists) a *finite join.* If A contains only two elements a and b, then we sometimes also write $a \vee b$ instead of $\bigvee\{a, b\}$ for our convenience; similarly $\bigvee A = a_1 \vee a_2 \vee \ldots \vee a_n$ where $A = \{a_1, a_2, \ldots, a_n\}(n \in \mathbb{N}$; $\mathbb{N}$ denotes the set of all natural numbers). A lattice is a poset $(L, \leq)$ in which both join and meet for every finite subset of L exist. For a lattice L, for all $a, b \in L$, $a \leq b$ if and only if $a \vee b = b$ if and only if $a \wedge b = a$. A lattice $L = (L, \leq)$ is called *complete* if every (possibly empty) subset of L admits an infimum (or equivalently if every subset of L admits a supremum). The symbol 1 denotes the top element (or supremum) of L and 0 denotes the bottom element (or infimum) of L (cf. [17]). A lattice L is said to be *σ-complete*, if every countable subset of L has a supremum and an infimum [3].

2.1 [1]. A lattice L is *locally complete* if it satisfies one of the following equivalent conditions:

(i) Every nonempty lower bounded subset of L admits an infimum.

(ii) Every nonempty upper bounded subset of L admits a supremum.

(iii) There exists a complete lattice, denoted by $\overline{L}$, with the bottom element 0 and top element 1, such that L is a sublattice of $\overline{L}$, $\overline{L} = L \cup \{0, 1\}$, $\inf L = 0$

and $\sup L = 1$.
It can be observed that every complete lattice is locally complete.

2.2. Let $\{a_n\}$ be a sequence in a lattice $L = (L, \leq)$. We call $a_n \uparrow a, (a \in L)$ if and only if $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$, $\bigvee a_n$ exists and $\bigvee a_n = a$. In this case we also write $a = \lim_{n\to\infty} a_n$. If $a_n \uparrow a$, $b_n \uparrow b$ and $a_n \leq b_n$, for all n, then we may deduce that $a \leq b$.

2.3 [2]. A lattice L is said to be *σ-continuous* if $a_n \uparrow a$ implies $a_n \wedge b \uparrow a \wedge b$ (or equivalently, $a_n \downarrow a$ implies $a_n \vee b \downarrow a \vee b$) for every $b \in L$. If L is σ-continuous then, for the sequences $\{a_n\}$ and $\{b_n\}$ in L such that $a_n \uparrow a$ and $b_n \uparrow b$, we have $a_n \wedge b_n \uparrow a \wedge b$ (or equivalently, $a_n \downarrow a, b_n \downarrow b$ implies $a_n \vee b_n \downarrow a \vee b$).

Every infinitely distributive lattice [17] L is σ-continuous.

For any set X, $(\mathfrak{P}(X), \subseteq)$, $(L^X, \leq)$ (where L is locally complete σ-continuous lattice) and $(\mathbb{I}, \leq)$ where ($\mathbb{I}$ is the closed unit interval [0, 1] of the real line $\mathbb{R}$) are locally complete σ-continuous lattices. For more examples of locally complete lattices, we refer to [1].

2.4. A function $\mu : L \to [0, \infty)$ is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$. We say that μ is *submodular* if for every $a, b \in L$, we have $\mu(a) + \mu(b) \geq \mu(a \vee b) + \mu(a \wedge b)$.

3. M_σ-Approachability and Extension of Nonadditive Measures

Let L be a locally complete σ-continuous lattice and let C be a nonempty subset of L, M be a sublattice of L, and N be a σ-complete sublattice of L containing M. We denote $M_\sigma = \{b \in L :$ there exists a sequence $\{a_n\}$ in M such that $a_n \uparrow b\}$. We may also describe M_σ as the family of all countable joins of elements from M. Then M_σ is a sublattice of L containing M.

