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ON INDEFINITE BINARY QUADRATIC FORMS AND QUADRATIC IDEALS

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Abstract. We consider some properties of indefinite binary quadratic forms $F(x,y) = ax^2+bxy-y^2$ of discriminant $\Delta = b^2+4a$, and quadratic ideals $I = [a, b-\sqrt{\Delta}\ ]$.

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1. Introduction

A real binary quadratic form (or just a form) F is a polynomial in two variables x, y of the type

$$F = F(x,y) = ax^2 + bxy + cy^2 \tag{1.1}$$

with real coefficients a, b, c. We denote F briefly by $F = (a,b,c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. A quadratic form F of discriminant Δ is called indefinite if $\Delta > 0$, and is called integral if and only if $a, b, c \in \mathbb{Z}$. An indefinite quadratic form $F = (a,b,c)$ of discriminant Δ is said to be reduced if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{1.2}$$

Most properties of quadratic forms (the most is equivalence of forms) can be given by the aid of extended modular group $\overline{\Gamma}$ (see [5]). Gauss defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$\begin{aligned} gF(x,y) &= \left(ar^2 + brs + cs^2\right)x^2 + (2art + bru + bts + 2csu)\,xy \\ &\quad + \left(at^2 + btu + cu^2\right)y^2 \end{aligned} \tag{1.3}$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$, that is, gF is obtained from F by making the substitution $x \to rx + tu$ and $y \to sx + uy$. Moreover, $\Delta(F) = \Delta(gF)$ for all $g \in \overline{\Gamma}$, that is, the action of $\overline{\Gamma}$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \overline{\Gamma}$. Let F and G be two forms. If there exists

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a $g \in \overline{\Gamma}$ such that $gF = G$, then F and G are called equivalent. If $\det g = 1$, then F and G are called properly equivalent, and if $\det g = -1$, then F and G are called improperly equivalent. If a form F is improperly equivalent to itself, then F is called ambiguous (for further details on binary quadratic forms see [1, 2, 3]).

Let $\rho(F)$ denote the normalization (it means that replacing F by its normalization, for further details see [1, p. 88]) of $(c, -b, a)$. We set

$$\rho^i(F) = (c, -b + 2cr_i, cr_i^2 - br_i + a), \tag{1.4}$$

where

$$r_i = r_i(F) = \begin{cases} sign(c) \left\lfloor \frac{b}{2|c|} \right\rfloor & \text{if } |c| \geq \sqrt{\Delta} \\ sign(c) \left\lfloor \frac{b+\sqrt{\Delta}}{2|c|} \right\rfloor & \text{if } |c| < \sqrt{\Delta} \end{cases} \tag{1.5}$$

for $i \geq 0$. Then the number r_i is called the reducing number and the form $\rho^i(F)$ is called the reduction of F. If $\rho^1(F)$ is not reduced, then we apply the reduction algorithm again and hence we get $\rho^2(F)$. If $\rho^2(F)$ is not reduced, then we apply the reduction algorithm again and hence we get $\rho^3(F)$. After a finite step $j \geq 1$, the form $\rho^j(F)$ is reduced. The form $\rho^j(F)$ is called the reducing type of F. Buchmann and Vollmer [1] proved that given an indefinite form F the algorithm reduction terminates with a correct result after at most $\frac{\log(|a|/\sqrt{\Delta})}{2} + 2$ reduction step. If F is reduced, then $\rho^i(F)$ is also reduced by (1.2). In fact, ρ^i is a permutation of the set of all reduced indefinite forms.

Now consider the following transformation

$$\tau(F) = \tau(a, b, c) = (-a, b, -c). \tag{1.6}$$

Then the cycle of F is the sequence $((\tau\rho)^i(G))$ for $i \in \mathbb{Z}$, where $G = (A, B, C)$ is a reduced form with $A > 0$ which is equivalent to F. We represent the cycle of F by its period

$$F_0 \sim F_1 \sim \cdots \sim F_{l-1}$$

of length l. We explain how the compute the cycle of F by the following theorem.

Theorem 1.1. *[1, Sec: 6.10, p. 106] Let $F = (a, b, c)$ be reduced indefinite quadratic form of discriminant Δ. Let $F_0 = F = (a_0, b_0, c_0)$,*

$$s_i = |s(F_i)| = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor \tag{1.7}$$

and

$$\begin{aligned} F_{i+1} &= (a_{i+1}, b_{i+1}, c_{i+1}) \\ &= \left(|c_i|,\ -b_i + 2s_i|c_i|,\ -(a_i + b_i s_i + c_i s_i^2)\right) \end{aligned} \tag{1.8}$$

for $1 \leq i \leq l-2$. Then the cycle of F is $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ of length l.

