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ON $4n$-DIMENSIONAL LIE GROUPS AS QUASI-KÄHLER MANIFOLDS WITH KILLING NORDEN METRIC[1]

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Abstract. A $4n$-parametric family of $4n$-dimensional quasi-Kähler manifolds with Killing Norden metric is constructed on a Lie group. This family is characterized geometrically.

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1. Introduction

It is a fundamental fact that on an almost complex manifold with Hermitian metric (almost Hermitian manifold), the action of the almost complex structure on the tangent space at each point of the manifold is isometry. There is another type of metric, called Norden metric or B-metric on an almost complex manifold, such as the action of the almost complex structure is anti-isometry with respect to the metric. Such a manifold is called an almost complex manifold with Norden metric [1] or with B-metric [2]. See also [6] for generalized B-manifolds. It is known [1] that these manifolds are classified into three basic classes, $\mathcal{W}_i$ $(i = 1, 2, 3)$, which give rise to eight classes in all.

Among the basic three classes of this classification, the almost complex structure is nonintegrable only in the class $\mathcal{W}_3$. This is the class of the so-called *quasi-Kähler manifolds with Norden metric*, which we call briefly $\mathcal{W}_3$*-manifolds*. We studied the geometry of manifolds belonging to this class in [5], [7], [8], [9], [10].

The purpose of the present paper is to show, by construction, almost complex structures with Norden metric on Lie groups as $4n$-manifolds, which are of the class $\mathcal{W}_3$. This $4n$-parametric family of manifolds is characterized geometrically.

The case of the initial dimension 4 is considered in [5] and [10].

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2. Almost complex manifolds with Norden metric

Let (M, J, g) be a $2n$-dimensional almost complex manifold with Norden metric, i. e. J is an almost complex structure and g is a metric on M such that

$$J^2X = -X, \qquad g(JX, JY) = -g(X, Y) \tag{2.1}$$

for all differentiable vector fields X, Y on M, i. e. $X, Y \in \mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of g on M given by $\tilde{g}(X, Y) = g(X, JY)$ for all $X, Y \in \mathfrak{X}(M)$ is a Norden metric, too. Both metrics are necessarily of signature (n, n). The manifold $(M, J, \tilde{g})$ is an almost complex manifold with Norden metric, too.

Further, X, Y, Z, U (x, y, z, u, respectively) will stand for arbitrary differentiable vector fields on M (vectors in T_pM, $p \in M$, respectively).

The Levi-Civita connection of g is denoted by ∇. The tensor filed F of type $(0, 3)$ on M is defined by

$$F(X, Y, Z) = g\big((\nabla_X J)\, Y, Z\big). \tag{2.2}$$

It has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ). \tag{2.3}$$

Further, let $\{e_i\}$ $(i = 1, 2, \dots, 2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$.

The eight classes of almost complex manifolds with Norden metric are determined in [1] according to the properties of F. The three basic classes are given as follows:

$$\begin{aligned}
\mathcal{W}_1\colon\ & F(x, y, z) = \tfrac{1}{4n}\,\{g(x, y)\theta(z) + g(x, z)\theta(y) \\
&\qquad\qquad + g(x, Jy)\theta(Jz) + g(x, Jz)\theta(Jy)\}\,; \\
\mathcal{W}_2\colon\ & \mathop{\mathfrak{S}}_{x,y,z} F(x, y, Jz) = 0, \quad \theta = 0; \\
\mathcal{W}_3\colon\ & \mathop{\mathfrak{S}}_{x,y,z} F(x, y, z) = 0,
\end{aligned}$$

where $\mathfrak{S}$ is the cyclic sum over three arguments and $\theta(z) = g^{ij}F(e_i, e_j, z)$. The special class $\mathcal{W}_0$ of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition $F = 0$.