Definition 3.1. *A function $\mu : C \to [0, \infty)$ is called a semi-continuous measure (or nonadditive measure, or simply a measure) on C, if it satisfies the following conditions:*

(i) $\mu(0) = 0$, whenever $0 \in C$,

(ii) (monotone) if $a \leq b$, $a, b \in C$, then $\mu(a) \leq \mu(b)$,

(iii) (semi-continuous from below) if $a_n \uparrow a$, $a \in C$, $a_n \in C\ (n \in \mathbb{N})$, then $\lim_{n\to\infty} \mu(a_n) = \mu(a)$,

(iv) (semi-continuous from above) if $a_n \downarrow a$, $a \in C$, $a_n \in C\ (n \in \mathbb{N})$, then $\lim_{n\to\infty} \mu(a_n) = \mu(a)$.

The function μ is said to be a lower semi-continuous measure (or lsc-measure) if it satisfies (i), (ii) and (iii), while μ is said to be an upper semi-continuous measure (or usc-measure) if it satisfies (i), (ii) and (iv).

Definition 3.2. *A nondecreasing function $\mu : C \to [0, \infty)$ is said to be lower (respectively, upper) consistent on C, if for every $b \in C$, $a_n \in C\ (n \in \mathbb{N})$, $a_n \uparrow a$ with $b \leq a$ we have $\lim_{n\to\infty} \mu(a_n) \geq \mu(b)$ (respectively, $a_n \downarrow a$ and $a \leq b$ we have $\lim_{n\to\infty} \mu(a_n) \leq \mu(b)$).*

We obtain the following:

Proposition 3.1. *Let $\mu : C \to [0, \infty)$ be a monotone function. If C is closed under the formation of finite meet (respectively, finite join), then μ is lower (respectively, upper) consistent if and only if μ is semi-continuous from below (respectively, semi-continuous from above).*

Lemma 3.1. *Let $\mu : C \to [0, \infty)$ be a monotone and semi-continuous from below function, where C is closed under the formation of finite meet. For $a, b \in L$, let $a_n \uparrow a$, $b_n \uparrow b$, $a_n, b_n \in C\,(n \in \mathbb{N})$. If $a \leq b$, then $\lim_{n\to\infty} \mu(a_n) \leq \lim_{n\to\infty} \mu(b_n)$.*

Theorem 3.1. *If μ is an lsc-measure on M, then μ can be extended uniquely to an lsc-measure on M_σ.*

Proof. For $b \in M_\sigma$, define $\widetilde{\mu}(b) = \lim_{n\to\infty} \mu(a_n)$ when $\{a_n\}$ is a sequence in M and $a_n \uparrow b$. In view of Lemma 3.1, $\widetilde{\mu}$ is well defined.

For monotonicity, suppose that $a, b \in M_\sigma$ and $a \leq b$. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ in M such that $a_n \uparrow a$ and $b_n \uparrow b$. Since L is σ-continuous, so $a_n \wedge b_n \uparrow a \wedge b = a$. Now

$$\widetilde{\mu}(b) = \lim_{n\to\infty} \mu(b_n) \geq \lim_{n\to\infty} \mu(a_n \wedge b_n) = \widetilde{\mu}(a).$$

Next, suppose that $\{a_n\}$ is a sequence in M_σ and $a_n \uparrow a, a \in M_\sigma$. Then there exists a sequence $\{a_{ni}\}_{i=1}^{\infty}$ in M such that $a_{ni} \uparrow a_n$ and $\widetilde{\mu}(a_n) = \lim_{i\to\infty} \mu(a_{ni})$, $(n \in \mathbb{N})$. For $i \in \mathbb{N}$, set $b_i = a_{1i} \vee a_{2i} \ldots \vee a_{ii}$. Then $b_i \in M$, $\{b_i\}$ is an increasing sequence and $b_i \leq a_i \leq a$ for all i, which yield that $b = \lim_{i\to\infty} b_i = \vee b_i \leq \vee a_i = a$. It may be noted that $b \in M_\sigma$.