Mollin [4, p. 4] considered the arithmetic of ideals in his book. Let $D \neq 1$ be a square free integer and let $\Delta = \frac{4D}{r^2}$, where $r = 2$ if $D \equiv 1 (\bmod 4)$ and $r = 1$ otherwise. If we set $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, then $\mathbb{K}$ is called a quadratic number field of discriminant Δ and O_Δ is the ring of integers of the quadratic field $\mathbb{K}$ of discriminant Δ. Let $I = [\alpha, \beta]$ denote the $\mathbb{Z}$-module $\alpha\mathbb{Z} \oplus \beta\mathbb{Z}$, i.e., the additive abelian group, with basis elements α and β consisting of $\{\alpha x + \beta y : x, y \in \mathbb{Z}\}$. Note that $O_\Delta = \left[1, \frac{1+\sqrt{D}}{r}\right]$. In this case $w_\Delta = \frac{r-1+\sqrt{D}}{r}$ is called the principal surd. Every principal surd $w_\Delta \in O_\Delta$ can be uniquely expressed as $w_\Delta = x\alpha + y\beta$, where $x, y \in \mathbb{Z}$ and $\alpha, \beta \in O_\Delta$. We call $[\alpha, \beta]$ an integral basis for $\mathbb{K}$. If $\frac{\alpha\overline{\beta} - \beta\overline{\alpha}}{\sqrt{\Delta}} > 0$, then α and β are called ordered basis elements.

Recall that two basis of an ideal are ordered if and only if they are equivalent under an element of $\overline{\Gamma}$. If I has ordered basis elements, then we say that I is simply ordered. If I is ordered, then

$$F(x, y) = \frac{N(\alpha x + \beta y)}{N(I)}$$

is a quadratic form of discriminant Δ (here $N(x)$ denotes the norm of x). In this case we say that F belongs to I and write $I \to F$. Conversely, let us assume that

$$G(x, y) = Ax^2 + Bxy + Cy^2 = d(ax^2 + bxy + cy^2)$$

be a quadratic form, where $d = \pm\gcd(A, B, C)$ and $b^2 - 4ac = \Delta$. If $B^2 - 4AC > 0$, then we get $d > 0$ and if $B^2 - 4AC < 0$, and choose d such that $a > 0$. If

$$I = [\alpha, \beta] = \begin{cases} \left[a, \frac{b-\sqrt{\Delta}}{2}\right] & \text{for } a > 0 \\ \\ \left[a, \frac{b-\sqrt{\Delta}}{2}\right]\sqrt{\Delta} & \text{for } a < 0 \text{ and } \Delta > 0, \end{cases}$$

then I is an ordered O_Δ-ideal. Note that if $a > 0$, then I is primitive and if $a < 0$, then $\frac{I}{\sqrt{\Delta}}$ is primitive. Thus to every form G corresponds an ideal I to which G belongs and we write $G \to I$. Hence we have a correspondence between ideals and quadratic forms (for further details see [4, p. 350].

Theorem 1.2. *[4, Sec: 1.2, p. 9] If $I = [a, b + cw_\Delta]$, then I is a non-zero ideal of O_Δ if and only if*

$$c|b,\ c|a \quad \textit{and} \quad ac|N(b + cw_\Delta).$$

Let δ denote a real quadratic irrational integer with trace $t = \delta + \overline{\delta}$ and norm $n = \delta\overline{\delta}$. Given a real quadratic irrational $\gamma \in \mathbb{Q}(\delta)$, there are rational integers P and Q such that $\gamma = \frac{P+\delta}{Q}$ with $Q|(\delta + P)(\overline{\delta} + P)$. Hence for each

$$\gamma = \frac{P + \delta}{Q} \tag{1.9}$$

there is a corresponding $\mathbb{Z}-$module

$$I_\gamma = [Q,\ P+\delta] \tag{1.10}$$

(in fact, this module is an ideal by Theorem 1.2), and an indefinite quadratic form

$$F_\gamma(x,y) = Q(x+\delta y)(x+\overline{\delta}y) \tag{1.11}$$

of discriminant $\Delta = t^2 - 4n$. The ideal I_γ in (1.10) is said to be reduced if and only if

$$P+\delta > Q \ \text{ and } \ -Q < P+\overline{\delta} < 0 \tag{1.12}$$

and is said to be ambiguous if and only if it contains both $\frac{P+\delta}{Q}$ and $\frac{P+\overline{\delta}}{Q}$, so if and only if $\frac{2P}{Q} \in \mathbb{Z}$.

Let $[m_0; \overline{m_1, m_2, \cdots, m_{l-1}}]$ denote continued fraction expansion of $\gamma = \frac{P+\delta}{Q}$ with a period length $l = l(I)$. Then the cycle of I_γ is $I_\gamma = I_\gamma^0 \sim I_\gamma^1 \sim \cdots \sim I_\gamma^{l-1}$ of length l, where

$$m_i = \left\lfloor \frac{P_i+\delta}{Q_i} \right\rfloor, \quad P_{i+1} = m_iQ_i - P_i \ \text{ and } \ Q_{i+1} = \frac{\delta^2 - P_{i+1}^2}{Q_i} \tag{1.13}$$

for $i \geq 0$.

2. Indefinite Binary Quadratic Forms

In [6, 7, 8], we considered some properties of quadratic irrationals γ, quadratic ideals I_γ and indefinite binary quadratic forms F_γ defined in (1.9), (1.10) and (1.11), respectively. In this section, we consider some properties of indefinite binary quadratic forms

$$F = (a, b, -1)$$

of the discriminant $\Delta = b^2 + 4a$. First we give the following theorem.

