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A GENERAL CLASS OF CONTRACTIONS: *A*-CONTRACTIONS

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Abstract. In this article we introduce a new class of contraction maps, called A-contractions, which includes the contractions studied by R. Bianchini, M. S. Khan, S. Reich and T. Kannen. It is shown that the class of A-contractions is proper super class of Kannan's and Khan's contractions. Several results due to B. Ahmad, F. U. Rehman, Z. Chuanyi, N. Shioji et al. are extended to the A-contractions. We also show that a metric space is complete if and only if it has a fixed point property for A-contractions.

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1. Introduction

Let R_+ denote the set of all non-negative real numbers and A be the set of all functions $\alpha : R_+^3 \to R_+$ satisfying

(i) α is continuous on the set R_+^3 (with respect to the Euclidean metric on R^3).

(ii) $a \le kb$ for some $k \in [0,1)$ whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$ or $a \le \alpha(b,b,a)$ for all a,b.

Now we introduce the class of contractions called A-contraction:

Definition 1. *A self-map T on a metric space X is said to be A-contraction if it satisfies the condition:*

$$(A) \qquad d(Tx,Ty) \le \alpha(d(x,y), d(x,Tx), d(y,Ty)) \quad \text{for all } x,y \in X \text{ and some } \alpha \in A.$$

We shall show that the class of A-contractions includes the classes of contractions studied by Kannan [4], Khan[6], Bianchini [2] and Reich [7].

Definition 2. *A self-map T on a metric space X is said to be*
(i) K-contraction if there exists a number $r \in [0,1/2)$ such that,

$$(K) \qquad d(Tx,Ty) \le r\{d(Tx,x) + d(Ty,y)\} \text{ for all } x,y \in X. \text{ (see [3, p. 116.])}$$

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(ii) M-contraction if there exists a number $h \in [0,1)$ such that,

$$(M) \qquad d(Tx,Ty) \le h\sqrt{d(Tx,x)\,d(Ty,y)} \text{ for all } x,y \in X \text{ (see [6]).}$$

2. Comparison of M and K-contractions with *A*-contraction

In this section we show that an M-contraction is a K-contraction and every K-contraction is an A-contraction; consequently every M-contraction is an A-contraction.

Theorem 1.

(i) Every M-contraction is K-contraction.

(ii) Every K-contraction is A-contraction and hence every M-contraction is A-contraction.

Proof.

(i) Let $T : X \to X$ be an M-contraction. Then there exists some $h \in [0,1)$ satisfying the condition (M).

We know that the geometric mean of two positive real numbers v, w always precedes their arithmetic mean, that is $\sqrt{vw} \le \frac{v+w}{2}$. So that $h\sqrt{vw} \le h\dfrac{v+w}{2}$ for all $h \in [0,1)$. Hence with $v = d(Tx,x)\,, w = d(y,Ty)$ we have

$$h\sqrt{d(Tx,x)\,d(Ty,y)} \le r\{d(Tx,x) + d(Ty,y)\}$$

for all $r \in [0,1/2)$ and for all $x,y \in X$. This, together with inequality (M), gives that

$$d(Tx,Ty) \le h\sqrt{d(Tx,x)\,d(Ty,y)} \le r\{d(Tx,x) + d(Ty,y)\}\,.$$

(ii) Let $T : X \to X$ be a K-contraction. Therefore there exists some $r \in [0,1/2)$ such that (K) holds for all x,y in X. Keeping one such r fixed, we define a map $\alpha : R_+^3 \to R_+$ as $\alpha(u,v,w) = r(v+w)$ for all $u,v,w \in R_+$. Since addition and multiplication of reals are continuous, so α is continuous.

Case I: if $u \le \alpha(u,v,v) = r(v+v)$ then $u \le kv$ with $k = 2r \in [0,1)$.

Case II: If $u \le \alpha(v,u,v) = r(u+v)$ *then* $u \le r(u+v)$ gives $u \le \frac{r}{1-r}v = kv$ with $k = \frac{r}{1-r} \in [0,1)$.

Similarly, for case III where $u \le \alpha(v,v,u)$ we have $u \le kv$ with $k = \frac{r}{1-r} \in [0,1)$. So, in any case $u \le kv$ for some $k \in [0,1)$. Hence $\alpha \in A$.

Now, by taking $u = d(x,y)\,, v = d(Tx,x)$ *and* $w = d(Ty,y)$ and using the inequality (K), we get that

$$d(Tx,Ty) \le a\{d(Tx,x) + d(Ty,y)\} = \alpha(d(x,y)\,, d(Tx,x)\,, d(Ty,y))$$

for all $x,y \in X$.