Also, $a_{ki} \leq b_i$ for $1 \leq k \leq i$. Therefore, $a_k = \lim_{i\to\infty} a_{ki} \leq \lim_{n\to\infty} b_n = b$. It follows that $a = \vee a_k \leq b$. Thus $a = b$. Now, from the monotonicity of $\widetilde{\mu}$,

$$\widetilde{\mu}(a) = \lim_{n\to\infty} \mu(b_n) = \lim_{n\to\infty} \widetilde{\mu}(b_n) \leq \lim_{n\to\infty} \widetilde{\mu}(a_n),$$

and the result follows. Obviously, the extension $\widetilde{\mu}$ is unique. □

Definition 3.3. *Let μ and ν be two measures on C. We say that μ is absolutely continuous with respect to ν (denoted by $\mu \ll \nu$), if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mu(c) - \mu(b)| < \varepsilon$, whenever $b, c \in C$ and $|\nu(c) - \nu(b)| < \delta$. (cf. [11])*

Theorem 3.2. *Let μ be a measure on M. Then μ can be extended to a measure $\widetilde{\mu}$ on M_σ, provided there exists a measure ν on M_σ such that $\mu \ll \nu$ on M. The extension is unique and $\widetilde{\mu} \ll \nu$ on M_σ.*

Proof. In view of Theorem 3.1, we need only to prove that $\widetilde{\mu}$ is semi-continuous from above. For this, let $\{a_n\}$ be a sequence in M_σ and $a_n \downarrow a_0$, $a_0 \in M_\sigma$. Then there exists a sequence $\{a_{ni}\}_{i=1}^{\infty}$ in M such that $a_{ni} \uparrow a_n$ $(n \in \mathbb{N} \cup \{0\})$. Let $\varepsilon > 0$. Since $\mu \ll \nu$ on M there exists $\delta > 0$ such that $|\mu(c) - \mu(b)| < \varepsilon/2$, whenever $b, c \in M$ and $|\nu(c) - \nu(b)| < \delta$. Since ν is a measure on M_σ and $a_n \downarrow a_0$ there exists $m \in \mathbb{N}$ such that

$$\nu(a_n) < \nu(a_0) + \delta/2, \text{ for all } n \geq m.$$

Since $a_{0i} \uparrow a_0$ and $a_{ni} \uparrow a_n$ $(n \in \mathbb{N})$ and ν is a measure on M_σ, there exists $k \in \mathbb{N}$ such that, for all $i \geq k$,

$$\nu(a_0) < \nu(a_{0i}) + \delta/2,$$
$$\nu(a_n) < \nu(a_{ni}) + \delta$$

and
$$\widetilde{\mu}(a_n) < \mu(a_{ni}) + \varepsilon/2,$$
(by definition of $\widetilde{\mu}$). So, for $n \geq m$, $i \geq k$ we have

$$\nu(a_{ni}) \leq \nu(a_n) < \nu(a_0) + \delta/2 < \nu(a_{0i}) + \delta,$$

and also

$$\nu(a_{0i}) \leq \nu(a_0) \leq \nu(a_n) < \nu(a_{ni}) + \delta,$$

which yield that $|\nu(a_{ni}) - \nu(a_{0i})| < \delta$. Since $\mu \ll \nu$, we have $|\mu(a_{ni}) - \mu(a_{0i})| < \varepsilon/2$. Therefore, we get (for $i \geq k$)

$$\widetilde{\mu}(a_n) < \mu(a_{ni}) + \varepsilon/2 < \mu(a_{0i}) + \varepsilon \leq \widetilde{\mu}(a_0) + \varepsilon, \text{ for all } n \geq m.$$

Since $a_0 \leq a_n$ and $\widetilde{\mu}$ is monotone, the result follows. Using similar argument we have $\widetilde{\mu} \ll \nu$ on M_σ. The uniqueness is proved in Theorem 3.1. □

Now to extend a measure from a sublattice M (containing 1, the largest element of L) to a σ-complete sublattice N, containing M, of a locally complete σ-continuous lattice L, we introduce a new concept of M_σ-approachability of a measure on N.