Theorem 2.1. *If $\Delta \equiv 0(mod 4)$, say $\Delta = 4k$ for an integer $k \geq 2$, then there exist $m-$indefinite binary quadratic forms of the type*

$$F_i = (a_i, b_i, c_i) = (k - i^2, 2i, -1), \quad 1 \leq i \leq m \tag{2.1}$$

of discriminant Δ, where $m = \lfloor\sqrt{k}\rfloor$.

Proof. Let $\Delta = 4k$ for $k \geq 2$. Then Δ is even. Let $F_i = (a_i, b_i, -1)$ be given a form of discriminant Δ. Then the coefficient b_i must be an even number since a_i must be an integer. Let $b_i = 2i$ for $i \geq 1$. Then

$$a_i = \frac{\Delta - b_i^2}{4} = \frac{4k - 4i^2}{4} = k - i^2.$$

By the assumption a_i must be positive. Therefore $k - i^2 > 0$, that is, $i < \sqrt{k}$. Hence we obtain the form $F_i = (k - i^2, 2i, -1)$ of discriminant $\Delta = 4k$ for $1 \leq i \leq m$. □

Let $S(F)$ denote the set of indefinite binary quadratic forms F_i defined in (2.1), that is,

$$S(F) = \left\{F_i : F_i = (k - i^2, 2i, -1), \quad 1 \leq i \leq m\right\}. \tag{2.2}$$

Then we have the following theorem.

Theorem 2.2. *F_m is the only reduced and ambiguous form in $S(F)$.*

Proof. Note that $F_m = (a_m, b_m, c_m) = (k - m^2, 2m, -1)$ by (2.2). We know that $m = \lfloor\sqrt{k}\rfloor$. So $m < \sqrt{k}$. Therefore $k - m^2 > 0$. Note that $\sqrt{k} - k + m^2$ is positive or negative. Nevertheless its absolute value is always smaller than m, that is, $|\sqrt{k} - k + m^2| < m$. Hence $\left|\sqrt{k} - k + m^2\right| < m < \sqrt{k}$ since $m < \sqrt{k}$. Therefore we conclude that F_m is reduced by (1.2) since

$$\begin{aligned}
\left|\sqrt{k} - k + m^2\right| < m < \sqrt{k} \quad &\Leftrightarrow \quad \left|\sqrt{k} - |k - m^2|\right| < m < \sqrt{k} \\
&\Leftrightarrow \quad 2\left|\sqrt{k} - |k - m^2|\right| < 2m < 2\sqrt{k} \\
&\Leftrightarrow \quad \left|2\sqrt{k} - 2|k - m^2|\right| < 2m < 2\sqrt{k} \\
&\Leftrightarrow \quad \left|\sqrt{4k} - 2|k - m^2|\right| < 2m < \sqrt{4k} \\
&\Leftrightarrow \quad \left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}.
\end{aligned}$$

The other forms $F_i = (a_i, b_i, c_i) = (t - i^2, 2i, -1)$ for $1 \leq i \leq m - 1$ are not reduced since for these forms $\left|\sqrt{\Delta} - 2|a_i|\right| > b_i$.

Now we show that $F_m = (k - m^2, 2m, -1)$ is ambiguous. Let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$. Then by (1.3), we have

$$\begin{aligned}
(k - m^2)r^2 + 2mrs - s^2 &= k - m^2 \\
2(k - m^2)rt + 2mru + 2mts - 2su &= 2m \\
(k - m^2)t^2 + 2mtu - u^2 &= -1.
\end{aligned}$$

This system of equations has a solution for $r = 1$, $s = 2m$, $t = 0$ and $u = -1$. Therefore $gF_m = F_m$ for

$$g = \begin{pmatrix} 1 & 2m \\ 0 & -1 \end{pmatrix}.$$

Hence F_m is improperly equivalent to itself since $\det g = -1$. So F_m is ambiguous by definition. □

We see as above that the forms $F_i = (k - i^2, 2i, -1)$ for $1 \leq i \leq m - 1$ are not reduced. But we can make them reduced using the reduction algorithm as we mentioned in Section 1.

Theorem 2.3. *Let* $F_i = (k-i^2, 2i, -1)$ *for* $1 \leq i \leq m-1$. *Then the reduction number is*

$$r_i = -(m+i),$$

and the reduction type of F_i *is*

$$\rho^1(F_i) = (-1, 2m, k-m^2).$$

Proof. Let $F_i = (a_i, b_i, c_i) = (k-i^2, 2i, -1)$ for $1 \leq i \leq m-1$. Note that $|-1| < \sqrt{4k}$. Then by (1.5), we get

$$r_i = sign(c_i)\left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor = -\left\lfloor \frac{2i+\sqrt{4k}}{2} \right\rfloor = -\left\lfloor i + \sqrt{k} \right\rfloor = -i-m.$$

Applying (1.4), we deduce that

$$\begin{aligned}
\rho^1(F_i) &= (c_i, -b_i + 2r_ic_i, c_ir_i^2 - b_ir_i + a_i) \\
&= \left(-1, -2i + 2(-m-i)(-1), (-1)(-i-m)^2 - 2i(-m-i) + k - i^2\right) \\
&= (-1, 2m, k-m^2).
\end{aligned}$$