The curvature tensor field R of ∇ is $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, and the corresponding tensor field of type $(0, 4)$ is determined by $R(X, Y, Z, U) = g(R(X, Y)Z, U)$. The Ricci tensor ρ and the scalar curvature τ are defined as usual by

$$\rho(y, z) = g^{ij}R(e_i, y, z, e_j), \qquad \tau = g^{ij}\rho(e_i, e_j). \tag{2.4}$$

It is well known that the Weyl tensor W on an m-dimensional pseudo-Riemannian manifold $(m \geq 3)$ is given by

$$W = R - \frac{1}{m-2}\Big(\psi_1(\rho) - \frac{\tau}{m-1}\pi_1\Big), \tag{2.5}$$

where

$$\begin{aligned}\psi_1(\rho)(x,y,z,u) &= g(y,z)\rho(x,u) - g(x,z)\rho(y,u)\\ &\quad + \rho(y,z)g(x,u) - \rho(x,z)g(y,u);\\ \pi_1 = \tfrac{1}{2}\psi_1(g) &= g(y,z)g(x,u) - g(x,z)g(y,u).\end{aligned}$$

Moreover, for $m \geq 4$ the Weyl tensor W is zero if and only if the manifold is conformally flat.

Let $\alpha = \{x, y\}$ be a non-degenerate 2-plane spanned by vectors $x, y \in T_pM$, $p \in M$. It means that $\pi_1(x,y,y,x) = g(x,x)g(y,y) - g(x,y)^2 \neq 0)$. Then, as is known, the sectional curvature of α is defined by the following equation

$$k(\alpha) = k(x,y) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)}. \tag{2.6}$$

The basic sectional curvatures in T_pM with an almost complex structure and a Norden metric g are:

- *holomorphic sectional curvatures* if $J\alpha = \alpha$;
- *totally real sectional curvatures* if $J\alpha \perp \alpha$ with respect to g.

In [4], a *holomorphic bisectional curvature* $h(x, y)$ for a pair of holomorphic 2-planes $\alpha_1 = \{x, Jx\}$ and $\alpha_2 = \{y, Jy\}$ is defined by

$$h(x,y) = -\frac{R(x,Jx,y,Jy)}{\sqrt{\pi_1(x,Jx,x,Jx)\pi_1(y,Jy,y,Jy)}}, \tag{2.7}$$

where x, y do not lie along the totally isotropic directions, i. e. both of the couples $\big(g(x,x), g(x,Jx)\big)$ and $\big(g(y,y), g(y,Jy)\big)$ are different from the couple $(0,0)$. The holomorphic bisectional curvature is invariant with respect to the basis of the 2-planes α_1 and α_2. In particular, if $\alpha_1 = \alpha_2$, then the holomorphic bisectional curvature coincides with the holomorphic sectional curvature of the 2-plane $\alpha_1 = \alpha_2$.

The square norm $\|\nabla J\|^2$ of ∇J is defined in [3] by

$$\|\nabla J\|^2 = g^{ij} g^{kl} g\big((\nabla_{e_i} J)\, e_k, \big(\nabla_{e_j} J\big)\, e_l\big).$$

Having in mind the definition (2.2) of the tensor F and the properties (2.3), we obtain the following equation for the square norm of ∇J

$$\|\nabla J\|^2 = g^{ij} g^{kl} g^{pq} F_{ikp} F_{jlq}, \tag{2.8}$$

where $F_{ikp} = F(e_i, e_k, e_p)$.

An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = 0$ is called *isotropic Kähler manifold with Norden metric* [9]. It is clear, if a manifold belongs to the class $\mathcal{W}_0$, then it is isotropic Kählerian, but the inverse statement is not always true.

3. A Lie group as a $4n$-dimensional $\mathcal{W}_3$-manifold

Let V be a $4n$-dimensional vector space and let us consider the structure of the Lie algebra $\mathfrak{g}$ defined by the brackets $[E_i, E_j] = C_{ij}^k E_k$, where $\{E_1, E_2, \dots, E_{4n}\}$ is a basis of V and $C_{ij}^k \in \mathbb{R}$.

Let G be the associated connected Lie group and $\{X_1, X_2, \dots, X_{4n}\}$ be a global basis of left invariant vector fields. Then the Jacobi identity holds:

$$\underset{X_i,X_j,X_k}{\mathfrak{S}} \big[[X_i, X_j], X_k\big] = 0. \tag{3.1}$$

Next we define an almost complex structure J by the conditions

$$\begin{array}{ll} JX_{4\alpha-3} = X_{4\alpha-1}, & JX_{4\alpha-2} = X_{4\alpha}, \\ JX_{4\alpha-1} = -X_{4\alpha-3}, & JX_{4\alpha} = -X_{4\alpha-2}, \end{array} \tag{3.2}$$

where $\alpha \in \{1, 2, \dots, n\}$.