Definition 3.4. *A measure μ on N is said to be M_σ-approachable if for $a \in N$ and for $\varepsilon > 0$ there exists $b \in M_\sigma$ such that $a \leq b$ and $\mu(b) < \mu(a) + \varepsilon$.*

Theorem 3.3. *A measure μ on M can be extended to an M_σ-approachable measure on N, provided there exists an M_σ-approachable measure ν on N such that $\mu \ll \nu$ on M. The extension is unique and it preserves the absolute continuity with respect to ν.*

Proof. From Theorem 3.2, we have $\widetilde{\mu} \ll \nu$ on M_σ and the extension $\widetilde{\mu}$ is unique. We define, for $a \in N$,

$$\widehat{\mu}(a) = \inf\{\widetilde{\mu}(b) : a \leq b, b \in M_\sigma\}.$$

Clearly, $\widetilde{\mu}(a) = \widehat{\mu}(a)$ for $a \in M_\sigma$, $\widehat{\mu}(0) = 0$ if $0 \in M$, and $\widehat{\mu}$ is monotone.

Now, to prove $\widehat{\mu}$ is semi-continuous from below, suppose that $\{a_n\}$ is a sequence N such that $a_n \uparrow a_0$, $a_0 \in N$. Let $\varepsilon > 0$. Since $\widetilde{\mu} \ll \nu$ on M_σ, there exists $\delta > 0$ such that $|\widetilde{\mu}(c) - \widetilde{\mu}(b)| < \varepsilon/2$ whenever $b, c \in M_\sigma$ and $|\nu(c) - \nu(b)| < \delta$. Since ν is a measure on N, there exists $m \in \mathbb{N}$ such that

$$\nu(a_0) < \nu(a_n) + \delta/2, \text{ for all } n \geq m,$$

and consequently, we obtain $b_m \in M_\sigma$ with $a_m \leq b_m$ and $\widetilde{\mu}(b_m) < \widehat{\mu}(a_m) + \varepsilon/2$. Since ν is M_σ-approachable on N, we get $b_0 \in M_\sigma$ with $a_0 \leq b_0$ such that $\nu(b_0) < \nu(a_0) + \delta/2$. Since M_σ is closed under the formation of finite meet, we may assume that $b_m \leq b_0$ (replace b_m by $(b_m \wedge b_0)$). Thus

$$\nu(b_0) < \nu(a_m) + \delta \leq \nu(b_m) + \delta,$$

which yields $|\nu(b_0) - \nu(b_m)| < \delta$. Since $\widetilde{\mu} \ll \nu$ on M_σ, we have $|\widetilde{\mu}(b_0) - \widetilde{\mu}(b_m)| < \varepsilon/2$. Now, we have

$$\widehat{\mu}(a_0) \leq \widehat{\mu}(b_0) = \widetilde{\mu}(b_0) < \widetilde{\mu}(b_m) + \varepsilon/2 < \widehat{\mu}(a_m) + \varepsilon.$$

Since $a_n \leq a_0$ and $\widehat{\mu}$ is monotone, it follows that $\widehat{\mu}$ is semi-continuous from below. Similarly we may prove that $\widehat{\mu}$ is semi-continuous from above, and also that $\widehat{\mu} \ll \nu$ on N, and $\widehat{\mu}$ is the unique extension. That $\widehat{\mu}$ is M_σ-approachable follows from its definition.

Proposition 3.2. *A function $\mu : M \to [0, \infty)$ is modular if and only if*

$$\mu(a_1) + \mu(b_1) = \mu(a_2) + \mu(b_2) \tag{1}$$

where $a_1, b_1, a_2, b_2 \in M$ with $a_1 \wedge b_1 = a_2 \wedge b_2$ and $a_1 \vee b_1 = a_2 \vee b_2$.