Note that $k \geq 2$. So $\sqrt{k} - 1 > 0$. Therefore $|\sqrt{k} - 1| = \sqrt{k} - 1$. Hence it is easily seen that the form $\rho^1(F_i)$ is reduced since

$$\begin{aligned}
\sqrt{k} - 1 < m < \sqrt{k} \quad &\Leftrightarrow \quad \left|\sqrt{k} - 1\right| < m < \sqrt{k} \\
&\Leftrightarrow \quad 2\left|\sqrt{k} - 1\right| < 2m < 2\sqrt{k} \\
&\Leftrightarrow \quad \left|\sqrt{4k} - 2|-1|\right| < 2m < \sqrt{4k} \\
&\Leftrightarrow \quad \left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}.
\end{aligned}$$

Therefore the reduction type of F_i is $\rho^1(F_i) = (-1, 2m, k-m^2)$, as we claimed. □

3. Cycles of Indefinite Binary Quadratic Forms

We see as above that the form $F_m = (k-m^2, 2m, -1)$ is reduced. Therefore we can consider its cycle. In this section we consider its cycle in four cases:

$$k = m^2 + 2m - 1, \;\; k = m^2 + 2m, \;\; k = m^2 + m \;\; \text{and} \;\; k = m^2 + 1.$$

Theorem 3.1. *Let* $F_m = (k-m^2, 2m, -1)$.

1. If $k = m^2 + 2m - 1$, *then the cycle of* $F_m = (2m-1, 2m, -1)$ *is*

$$\begin{aligned} F_m^0 &= (2m-1, 2m, -1) \sim F_m^1 = (1, 2m, 1-2m) \sim \\ F_m^2 &= (2m-1, 2m-2, -2) \sim F_m^3 = (2, 2m-2, 1-2m). \end{aligned}$$

2. If $k = m^2 + 2m$, *then the cycle of* $F_m = (2m, 2m, -1)$ *is*

$$F_m^0 = (2m, 2m, -1) \sim F_m^1 = (1, 2m, -2m).$$

3. If $k = m^2 + m$, *then the cycle of* $F_m = (m, 2m, -1)$ *is*

$$F_m^0 = (m, 2m, -1) \sim F_m^1 = (1, 2m, -m).$$

4. If $k = m^2 + 1$, *then the cycle of* $F_m = (1, 2m, -1)$ *is*

$$F_m^0 = (1, 2m, -1).$$

Proof. (1) Let $k = m^2 + 2m - 1$. Then $F_m = (2m-1, 2m, -1)$. Hence by (1.7), we get

$$s_0 = \left\lfloor \frac{b_0 + \sqrt{\Delta}}{2|c_0|} \right\rfloor = \left\lfloor \frac{2m + \sqrt{4(m^2+2m-1)}}{2|-1|} \right\rfloor = 2m$$

and from (1.8)

$$\begin{aligned} F_m^1 &= (a_1, b_1, c_1) \\ &= \left(|c_0|,\ -b_0 + 2s_0|c_0|,\ -a_0 - b_0 s_0 - c_0 s_0^2\right) \\ &= \left(1,\ -2m + 2.2m,\ 1 - 2m - 2m.2m + 4m^2\right) \\ &= (1, 2m, 1-2m)\,. \end{aligned}$$

For $i = 1$ we have

$$s_1 = \left\lfloor \frac{b_1 + \sqrt{\Delta}}{2|c_1|} \right\rfloor = \left\lfloor \frac{2m + \sqrt{4(m^2+2m-1)}}{2|1-2m|} \right\rfloor = 1$$

and hence

$$\begin{aligned} F_m^2 &= (a_2, b_2, c_2) \\ &= \left(|c_1|,\ -b_1 + 2s_1|c_1|,\ -a_1 - b_1 s_1 - c_1 s_1^2\right) \\ &= (2m-1,\ -2m + 2.(2m-1),\ -1 - 2m - (1-2m)) \\ &= (2m-1, 2m-2, -2)\,. \end{aligned}$$

For $i = 2$ we have

$$s_2 = \left\lfloor \frac{b_2 + \sqrt{\Delta}}{2|c_2|} \right\rfloor = \left\lfloor \frac{2m - 2 + \sqrt{4(m^2+2m-1)}}{2|-2|} \right\rfloor = m - 1$$

and hence

$$\begin{aligned} F_m^3 &= (a_3, b_3, c_3) \\ &= \left(|c_2|,\ -b_2 + 2s_2|c_2|,\ -a_2 - b_2 s_2 - c_2 s_2^2\right) \\ &= \left(2,\ 2 - 2m + 2(m-1).2, 1 - 2m - (2m-2)(m-1) + 2(m-1)^2\right) \\ &= (2, 2m-2, 1-2m)\,. \end{aligned}$$