This shows that T is an A-contraction whenever T is K-contraction. This, together with (i), gives that every M-contraction is an A-contraction. □

Next we show that an A-contraction may not be K-contraction; hence M and K-contractions are proper sub-classes of A-contractions. For this, we need the following result.

Theorem 2. *The self-map T on the metric space X satisfying*

$$d(Tx,Ty) \leq \beta \max\{d(Tx,x)+d(Ty,y), d(Ty,y)+d(x,y), d(Tx,x)+d(x,y)\}$$

for all x,y in X and some $\beta \in [0,1/2)$ is an A-contraction.

Proof. Define the map $\alpha : R_+^3 \to R_+$ as

$$\alpha(u,v,w) = \beta \max\{u+v, v+w, u+w\}$$

for all u,v,w in R_+, where β is any fixed number in $[0,1/2)$. Then $\alpha \in A$ because,

1. Clearly α is continuous.

2. For $u \leq \alpha(u,v,v) = \beta \max\{u+v, v+u, v+v\}$, we consider the following cases.

Case I. $\max\{u+v, v+u, v+v\} = u+v$. In this case, $u \leq \frac{\beta}{1-\beta} v \leq kv$, with $k = \frac{\beta}{1-\beta} \in [0,1)$.

Case II. $\max\{u+v, v+u, v+v\} = v+v$. In this case, $u \leq kv$, with $k = 2\beta \in [0,1)$. Similarly, for $u \leq \alpha(v,u,v)$ *or* $u \leq \alpha(v,v,u)$ we have $u \leq kv$ for some $k \in [0,1)$. Hence

$$\begin{aligned} d(Tx,Ty) &\leq \beta \max\{d(Tx,x)+d(Ty,y), d(Ty,y)+d(x,y), \\ &\quad d(Tx,x)+d(x,y)\} \\ &= \alpha(d(x,y), d(Tx,x), d(Ty,y)) \end{aligned}$$

by the construction of α. Thus T is an A-contraction. □

The following example, together with Theorem 2, shows that the class of A-contraction is a proper super-class of K-contractions, and hence so is of M-contraction.

Example 1. *Consider $X = \{0,1,2,3,4\}$ with usual metric relative to real line. T be a self-map on X, given by*

$$Tx = \begin{cases} 2 & \text{if } x = 0; \\ 1 & \text{otherwise.} \end{cases}$$

We observe that the condition K and hence condition M are not satisfied by T because with $x=0, y=1$ we have

$$1 = d(Tx,Ty) \leq r\{d(Tx,x)+d(Ty,y)\} = r(2+0) = 2r < 1$$

for all $r \in [0,1/2)$; a contradiction.

However, one can easily verify that T satisfies

$$d(Tx,Ty) \leq \beta \max\{d(Tx,x)+d(Ty,y), d(Ty,y)+d(x,y), d(Tx,x)+d(x,y)\}$$

for all $x,y \in X$ and some $\beta \in [0,1/2)$. Hence, by Theorem 2, T must be an A-contraction.

3. Comparison of A-contractions with some other contractions

In this section we investigate comparison of an A-contraction with the contraction maps studied by Bianchini [2] and Reich [7].

Definition 3. *A self-map T on a metric space X is said to be*

(i) B-contraction if there exists a number $b \in [0,1)$ such that

(B) $d(Tx,Ty) \leq b \max\{d(x,Tx), d(y,Ty)\}$ *for all* $x,y \in X$.

(ii) R-contraction if there exist non-negative numbers a,b,c satisfying $a+b+c \leq 1$ such that

(R) $d(Tx,Ty) \leq ad(Tx,x) + bd(Ty,y) + cd(x,y)$ *for all* $x,y \in X$ *(see [2], [7]).*

Theorem 3. *Every B-contraction is an A-contraction on any metric space.*

Proof. Assume that T on the metric space X is B-contraction. Define $\alpha : R_+^3 \to R_+$ by $\alpha(u,v,w) = h \max\{v,w\}$ for all $u,v,w \in R_+$ with some fixed $h \in [0,1)$. Next we show that $\alpha \in A$.

(i) Clearly α is continuous.