Let us consider the left invariant metric g defined in the following way

$$\begin{aligned} g(X_{4\alpha-3}, X_{4\alpha-3}) = g(X_{4\alpha-2}, X_{4\alpha-2}) &= -g(X_{4\alpha-1}, X_{4\alpha-1}) \\ &= -g(X_{4\alpha}, X_{4\alpha}) = 1, \\ g(X_i, X_j) = 0 \quad \text{for} \quad i \neq j. & \end{aligned} \tag{3.3}$$

The introduced metric is a Norden metric because of (3.2).

In this way, the induced $4n$-dimensional manifold (G, J, g) is an almost complex manifold with Norden metric, in short *almost Norden manifold*.

From this point on, until the end of this paper, we shall consider almost Norden manifolds (G, J, g) with *Killing metric* g. This means that g satisfies the following condition for arbitrary $X, Y, Z \in \mathfrak{g}$

$$g\left([X, Y], Z\right) + g\left([X, Z], Y\right) = 0.$$

In [7] is shown that each almost Norden manifold with Killing metric is a locally symmetric $\mathcal{W}_3$-manifold. Moreover, the following formulae are valid:

$$\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j], \tag{3.4}$$

$$F(X_i, X_j, X_k) = \frac{1}{2}\Big\{g\big([X_i, JX_j], X_k\big) - g\big([X_i, X_j], JX_k\big)\Big\}, \tag{3.5}$$

$$R(X_i, X_j, X_k, X_l) = -\frac{1}{4} g\Big([X_i, X_j], [X_k, X_l]\Big), \tag{3.6}$$

$$[X_i, X_j] \perp \operatorname{span}\{X_i, X_j\}, \tag{3.7}$$

where $i, j, k, l \in \{1, 2, \dots, 4n\}$.

Since g is a Killing metric, the structural constants C_{ij}^k are specialized so that the commutators have the following decompositions:

$$
\begin{aligned}
&[X_{4\alpha-3}, X_{4\alpha-1}] = \lambda_{4\alpha-2}X_{4\alpha-2} + \lambda_{4\alpha}X_{4\alpha},\\
&[X_{4\alpha-2}, X_{4\alpha}] = \lambda_{4\alpha-3}X_{4\alpha-3} + \lambda_{4\alpha-1}X_{4\alpha-1},\\
&[X_{4\alpha-2}, X_{4\alpha-1}] = -\lambda_{4\alpha-2}X_{4\alpha-3} - \lambda_{4\alpha-1}X_{4\alpha},\\
&[X_{4\alpha-1}, X_{4\alpha}] = -\lambda_{4\alpha}X_{4\alpha-3} + \lambda_{4\alpha-1}X_{4\alpha-2},\\
&[X_{4\alpha}, X_{4\alpha-3}] = \lambda_{4\alpha-3}X_{4\alpha-2} + \lambda_{4\alpha}X_{4\alpha-1},\\
&[X_{4\alpha-2}, X_{4\alpha-3}] = -\lambda_{4\alpha-2}X_{4\alpha-1} + \lambda_{4\alpha-3}X_{4\alpha},
\end{aligned}
\tag{3.8}
$$

where $\lambda_{4\alpha-i} \in \mathbb{R}$ $(i = 1, 2, 3, 4; \alpha = 1, 2, \ldots, n)$. The other commutators are zero.

By direct verification we prove that the commutators from (3.8) satisfy the Jacobi identity (3.1). The Lie groups G thus obtained are a family which is characterized by $4n$ real parameters $\lambda_{4\alpha-i}$.

Vice versa, let the condition (3.8) is valid for an almost Norden manifold (G, J, g) with a structure J and a metric g determined by (3.2) and (3.3), respectively. Then we verify directly that g is a Killing metric, i.e. (G, J, g) is locally symmetric $\mathcal{W}_3$-manifold.

Therefore we establish the truthfulness of the following theorem.