For $i = 3$ we have

$$s_3 = \left\lfloor \frac{b_3 + \sqrt{\Delta}}{2|c_3|} \right\rfloor = \left\lfloor \frac{2m - 2 + \sqrt{4(m^2 + 2m - 1)}}{2|1 - 2m|} \right\rfloor = 1$$

and hence

$$\begin{aligned} F_m^4 &= (a_4, b_4, c_4) \\ &= \left(|c_3|,\ -b_3 + 2s_3|c_3|,\ -a_3 - b_3 s_3 - c_3 s_3^2\right) \\ &= (2m-1, 2 - 2m + 2(2m-1), -2 - (2m-2) - (1-2m)) \\ &= (2m-1, 2m, -1) \\ &= F_m^0. \end{aligned}$$

Therefore the cycle of F_m is completed and is $F_m^0 = (2m-1, 2m, -1) \sim F_m^1 = (1, 2m, 1-2m) \sim F_m^2 = (2m-1, 2m-2, -2) \sim F_m^3 = (2, 2m-2, 1-2m)$.

(2) Let $k = m^2 + 2m$. Then $F_m = (2m, 2m, -1)$. Then by (1.7), we get

$$s_0 = \left\lfloor \frac{b_0 + \sqrt{\Delta}}{2|c_0|} \right\rfloor = \left\lfloor \frac{2m + \sqrt{4(m^2 + 2m)}}{2|-1|} \right\rfloor = 2m$$

and hence by (1.8)

$$\begin{aligned} F_m^1 &= (a_1, b_1, c_1) \\ &= \left(|c_0|,\ -b_0 + 2s_0|c_0|,\ -a_0 - b_0 s_0 - c_0 s_0^2\right) \\ &= \left(1,\ -2m + 2.2m, -2m - 2m.2m + 4m^2\right) \\ &= (1, 2m, -2m)\,. \end{aligned}$$

For $i = 1$ we have

$$s_1 = \left\lfloor \frac{b_1 + \sqrt{\Delta}}{2|c_1|} \right\rfloor = \left\lfloor \frac{2m + \sqrt{4(m^2 + 2m)}}{2|-2m|} \right\rfloor = 1$$

and hence

$$\begin{aligned} F_m^2 &= (a_2, b_2, c_2) \\ &= \left(|c_1|,\ -b_1 + 2s_1|c_1|,\ -a_1 - b_1 s_1 - c_1 s_1^2\right) \\ &= (2m,\ -2m + 2.2m,\ -1 - 2m + 2m) \\ &= (2m, 2m-1) \\ &= F_m^0. \end{aligned}$$

Therefore the cycle of F_m is completed and is $F_m^0 = (2m, 2m, -1) \sim F_m^1 = (1, 2m, -2m)$.

(3) Let $t = m^2 + m$. Then $F_m = (m, 2m, -1)$ and hence by (1.7)

$$s_0 = \left\lfloor \frac{b_0 + \sqrt{\Delta}}{2|c_0|} \right\rfloor = \left\lfloor \frac{2m + \sqrt{4(m^2+m)}}{2|-1|} \right\rfloor = 2m.$$

So by (1.8)

$$\begin{aligned} F_m^1 &= (a_1, b_1, c_1) \\ &= \left(|c_0|, -b_0 + 2s_0|c_0|, -a_0 - b_0 s_0 - c_0 s_0^2\right) \\ &= \left(1, -2m + 2.2m, -m - 2m.2m + 4m^2\right) \\ &= (1, 2m, -m)\,. \end{aligned}$$

For $i = 1$ we have

$$s_1 = \left\lfloor \frac{b_1 + \sqrt{\Delta}}{2|c_1|} \right\rfloor = \left\lfloor \frac{2m + \sqrt{4(m^2+m)}}{2|-m|} \right\rfloor = 2$$

and hence

$$\begin{aligned} F_m^2 &= (a_2, b_2, c_2) \\ &= \left(|c_1|, -b_1 + 2s_1|c_1|, -a_1 - b_1 s_1 - c_1 s_1^2\right) \\ &= (m, -2m + 2.2.m, -1 - 2m.2 + 4m) \\ &= (m, 2m, -1) \\ &= F_m^0. \end{aligned}$$

Therefore the cycle of F_m is completed and is $F_m^0 = (m, 2m, -1) \sim F_m^1 = (1, 2m, -m)$.

(4) Let $k = m^2 + 1$. Then $F_m = (1, 2m, -1)$ and hence

$$s_0 = \left\lfloor \frac{b_0 + \sqrt{\Delta}}{2|c_0|} \right\rfloor = \left\lfloor \frac{2m + \sqrt{4(m^2+1)}}{2|-1|} \right\rfloor = 2m.$$

So

$$\begin{aligned} F_m^1 &= (a_1, b_1, c_1) \\ &= \left(|c_0|, -b_0 + 2s_0|c_0|, -a_0 - b_0 s_0 - c_0 s_0^2\right) \\ &= \left(1, -2m + 2.2m, -1 - 2m.2m + 4m^2\right) \\ &= (1, 2m, -1) \\ &= F_m^0. \end{aligned}$$

Therefore the cycle of F_m is completed and is $F_m^0 = (1, 2m, -1)$. □

4. Cycle of Ideals $I = [a, b - \sqrt{\Delta}]$

In the previous section, we considered the cycles of the form $F_m = (a_m, b_m, -1) = (k - m^2, 2m, -1)$ of discriminant $\Delta = b_m^2 + 4a_m$ in four cases. Similarly, in this section we consider the cycles of ideals $I = [a, b - \sqrt{\Delta}]$ in four cases.