(ii) If $u \leq \alpha(u,v,v)$ then $u \leq h \max\{v,v\} = kv$ with $k = h \in [0,1)$. If $u \leq \alpha(v,u,v)$ then $u \leq h \max\{u,v\} = hv$ because $h < 1$, so that $u \leq kv$ with $k = h \in [0,1)$. Similarly, if $u \leq \alpha(v,v,u)$ then $u \leq kv$ for some $k = h \in [0,1)$. So we deduce that $\alpha \in A$. Further, since T is a B-contraction, we get from the construction of α that

$$d(Tx,Ty) \leq h \max\{d(x,Tx), d(y,Ty)\} = \alpha(d(x,y), d(x,Tx), d(y,Ty))$$

for all $x,y \in X$. We conclude that T is an A-contraction.

Next theorem establishes the fact that the class of A-contractions includes all R-contractions:

Theorem 4. *Every R-contraction is an A-contraction on a metric space X.*

Proof. Assume that $T : X \to X$ is an R-contraction. Then by definition, (R) holds for all x,y in X and $a+b+c \leq 1$. Let us define $\alpha : R_+^3 \to R_+$ by $\alpha(u,v,w) = au + bv + cw$ for all u,v,w $\in R_+$. Then α is continuous.

Further, $u \leq \alpha(u,v,v) = au + bv + cv$, implies $(1-a)u \leq (b+c)v$ and so $u \leq kv$ with $k = \frac{b+c}{1-a} \in [0,1)$.

Similarly, $u \leq \alpha(v,u,v) = av + bu + cv$ implies $(1-b)u \leq (a+c)v$, which gives $u \leq kv$ with $k = \frac{a+c}{1-b} \in [0,1)$, and $u \leq \alpha(v,v,u) = av + bv + cu$ gives $u \leq kv$ with $k = \frac{a+b}{1-c} \in [0,1)$.

So, $\alpha \in A$. Moreover, by taking $u = d(Tx,x)$, $v = d(Ty,y)$ *and* $w = d(x,y)$ we get that

$$d(Tx,Ty) \leq ad(Tx,x) + bd(Ty,y) + cd(x,y) = \alpha(d(x,Tx), d(y,Ty), d(x,y))$$

by (R). □

Thus T is an A-contraction whenever it is R-contraction.

4. Some fixed point theorems using *A*-contractions

In this section we give some results on fixed points of A-contractions. These include the analogues of certain results in [1], [3] and [6].

Theorem 5. *Let T be an A-contraction on a complete metric space X. Then T has a unique fixed point in X such that the sequence $\{T^n x_0\}$ converges to the fixed point, for any $x_0 \in X$.*

Proof. Fix $x_0 \in X$ and define the iterative sequence $\{x_n\}$ by $x_n = T^n x_0$ (equivalently, $x_{n+1} = Tx_n$) where T^n stands for the map obtained by n-time composition of T with T. Since T is an A-contraction, $\exists\ \alpha \in A$ s.t (A) of Definition 1 holds, i.e.

$$(A) \qquad d(Tx, Ty) \leq \alpha\left(d(x, Ty), d(x, Tx), d(y, Ty)\right)$$

for all x, y in X.

Replacing x by x_{n+1} and y by x_n in (A), we (by construction of α in A) get the existence of $k \in [0, 1)$ satisfying

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha\left(d(x_{n-1}, x_n), d(Tx_{n-1}, x_{n-1}), d(Tx_n, x_n)\right) \\ &\leq \alpha\left(d(x_{n-1}, x_n), d(x_n, x_{n-1}), d(x_{n+1}, x_n)\right) \\ &\leq kd(x_{n-1}, x_n). \end{aligned}$$

Continuing this way we get

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \ldots \leq k^n d(x_0, x_1)$$

so that $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$ for some $k \in [0, 1)$. Thus x_n is a Cauchy sequence in X. Since X is complete, there exists $x' \in X$ such that $x_n \to x'$ as $n \to \infty$.

Again, with $x = x'$ and $y = x_n$, the inequality (A) gives

$$\begin{aligned} d(Tx, x_{n+1}) &= d(Tx', Tx_n) \\ &\leq \alpha\left(d(x', x_n), d(x', Tx'), d(x_n, Tx_n)\right) \\ &= \alpha\left(d(x', x_n), d(x', Tx'), d(x_n, x_{n+1})\right), \end{aligned}$$

for all $n \in N$.

By allowing $n \to \infty$ and using the continuity of α and metric d, we get

$$d(Tx', x') \leq \alpha\left(d(x', x'), d(x', Tx'), d(x', x')\right)$$

and hence $d(Tx', x') \leq k0 = 0$. Thus $Tx' = x'$.