Theorem 3.1. *Let (G, J, g) be a $4n$-dimensional almost Norden manifold, where G is a connected Lie group with corresponding Lie algebra $\mathfrak{g}$ determined by the global basis of left invariant vector fields $\{X_1, X_2, \ldots, X_{4n}\}$; J is an almost complex structure defined by* (3.2) *and g is a Norden metric determined by* (3.3)*. Then (G, J, g) is a $\mathcal{W}_3$-manifold with Killing metric g if and only if G belongs to the $4n$-parametric family of Lie groups determined by the conditions* (3.8)*.*

4. Geometric characteristics of the constructed manifold

Let (G, J, g) be the $4n$-dimensional quasi-Kähler manifold with Norden metric introduced in the previous section. Let us introduce the following index denotations: $\bar{1} = 4\alpha - 3$, $\bar{2} = 4\alpha - 2$, $\bar{3} = 4\alpha - 1$, $\bar{4} = 4\alpha$ for any fixed $\alpha \in \{1, 2, \ldots, n\}$.

Having in mind (3.5), (3.2), (3.3) and (3.8), we obtain immediately the nonzero components of the tensor F as follows:

$$
\begin{aligned}
-F_{\bar{1}\bar{2}\bar{2}} = -F_{\bar{1}\bar{4}\bar{4}} = 2F_{\bar{2}\bar{1}\bar{2}} = 2F_{\bar{2}\bar{3}\bar{4}} = 2F_{\bar{4}\bar{1}\bar{4}} = -2F_{\bar{4}\bar{2}\bar{3}} = \lambda_{\bar{1}},\\
2F_{\bar{1}\bar{1}\bar{2}} = 2F_{\bar{1}\bar{3}\bar{4}} = -2F_{\bar{2}\bar{1}\bar{1}} = -2F_{\bar{2}\bar{3}\bar{3}} = -2F_{\bar{3}\bar{1}\bar{4}} = 2F_{\bar{3}\bar{2}\bar{3}} = \lambda_{\bar{2}},\\
2F_{\bar{2}\bar{1}\bar{4}} = -2F_{\bar{2}\bar{2}\bar{3}} = F_{\bar{3}\bar{2}\bar{2}} = F_{\bar{3}\bar{4}\bar{4}} = -2F_{\bar{4}\bar{1}\bar{2}} = -2F_{\bar{4}\bar{3}\bar{4}} = \lambda_{\bar{3}},\\
-2F_{\bar{1}\bar{1}\bar{4}} = 2F_{\bar{1}\bar{2}\bar{3}} = -2F_{\bar{3}\bar{1}\bar{2}} = -2F_{\bar{3}\bar{3}\bar{4}} = F_{\bar{4}\bar{1}\bar{1}} = F_{\bar{4}\bar{3}\bar{3}} = \lambda_{\bar{4}}.
\end{aligned}
\tag{4.1}
$$

The other nonzero components are obtained from the property $F_{\bar{i}\bar{j}\bar{k}} = F_{\bar{i}\bar{k}\bar{j}}$.

Let N be the Nijenhuis tensor of the almost complex structure J on G, i.e.

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \quad X,Y \in \mathfrak{g}.$$

Having in mind (3.8) and (3.2) we obtain the nonzero components $N_{\bar{i}\bar{j}} = N(X_{\bar{i}}, X_{\bar{j}})$ as follows

$$\begin{aligned} N_{\bar{1}\bar{2}} &= -N_{\bar{3}\bar{4}} = 2\left(\lambda_{\bar{4}}X_{\bar{1}} - \lambda_{\bar{3}}X_{\bar{2}} + \lambda_{\bar{2}}X_{\bar{3}} - \lambda_{\bar{1}}X_{\bar{4}}\right), \\ N_{\bar{1}\bar{4}} &= -N_{\bar{2}\bar{3}} = 2\left(\lambda_{\bar{2}}X_{\bar{1}} - \lambda_{\bar{1}}X_{\bar{2}} - \lambda_{\bar{4}}X_{\bar{3}} + \lambda_{\bar{3}}X_{\bar{4}}\right). \end{aligned} \tag{4.2}$$