Theorem 4.1. *Let* $I = [a, b - \sqrt{\Delta}]$.

1. *If* $a = b - 1$ *and if* $a = 4k + 1$ *for an integer* $k \geq 1$, *then the continued fraction expansion of* $\gamma = \frac{4k+2-\sqrt{16k^2+32k+8}}{4k+1}$ *is* $[-1; 1, 2k, \overline{2, k, 2, 2k+1}]$, *and the cycle of* $I = [4k+1, 4k+2 - \sqrt{16k^2 + 32k + 8}]$ *is*

$$\begin{aligned}
I_0 &= [4k+1, 4k+2 - \sqrt{16k^2+32k+8}] \sim \\
I_1 &= [-1-12k, -3-8k - \sqrt{16k^2+32k+8}] \sim \\
I_2 &= [-4, 2-4k - \sqrt{16k^2+32k+8}] \sim \\
I_3 &= [-1-4k, -2-2k - \sqrt{16k^2+32k+8}] \sim \\
I_4 &= [-8, -4k - \sqrt{16k^2+32k+8}] \sim \\
I_5 &= [-1-4k, -4k - \sqrt{16k^2+32k+8}] \sim \\
I_6 &= [-4, -2-2k - \sqrt{16k^2+32k+8}].
\end{aligned}$$

2. *If* $a = b = 2k$ *for an integer* $k > 3$, *then the continued fraction expansion of* $\gamma = \frac{2k-\sqrt{4k^2+8k}}{2k}$ *is* $[-1; 1, k-1, \overline{2, k}]$, *and the cycle of* $I = [2k, 2k - \sqrt{4k^2 + 8k}]$ *is*

$$\begin{aligned}
&I_0 = [2k, 2k - \sqrt{4k^2+8k}] \sim I_1 = [4-6k, -4k - \sqrt{4k^2+8k}] \sim \\
&I_2 = [-4, 4-2k - \sqrt{4k^2+8k}] \sim I_3 = [-2k, -2k - \sqrt{4k^2+8k}] \sim \\
&I_4 = [-4, -2k - \sqrt{4k^2+8k}].
\end{aligned}$$

3. *If* $b = 2a$, *then the continued fraction expansion of* $\gamma = \frac{2a-\sqrt{4a^2+4a}}{a}$ *is* $[-1; 1, a-1, \overline{4, a}]$, *and the cycle of* $I = [a, 2a - \sqrt{4a^2 + 4a}]$ *is*

$$\begin{aligned}
&I_0 = [a, 2a - \sqrt{4a^2+4a}] \sim I_1 = [4-5a, -3a - \sqrt{4a^2+4a}] \sim \\
&I_2 = [-4, 4-2a - \sqrt{4a^2+4a}] \sim I_3 = [-a, -2a - \sqrt{4a^2+4a}] \sim \\
&I_4 = [-4, -2a - \sqrt{4a^2+4a}].
\end{aligned}$$

4. *If* $a = 1$ *and* $b = 2k$ *for an integer* $k \geq 1$, *then the continued fraction expansion of* $\gamma = \frac{2k-\sqrt{4k^2+4}}{1}$ *is* $[-1; 1, k-1, \overline{4k, k}]$, *and the cycle of* $I = [1, 2k - \sqrt{4k^2 + 4}]$ *is*

$$\begin{aligned}
&I_0 = [1, 2k - \sqrt{4k^2+4}] \sim I_1 = [3-4k, -1-2k - \sqrt{4k^2+4}] \sim \\
&I_2 = [-4, 4-2k - \sqrt{4k^2+4}] \sim I_3 = [-1, -2k - \sqrt{4k^2+4}] \sim \\
&I_4 = [-4, -2k - \sqrt{4k^2+4}].
\end{aligned}$$

Proof. (1) Let $I = I_0 = [4k+1, 4k+2-\sqrt{16k^2+32k+8}]$. Then by (1.13) we get $m_0 = -1$ and hence

$$\begin{aligned} P_1 &= m_0Q_0 - P_0 = -1(4k+1) - (4k+2) = -8k-3 \\ Q_1 &= \frac{D-P_1^2}{Q_0} = \frac{16k^2+32k+8-(-8k-3)^2}{4k+1} = -1-12k. \end{aligned}$$

For $i=1$ we have $m_1 = 1$ and hence

$$\begin{aligned} P_2 &= m_1Q_1 - P_1 = 1(-1-12k) - (-3-8k) = 2-4k \\ Q_2 &= \frac{D-P_2^2}{Q_1} = \frac{16k^2+32k+8-(2-4k)^2}{-1-12k} = -4. \end{aligned}$$

For $i=2$ we have $m_2 = 2k$ and hence

$$\begin{aligned} P_3 &= m_2Q_2 - P_2 = 2k(-4) - (2-4k) = -2-4k \\ Q_3 &= \frac{D-P_3^2}{Q_2} = \frac{16k^2+32k+8-(-2-4k)^2}{-4} = -1-4k. \end{aligned}$$

