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CHARACTERIZATION OF A SLANT SUBMANIFOLD OF A KENMOTSU MANIFOLD

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Abstract. Recently, we have studied and obtained some results on slant submanifolds of Kenmotsu manifolds [9, 10]. In this paper we study the slant submanifold of a Kenmotsu manifold when structure tensor field φ is Killing and obtain some interesting results.

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1. Introduction

The notion of slant submanifolds of an almost Hermitian manifold were introduced by Chen [5, 6]. Examples of slant submanifolds of C^2 and C^4 were given by Chen and Tazawa [6, 7, 8]. It was A. Lotta [12] who has defined and studied slant submanifolds of an almost contact metric manifold. Moreover, he has also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-Contact manifolds [13]. Latter, L. Cabrerizo and others investigated slant submanifolds of a Sasakian manifolds and obtained many interesting results [3, 4]. Recently, we have studied the structure on a slant submanifold of a Kenmotsu manifold [9].

2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors $\{\varphi, \xi, \eta, g\}$, where φ is a (1,1) tensor field, ξ a vector field, η a 1-form and g the Riemannian metric on $\overline{M}$. These tensors satisfy [1]

$$\begin{aligned}(2.1)\qquad &\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0 \quad \text{and}\\ &g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi)\end{aligned}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$. An almost contact metric manifold is called a Kenmotsu manifold if [11]

$$(2.2)\qquad (\overline{\nabla}_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X \text{ and } \overline{\nabla}_X \xi = X - \eta(X)\xi$$

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where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We denote by TM the Lie algebra of vector fields on M and by $T^{\perp}M$ the set of all vector fields normal to M. For any $X \in TM$ and $N \in T^{\perp}M$, we write

$$\varphi X = PX + FX, \quad \varphi N = tN + fN \tag{2.3}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN.

In what follows, we suppose that the structure vector field ξ is tangent to M. Hence if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \xi$.

For each non-zero X tangent to M at x such that X is not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and T_xM. The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$ [1] . The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.4}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.5}$$

for $X, Y \in TM$ and $N \in T^{\perp}M$ of $\overline{M}$; h and A_N are the second fundamental forms related by

$$g(A_N X, Y) = g(h(X,Y), N) \tag{2.6}$$

and $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$ of M.

The mean curvature vector H is defined by $H = (\frac{1}{m})\, trace\ h$. We say that M is minimal if H vanishes identically.

If P is the endomorphism defined by (2.3), then

$$g(PX,Y) + g(X,PY) = 0 \tag{2.7}$$

Thus P^2, which is denoted by Q, is self-adjoint.

On the other hand, Gauss and Weingarten formulae, together with (2.2) and (2.3) imply

$$(\nabla_X P)Y = A_{FY}X + th(X,Y) + g(Y,PX)\xi - \eta(Y)PX \tag{2.8}$$

$$(\nabla_X F)Y = fh(X,Y) - h(X,PY) - \eta(Y)FX \tag{2.9}$$

for any $X, Y \in TM$.

A tensor field φ is said to be Killing [2], if

$$(\overline{\nabla}_X \varphi)Y + (\overline{\nabla}_Y \varphi)X = 0 \tag{2.10}$$

We mention the following results for latter use:

Theorem A.[10] *Let M be a 3-dimensional submanifold of a Kenmotsu manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if*

$$(\nabla_X P)Y = -\eta(Y)PX + g(Y,PX) \tag{2.11}$$

Theorem B.[3] *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi) \tag{2.12}$$

Furthermore, if θ is the slant angle of M, then $\lambda = cos^2\theta$.

3. Submanifolds of Kenmotsu manifolds with Killing structure tensor field φ

In this section we will consider slant submanifolds of Kenmotsu manifolds with Killing structure tensor field φ of type (1, 1).