The other nonzero components are obtained from the property $N_{\bar{i}\bar{j}} = -N_{\bar{j}\bar{i}}$. Hence its square norm $\|N\|^2 = g^{ik}g^{ks}g(N_{ij}, N_{ks})$ for $i,j,k,s \in \{1,2,\dots,4n\}$ has the form

$$\|N\|^2 = -32\sum_{\alpha=1}^{n}\left(\lambda^2_{4\alpha-3} + \lambda^2_{4\alpha-2} - \lambda^2_{4\alpha-1} - \lambda^2_{4\alpha}\right), \tag{4.3}$$

where the inverse matrix of g has the form

$$\left(g^{ij}\right) = \begin{pmatrix} \tilde{E}_4 & 0 & \dots & 0 \\ 0 & \tilde{E}_4 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \tilde{E}_4 \end{pmatrix}, \quad \tilde{E}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.4}$$

According to (3.2), (3.3), (3.8) and (4.4), from (2.8) we obtain the square norm of ∇J as

$$\|\nabla J\|^2 = 4\sum_{\alpha=1}^{n}\left(\lambda^2_{4\alpha-3} + \lambda^2_{4\alpha-2} - \lambda^2_{4\alpha-1} - \lambda^2_{4\alpha}\right). \tag{4.5}$$

From (3.6) and (3.8) we get the nonzero components of R as follows

$$\begin{aligned} R_{\bar{1}\bar{2}\bar{2}\bar{1}} &= -\tfrac{1}{4}\left(\lambda^2_{\bar{1}} + \lambda^2_{\bar{2}}\right), & R_{\bar{1}\bar{3}\bar{3}\bar{1}} &= \tfrac{1}{4}\left(\lambda^2_{\bar{2}} - \lambda^2_{\bar{4}}\right), \\ R_{\bar{1}\bar{4}\bar{4}\bar{1}} &= -\tfrac{1}{4}\left(\lambda^2_{\bar{1}} - \lambda^2_{\bar{4}}\right), & R_{\bar{2}\bar{3}\bar{3}\bar{2}} &= \tfrac{1}{4}\left(\lambda^2_{\bar{2}} - \lambda^2_{\bar{3}}\right), \\ R_{\bar{2}\bar{4}\bar{4}\bar{2}} &= \tfrac{1}{4}\left(\lambda^2_{\bar{1}} - \lambda^2_{\bar{3}}\right), & R_{\bar{3}\bar{4}\bar{4}\bar{3}} &= \tfrac{1}{4}\left(\lambda^2_{\bar{3}} + \lambda^2_{\bar{4}}\right), \\ R_{\bar{1}\bar{3}\bar{4}\bar{1}} &= R_{\bar{2}\bar{3}\bar{4}\bar{2}} = -\tfrac{1}{4}\lambda_{\bar{1}}\lambda_{\bar{2}}, & R_{\bar{2}\bar{1}\bar{3}\bar{2}} &= -R_{\bar{4}\bar{1}\bar{3}\bar{4}} = \tfrac{1}{4}\lambda_{\bar{1}}\lambda_{\bar{3}}, \\ R_{\bar{1}\bar{2}\bar{3}\bar{1}} &= -R_{\bar{4}\bar{2}\bar{3}\bar{4}} = \tfrac{1}{4}\lambda_{\bar{1}}\lambda_{\bar{4}}, & R_{\bar{2}\bar{1}\bar{4}\bar{2}} &= -R_{\bar{3}\bar{1}\bar{4}\bar{3}} = \tfrac{1}{4}\lambda_{\bar{2}}\lambda_{\bar{3}}, \\ R_{\bar{1}\bar{2}\bar{4}\bar{1}} &= -R_{\bar{3}\bar{2}\bar{4}\bar{3}} = \tfrac{1}{4}\lambda_{\bar{2}}\lambda_{\bar{4}}, & R_{\bar{3}\bar{1}\bar{2}\bar{3}} &= R_{\bar{4}\bar{1}\bar{2}\bar{4}} = \tfrac{1}{4}\lambda_{\bar{3}}\lambda_{\bar{4}}. \end{aligned} \tag{4.6}$$

The other nonzero components of R are obtained from the properties $R_{\bar{i}\bar{j}\bar{k}\bar{s}} = R_{\bar{k}\bar{s}\bar{i}\bar{j}}$ and $R_{\bar{i}\bar{j}\bar{k}\bar{s}} = -R_{\bar{j}\bar{i}\bar{k}\bar{s}} = -R_{\bar{i}\bar{j}\bar{s}\bar{k}}$.