For $i=3$ we have $m_3 = 2$ and hence

$$\begin{aligned} P_4 &= m_3Q_3 - P_3 = 2(-1-4k) - (-2-4k) = -4k \\ Q_4 &= \frac{D-P_4^2}{Q_3} = \frac{16k^2+32k+8-(-4k)^2}{-1-4k} = -8. \end{aligned}$$

For $i=4$ we have $m_4 = k$ and hence

$$\begin{aligned} P_5 &= m_4Q_4 - P_4 = k(-8) - (-4k) = -4k \\ Q_5 &= \frac{D-P_5^2}{Q_4} = \frac{16k^2+32k+8-(-4k)^2}{-8} = -1-4k. \end{aligned}$$

For $i=5$ we have $m_5 = 2$ and hence

$$\begin{aligned} P_6 &= m_5Q_5 - P_5 = 2(-1-4k) - (-4k) = -2-4k \\ Q_6 &= \frac{D-P_6^2}{Q_5} = \frac{16k^2+32k+8-(-2-4k)^2}{-1-4k} = -4. \end{aligned}$$

For $i=6$ we have $m_6 = 2k+1$ and hence

$$\begin{aligned} P_7 &= m_6Q_6 - P_6 = (2k+1)(-4) - (-2-4k) = -2-4k = P_3 \\ Q_7 &= \frac{D-P_7^2}{Q_6} = \frac{16k^2+32k+8-(-2-4k)^2}{-4} = -1-4k = Q_3. \end{aligned}$$

For $i=7$ we have $m_7 = 2 = m_3$. Therefore the continued fraction expansion of γ is $[-1; 1, 2k, \overline{2, k, 2, 2k+1}]$, and the cycle of I is $I_0 = [4k+1, 4k+2-\sqrt{16k^2+32k+8}] \sim I_1 = [-1-12k, -3-8k-\sqrt{16k^2+32k+8}] \sim I_2 = [-4, 2-4k-\sqrt{16k^2+32k+8}] \sim I_3 = [-1-4k, -2-2k-\sqrt{16k^2+32k+8}] \sim$

$I_4 = [-8, -4k - \sqrt{16k^2 + 32k + 8}] \sim I_5 = [-1 - 4k, -4k - \sqrt{16k^2 + 32k + 8}] \sim I_6 = [-4, -2 - 2k - \sqrt{16k^2 + 32k + 8}]$.

(2) Let $I = I_0 = [2k, 2k - \sqrt{4k^2 + 8k}]$. Then by (1.13) we get $m_0 = -1$ and hence

$$\begin{aligned} P_1 &= m_0Q_0 - P_0 = -1(2k) - (2k) = -4k \\ Q_1 &= \frac{D - P_1^2}{Q_0} = \frac{4k^2 + 8k - (-4k)^2}{2k} = \frac{2k(4 - 6k)}{2k} = 4 - 6k. \end{aligned}$$

For $i = 1$ we have $m_1 = 1$ and hence

$$\begin{aligned} P_2 &= m_1Q_1 - P_1 = 1.(4 - 6k) - (-4k) = 4 - 2k \\ Q_2 &= \frac{D - P_2^2}{Q_1} = \frac{4k^2 + 8k - (4 - 2k)^2}{4 - 6k} = \frac{-4(4 - 6k)}{4 - 6k} = -4. \end{aligned}$$

For $i = 2$ we have $m_2 = k - 1$ and hence

$$\begin{aligned} P_3 &= m_2Q_2 - P_2 = (k - 1)(-4) - (4 - 2k) = -2k \\ Q_3 &= \frac{D - P_3^2}{Q_2} = \frac{4k^2 + 8k - (-2k)^2}{-4} = \frac{8k}{-4} = -2k. \end{aligned}$$

For $i = 3$ we have $m_3 = 2$ and hence

$$\begin{aligned} P_4 &= m_3Q_3 - P_3 = 2(-2k) - (-2k) = -2k \\ Q_4 &= \frac{D - P_4^2}{Q_3} = \frac{4k^2 + 8k - (-2k)^2}{-2k} = \frac{8k}{-2k} = -4. \end{aligned}$$

For $i = 4$ we have $m_4 = k$ and hence

$$\begin{aligned} P_5 &= m_4Q_4 - P_4 = k(-4) - (-2k) = -2k = P_3 \\ Q_5 &= \frac{D - P_5^2}{Q_4} = \frac{4k^2 + 8k - (-2k)^2}{-4} = \frac{8k}{-4} = -2k = Q_3. \end{aligned}$$

For $i = 5$ we have $m_5 = 2 = m_3$. Therefore the continued fraction expansion of γ is $[-1; 1, k - 1, \overline{2, k}]$, and the cycle of I is $I_0 = [2k, 2k - \sqrt{4k^2 + 8k}] \sim I_1 = [4 - 6k, -4k - \sqrt{4k^2 + 8k}] \sim I_2 = [-4, 4 - 2k - \sqrt{4k^2 + 8k}] \sim I_3 = [-2k, -2k - \sqrt{4k^2 + 8k}] \sim I_4 = [-4, -2k - \sqrt{4k^2 + 8k}]$.