Proof. Let μ be modular. For $a_1, b_1, a_2, b_2 \in M$ such that $a_1 \wedge b_1 = a_2 \wedge b_2$ and $a_1 \vee b_1 = a_2 \vee b_2$, we have $\mu(a_1 \vee b_1) = \mu(a_2 \vee b_2)$, and so,

$$\mu(a_1) + \mu(b_1) - \mu(a_1 \wedge b_1) = \mu(a_2) + \mu(b_2) - \mu(a_2 \wedge b_2).$$

Since $\mu(a_1 \wedge b_1) = \mu(a_2 \wedge b_2)$ we have equation (1). Conversely, let (1) hold. Let $a, b \in M$. Since $((a \vee b) \vee (a \wedge b)) = a \vee b$ and $((a \vee b) \wedge (a \wedge b)) = a \wedge b$. Then from (1), we have

$$\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b).$$

Hence μ is modular.

Theorem 3.4. *Let μ be an lsc-measure on M (containing 1). Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*

(i) μ is submodular.
(ii) $\widetilde{\mu}$ is submodular.
(iii) $\widehat{\mu}$ is submodular.

Proof. (i)⇒(ii). Let μ be submodular. Let $a, b \in M_\sigma$. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ in M such that $a_n \uparrow a, b_n \uparrow b$, $\widetilde{\mu}(a) = \lim_{n\to\infty} \mu(a_n)$ and $\widetilde{\mu}(b) = \lim_{n\to\infty} \mu(b_n)$. Since L is σ-continuous, so $a_n \wedge b_n \uparrow a \wedge b$ and also we have $a_n \vee b_n \uparrow a \vee b$. Therefore

$$\begin{aligned}\widetilde{\mu}(a) + \widetilde{\mu}(b) &= \lim_{n\to\infty} \mu(a_n) + \lim_{n\to\infty} \mu(b_n)\\ &= \lim_{n\to\infty} (\mu(a_n) + \mu(b_n))\\ &\geq \lim_{n\to\infty} (\mu(a_n \vee b_n) + \mu(a_n \wedge b_n))\\ &= \lim_{n\to\infty} \mu(a_n \vee b_n) + \lim_{n\to\infty} \mu(a_n \wedge b_n)\\ &= \widetilde{\mu}(a \vee b) + \widetilde{\mu}(a \wedge b).\end{aligned}$$

(ii)⇒(iii). Let $a, b \in N$. For $\varepsilon > 0$, we have $c, d \in M_\sigma$ such that $a \leq c, b \leq d$, $\widehat{\mu}(a) > \widetilde{\mu}(c) - \varepsilon/2$ and $\widehat{\mu}(b) > \widetilde{\mu}(d) - \varepsilon/2$. Thus,

$$\begin{aligned}\widehat{\mu}(a) + \widehat{\mu}(b) + \varepsilon &> \widetilde{\mu}(c) + \widetilde{\mu}(d)\\ &\geq \widetilde{\mu}(c \vee d) + \widetilde{\mu}(c \wedge d)\\ &\geq \widehat{\mu}(a \vee b) + \widehat{\mu}(a \wedge b).\end{aligned}$$

Since ε is arbitrary, we have

$$\widehat{\mu}(a) + \widehat{\mu}(b) \geq \widehat{\mu}(a \vee b) + \widehat{\mu}(a \wedge b). \qquad \square$$

Concluding Remark

If we take M to be a sublattice of L (L being locally complete σ-continuous lattice) and consider a nonnegative extended real-valued μ, then Lemma 3.1, Theorem 3.1, Theorem 3.2 remain valid provided ν is a finite measure (obviously then $\mu(a_1) < \infty$ would be needed in Definition 3.1(iv)). However, for the extension of μ on M to N we need to take the top element 1 in M. In view of Theorem 3.4, if μ is submodular then $\widetilde{\mu}$ is submodular.

Since $(\mathfrak{P}(X), \subseteq)$, $(\mathbb{I}, \leq)$ (where $\mathbb{I}$ is the closed unit interval [0, 1] of the real line $\mathbb{R}$), and $(L^X, \leq)$ (provided L is a locally complete σ-continuous lattice) are locally complete σ-continuous lattices, the present study provides a unified approach for classical theory, popular fuzzy theory and theory of L-fuzzy sets.

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