Having in mind (2.4), (4.4) and (4.6), we obtain the components $\rho_{\bar{i}\bar{j}} = \rho(X_{\bar{i}}, X_{\bar{j}})$ of the Ricci tensor ρ and the scalar curvature τ as follows:

$$
\begin{aligned}
&\rho_{\bar{1}\bar{1}} = -\tfrac{1}{2}\left(\lambda_{\bar{1}}^2 + \lambda_{\bar{2}}^2 - \lambda_{\bar{4}}^2\right), && \rho_{\bar{2}\bar{2}} = -\tfrac{1}{2}\left(\lambda_{\bar{1}}^2 + \lambda_{\bar{2}}^2 - \lambda_{\bar{3}}^2\right),\\
&\rho_{\bar{3}\bar{3}} = \tfrac{1}{2}\left(\lambda_{\bar{2}}^2 - \lambda_{\bar{3}}^2 - \lambda_{\bar{4}}^2\right), && \rho_{\bar{4}\bar{4}} = \tfrac{1}{2}\left(\lambda_{\bar{1}}^2 - \lambda_{\bar{3}}^2 - \lambda_{\bar{4}}^2\right),\\
&\rho_{\bar{1}\bar{2}} = \rho_{\bar{2}\bar{1}} = -\tfrac{1}{2}\lambda_{\bar{3}}\lambda_{\bar{4}}, && \rho_{\bar{1}\bar{3}} = \rho_{\bar{3}\bar{1}} = \tfrac{1}{2}\lambda_{\bar{1}}\lambda_{\bar{3}},\\
&\rho_{\bar{1}\bar{4}} = \rho_{\bar{4}\bar{1}} = \tfrac{1}{2}\lambda_{\bar{2}}\lambda_{\bar{3}}, && \rho_{\bar{2}\bar{3}} = \rho_{\bar{3}\bar{2}} = \tfrac{1}{2}\lambda_{\bar{1}}\lambda_{\bar{4}},\\
&\rho_{\bar{2}\bar{4}} = \rho_{\bar{4}\bar{2}} = \tfrac{1}{2}\lambda_{\bar{2}}\lambda_{\bar{4}}, && \rho_{\bar{3}\bar{4}} = \rho_{\bar{4}\bar{3}} = -\tfrac{1}{2}\lambda_{\bar{1}}\lambda_{\bar{2}};
\end{aligned} \tag{4.7}
$$

$$
\tau = -\frac{3}{2}\sum_{\alpha=1}^{n}\left(\lambda_{4\alpha-3}^2 + \lambda_{4\alpha-2}^2 - \lambda_{4\alpha-1}^2 - \lambda_{4\alpha}^2\right). \tag{4.8}
$$

Taking into account (3.3), (4.6), (4.7), (4.8) and (2.5) for $m = 4n$, we establish that the Weyl tensor vanishes. Then (G, J, g) is a conformally flat manifold.

For the sectional curvatures $k_{\bar{i}\bar{j}} = k(\alpha_{\bar{i}\bar{j}})$ of the basic 2-planes $\alpha_{\alpha_{\bar{i}\bar{j}}} = \{X_{\bar{i}}, X_{\bar{j}}\}$, according to (2.6), (3.3) and (4.6), we have:

$$
\begin{aligned}
&k_{\bar{1}\bar{3}} = -\tfrac{1}{4}\left(\lambda_{\bar{2}}^2 - \lambda_{\bar{4}}^2\right), && k_{\bar{2}\bar{4}} = -\tfrac{1}{4}\left(\lambda_{\bar{1}}^2 - \lambda_{\bar{3}}^2\right),\\
&k_{\bar{1}\bar{2}} = -\tfrac{1}{4}\left(\lambda_{\bar{1}}^2 + \lambda_{\bar{2}}^2\right), && k_{\bar{1}\bar{4}} = -\tfrac{1}{4}\left(\lambda_{\bar{1}}^2 - \lambda_{\bar{4}}^2\right),\\
&k_{\bar{2}\bar{3}} = -\tfrac{1}{4}\left(\lambda_{\bar{2}}^2 - \lambda_{\bar{3}}^2\right), && k_{\bar{3}\bar{4}} = \tfrac{1}{4}\left(\lambda_{\bar{3}}^2 + \lambda_{\bar{4}}^2\right).
\end{aligned} \tag{4.9}
$$

The obtained geometric characteristics of the considered manifold we generalize in the following theorem.