(3) Let $b = 2a$ and let $I = I_0 = [a, 2a - \sqrt{4a^2 + 4a}]$. Then by (1.13) we get $m_0 = -1$ and hence

$$\begin{aligned} P_1 &= m_0Q_0 - P_0 = -1(a) - (2a) = -3a \\ Q_1 &= \frac{D - P_1^2}{Q_0} = \frac{4a^2 + 4a - (-3a)^2}{a} = \frac{a(4 - 5a)}{a} = 4 - 5a. \end{aligned}$$

For $i = 1$ we have $m_1 = 1$ and hence

$$\begin{aligned} P_2 &= m_1Q_1 - P_1 = 1.(4 - 5a) - (-3a) = 4 - 2a \\ Q_2 &= \frac{D - P_2^2}{Q_1} = \frac{4a^2 + 4a - (4 - 2a)^2}{4 - 5a} = \frac{-4(4 - 5a)}{4 - 5a} = -4. \end{aligned}$$

For $i=2$ we have $m_2=a-1$ and hence

$$\begin{aligned} P_3 &= m_2Q_2-P_2=(a-1)(-4)-(4-2a)=-2a \\ Q_3 &= \frac{D-P_3^2}{Q_2}=\frac{4a^2+4a-(-2a)^2}{-4}=\frac{4a}{-4}=-a. \end{aligned}$$

For $i=3$ we have $m_3=4$ and hence

$$\begin{aligned} P_4 &= m_3Q_3-P_3=4(-a)-(-2a)=-2a \\ Q_4 &= \frac{D-P_4^2}{Q_3}=\frac{4a^2+4a-(-2a)^2}{-a}=\frac{4a}{-a}=-4. \end{aligned}$$

For $i=4$ we have $m_4=a$ and hence

$$\begin{aligned} P_5 &= m_4Q_4-P_4=a(-4)-(-2a)=-2a=P_3 \\ Q_5 &= \frac{D-P_5^2}{Q_4}=\frac{4a^2+4a-(-2a)^2}{-4}=\frac{4a}{-4}=-a=Q_3. \end{aligned}$$

For $i=5$ we have $m_5=4=m_3$. Therefore the continued fraction expansion of γ is $[-1;1,a-1,\overline{4,a}]$, and the cycle of I is $I_0=[a,2a-\sqrt{4a^2+4a}\sim I_1=[4-5a,-3a-\sqrt{4a^2+4a}]\sim I_2=[-4,4-2a-\sqrt{4a^2+4a}]\sim I_3=[-a,-2a-\sqrt{4a^2+4a}]\sim I_4=[-4,-2a-\sqrt{4a^2+4a}]$.

(4) Let $a=1$, let $b=2k$, and let $I=I_0=[1,2k-\sqrt{4k^2+4}]$. Then by (1.13) we get $m_0=-1$ and hence

$$\begin{aligned} P_1 &= m_0Q_0-P_0=-1(1)-(2k)=-1-2k \\ Q_1 &= \frac{D-P_1^2}{Q_0}=\frac{4k^2+4-(-1-2k)^2}{1}=3-4k. \end{aligned}$$

For $i=1$ we have $m_1=1$ and hence

$$\begin{aligned} P_2 &= m_1Q_1-P_1=1.(3-4k)-(-1-2k)=4-2k \\ Q_2 &= \frac{D-P_2^2}{Q_1}=\frac{4k^2+4-(4-2k)^2}{3-4k}=-4. \end{aligned}$$

For $i=2$ we have $m_2=k-1$ and hence

$$\begin{aligned} P_3 &= m_2Q_2-P_2=(k-1)(-4)-(4-2k)=-2k \\ Q_3 &= \frac{D-P_3^2}{Q_2}=\frac{4k^2+4-(-2k)^2}{-4}=-1. \end{aligned}$$

For $i=3$ we have $m_3=4k$ and hence

$$\begin{aligned} P_4 &= m_3Q_3-P_3=4k(-1)-(-2k)=-2k \\ Q_4 &= \frac{D-P_4^2}{Q_3}=\frac{4k^2+4-(-2k)^2}{-1}=-4. \end{aligned}$$

For $i = 4$ we have $m_4 = k$ and hence

$$\begin{aligned} P_5 &= m_4Q_4 - P_4 = k(-4) - (-2k) = -2k = P_3 \\ Q_5 &= \frac{D - P_5^2}{Q_4} = \frac{4k^2 + 4 - (-2k)^2}{-4} = -1 = Q_3. \end{aligned}$$

For $i = 5$ we have $m_5 = 4k = m_3$. Therefore the continued fraction expansion of γ is $[-1; 1, k-1, \overline{4k, k}]$, and the cycle of I is $I_0 = [1, 2k - \sqrt{4k^2+4}] \sim I_1 = [3-4k, -1-2k-\sqrt{4k^2+4}] \sim I_2 = [-4, 4-2k-\sqrt{4k^2+4}] \sim I_3 = [-1, -2k-\sqrt{4k^2+4}] \sim I_4 = [-4, -2k-\sqrt{4k^2+4}]$. □

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