Theorem 3.1. *Let M be a 3-dimensional submanifold of a Kenmotsu manifold with Killing structure tensor field φ . Then M is slant if and only if*

$$\eta(Y)PX + \eta(X)PY = 0 \tag{3.1}$$

and

(3.2) (3.2)
$$\eta(Y)FX + \eta(X)FY = 0$$

Proof. From equation (2.2), we have

$$(\overline{\nabla}_X \varphi)(Y) = -g(\varphi Y, X)\xi - \eta(Y)\varphi X \tag{3.3}$$

Interchanging X and Y in the above equation, we get

$$(\overline{\nabla}_Y \varphi)(X) = g(\varphi Y, X)\xi - \eta(X)\varphi Y \tag{3.4}$$

Combining equations (3.3) and (3.4), we get

$$(\overline{\nabla}_X \varphi)Y + (\overline{\nabla}_Y \varphi)X = -\eta(Y)\varphi(X) - \eta(X)\varphi(Y)$$

Using (2.10) in the above equation, we get

$$\eta(Y)\varphi X + \eta(X)\varphi Y = 0 \tag{3.5}$$

Using (2.3) in (3.5) and comparing tangential and normal components we get the result. Converse follows by straightforward computation. □

Theorem 3.2. *Let M be a 3-dimensional submanifold of a Kenmotsu manifold $\overline{M}$ with Killing tensor field φ such that $\xi \in TM$. Then M is slant if and only if P satisfies*

$$(\nabla_X P)Y + (\nabla_Y P)X = 0. \tag{3.6}$$

Proof. From (2.11), we have

$$(\nabla_X P)Y = -\eta(Y)PX - g(PY, X)\xi \tag{3.7}$$

for any $X, Y \in TM$

Interchanging X and Y in the above equation we get

$$(\nabla_Y P)X = -\eta(X)PY + g(PY, X)\xi \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$(\nabla_X P)Y + (\nabla_Y P)X = -\eta(Y)PX - \eta(X)PY$$

Then using (3.1) in the above equation, we get the result. □

Lemma 3.1. *Let M be a 3-dimensional submanifold of a Kenmotsu manifold $\overline{M}$ with Killing structure tensor field φ. Then*

$$A_{FX}Y + A_{FY}X + 2\,th(X, Y) = 0 \tag{3.9}$$

for any $X, Y \in TM$.

Proof. From (2.8) we can write

$$(\nabla_Y P)X = A_{FX}Y + th(X, Y) + g(X, PY)\xi - \eta(X)PY \tag{3.10}$$

Now combining (2.8) and (3.10) and making use of equation (3.1), we get

$$(\nabla_Y P)X + (\nabla_X P)Y = A_{FX}Y + A_{FY}X + 2th(X, Y)$$

and since P satisfies equation (3.6), we get the result. □

Theorem 3.3. *Let M be a 3-dimensional submanifold of a Kenmotsu manifold $\overline{M}$. Then*

$$(\nabla_X F)Y + (\nabla_Y F)X = 0$$

if and only if

$$2fh(X, Y) = h(X, PY) + h(Y, PX). \tag{3.11}$$

Proof. From (2.9), we have

$$(\nabla_Y F)X = fh(X,Y) - h(Y,PX) - \eta(X)FY. \tag{3.12}$$

Combining (2.9) and (3.12), we get

$$(\nabla_X F)Y + (\nabla_Y F)X = 2fh(X,Y) - h(X,PY) - h(Y,PX) - \eta(Y)FX - \eta(X)FY$$

Using equation (3.2) in the above equation we get

$$(\nabla_X F)Y + (\nabla_Y F)X = 2fh(X,Y) - h(X,PY) - h(Y,PX)$$

which proves the result. □

Lemma 3.2. *Let M be a 3-dimensional submanifold of a 5-dimensional Kenmotsu manifold $\overline{M}$ with Killing tensor field φ such that $\xi \in TM$. Then M is minimal proper slant submanifold of $\overline{M}$ if and only if*

$$(\nabla_X F)Y + (\nabla_Y F)X = 0.$$

Proof. From theorem (3.2) of [10], we have

$$(\nabla_X F)Y = -\eta(Y)FX \tag{3.13}$$

Now interchanging X and Y, we get

$$\nabla_Y F)X = -\eta(X)FY. \tag{3.14}$$

Combining (3.13) and (3.14) and making use of equation (3.2), we get

$$(\nabla_X F)Y + (\nabla_Y F)X = 0.$$

It can be easily seen that M is also minimal. □

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