Theorem 4.1. *Let (G, J, g) be a $4n$-dimensional almost Norden manifold, where G is a connected Lie group with corresponding Lie algebra $\mathfrak{g}$ determined by the global basis of left invariant vector fields $\{X_1, X_2, \dots, X_{4n}\}$; J is an almost complex structure defined by* (3.2) *and g is a Norden metric determined by* (3.3)*. Then*

(i) (G, J, g) is a locally symmetric conformally flat $\mathcal{W}_3$-manifold with Killing metric g;

(ii) The nonzero components of the basic tensor F, the Nijenhuis tensor N, the curvature tensor R and the Ricci tensor ρ are (4.1), (4.2), (4.6) *and* (4.7)*, respectively;*

(iii) The square norms of the Nijenhuis tensor N and ∇J are (4.3) *and* (4.5)*, respectively;*

(iv) The scalar curvature τ and the sectional curvatures $k_{\bar{i}\bar{j}}$ of the basic 2-planes are (4.8) *and* (4.9)*, respectively.*

The last theorem implies immediately the following corollary.

Corollary 4.2. *Let (G, J, g) be a $4n$-dimensional almost Norden manifold, where G is a connected Lie group with corresponding Lie algebra $\mathfrak{g}$ determined by the global basis of left invariant vector fields $\{X_1, X_2, \dots, X_{4n}\}$; J is an almost complex structure defined by* (3.2) *and g is a Norden metric determined by* (3.3)*. Then the following propositions are equivalent:*

(i) (G, J, g) *is an isotropic Kähler manifold;*

(ii) (G, J, g) *is a scalar flat manifold;*

(iii) The Nijenhuis tensor is isotropic;

(iv) The condition $\sum_{\alpha=1}^{n} \left(\lambda_{4\alpha-3}^2 + \lambda_{4\alpha-2}^2 - \lambda_{4\alpha-1}^2 - \lambda_{4\alpha}^2\right) = 0$ *holds.*

The condition (iv) of the last theorem means that the set of vectors with the coordinates $(\lambda_1, \lambda_2, \dots, \lambda_{4n})$ at an arbitrary point $p \in G$ describes the isotropic cone in T_pG with respect to the Norden metric g.

Let us remark that the 2-planes $\alpha_{\bar{1}\bar{3}}$ and $\alpha_{\bar{2}\bar{4}}$ are holomorphic 2-planes and the 2-planes $\alpha_{\bar{1}\bar{2}}$, $\alpha_{\bar{1}\bar{4}}$, $\alpha_{\bar{2}\bar{3}}$, $\alpha_{\bar{3}\bar{4}}$ are totally real 2-planes. Taking into account (2.7), (3.3) and (4.6), we obtain that the holomorphic bisectional curvature of the unique pair of basis holomorphic 2-planes $\{\alpha_{\bar{1}\bar{3}}, \alpha_{\bar{2}\bar{4}}\}$ vanishes. Moreover, the equalities (4.9) imply the following

Theorem 4.3. *Let (G, J, g) be a $4n$-dimensional almost Norden manifold, where G is a connected Lie group with corresponding Lie algebra $\mathfrak{g}$ determined by the global basis of left invariant vector fields $\{X_1, X_2, \dots, X_{4n}\}$; J is an almost complex structure defined by* (3.2) *and g is a Norden metric determined by* (3.3)*. Then*

(i) (G, J, g) *is of constant holomorphic sectional curvatures iff*

$$\lambda_{\bar{1}}^2 + \lambda_{\bar{4}}^2 = \lambda_{\bar{2}}^2 + \lambda_{\bar{3}}^2;$$

(ii) (G, J, g) *does not admit constant totally real sectional curvatures.*

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