Now, if $w \in X$ satisfies, $Tw = w$, then by taking $x = w$ and $y = x'$ in (A) we get

$$\begin{aligned} d(w, x') &= d(Tw, x') \\ &\leq \alpha(d(w, x'), d(Tw, w), d(Tx', x')) \\ &\leq \alpha(d(w, x'), d(w, w), d(x', x')) \\ &\leq \alpha(d(w, x'), 0, 0) \\ &\leq 0. \end{aligned}$$

So that $w = x'$. □

Corollary 1. *A metric space (X, d) is complete if and only if every A-contraction on X has a fixed point in X.*

Proof. If the space X is complete then by the above theorem every A-contraction on X has a fixed point in X.

Conversely, if every A-contraction on a metric space X has a fixed point, then, in particular, every K-contraction on X has a fixed point (Notice that our term K-contraction is called Kannan contraction in [4]). Hence by the argument given in the proof of Theorem 2 of [3], the space X must be complete. □

Our next theorem extends Theorem 3 of [1] as follows.

Theorem 6. *Let $\alpha \in A$ and $\{T_n\}_{n=1}^{\infty}$ be a sequence of self-maps on the complete metric space (X, d) such that*

$$(A') \quad d(T_i x, T_j y) \leq \alpha(d(x, y), d(T_i x, x), d(T_j y, y))$$

for all x, y in X. Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X.

Proof. Taking any $x_0 \in X$. For each $n \in N$, we define $x_n = T_n x_{n-1}$. Since $\alpha \in A$, we get from (A') that

$$(1) \quad \begin{aligned} d(x_1, x_2) &= d(T_1 x_0, T_2 x_1) \\ &\leq \alpha(d(x_0, x_1), d(x_0, T_1 x_0), d(x_1, T_2 x_1)) \\ &= \alpha(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2)) \\ &\leq k d(x_0, x_1) \end{aligned}$$

for some $k \in [0, 1)$. Similarly,

$$(2) \quad \begin{aligned} d(x_2, x_3) &= d(T_2 x_1, T_3 x_2) \\ &\leq \alpha(d(x_1, x_2), d(x_1, T_2 x_1), d(x_2, T_3 x_2)) \\ &= \alpha(d(x_1, x_2), d(x_1, x_2), d(x_2, x_3)) \\ &\leq k d(x_1, x_2). \end{aligned}$$

We get from (1) and (2) that

$$d(x_1, x_2) \leq k^2 d(x_0, x_1).$$

In general, we get

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

for some $k \in [0, 1)$. Therefore, $\{x_n\}$ is a Cauchy sequence in the complete metric space X, so it converges to x' in X. Next,

$$\begin{aligned} d(x', T_n x') &\leq d(x', x_{m+1}) + d(x_{m+1}, T_n x') \\ &= d(x', x_{m+1}) + d(T_{m+1} x_m, T_n x') \\ &\leq d(x', x_{m+1}) + \alpha(d(x_m, x') + d(T_{m+1} x_m, x_m), d(T_n x', x')) \text{ (by } (A')) \\ &\leq d(x', x_{m+1}) + \alpha(d(x_m, x'), d(x_{m+1}, x_m), d(T_n x', x')) \end{aligned}$$

for all m, n in N. If m tends to ∞ then the above inequalities give that

$$\begin{aligned} d(x', T_n x') &\leq d(x', x') + \alpha(d(x', x'), d(x', x'), d(T_n x', x')) \\ &\leq \alpha(0, 0, d(T_n x', x')) \\ &\leq 0 \end{aligned}$$

and hence $T_n x' = x'$, $\forall n \in N$.

For uniqueness of the fixed point x', we suppose $T_n y = y$ for some $y \in X$. Then by (A'),

$$\begin{aligned} d(x', y) &= d(T_i x', T_j y) \\ &\leq \alpha(d(x', y), d(T_i x', x'), d(y, T_j y)) \\ &= \alpha(d(x', y), 0, 0) \\ &\leq 0 \end{aligned}$$

implies $x' = y$. □

Next theorem describes common fixed point of two self-maps on X having two related metrics. This result generalizes Theorem 4 of [1].

Theorem 7. *Let X be a set with two metrics d and δ satisfying the following conditions:*

(i) $d(x, y) \leq \delta(x, y)$ for all x, y in X;

(ii) X is complete with respect to d;

(iii) S, T are self-maps on X such that T is continuous with respect to d and

$$\delta(Tx, Sy) \leq \alpha(\delta(x, y), \delta(x, Tx), \delta(y, Ty))$$

for all x, y in X and for some $\alpha \in A$.

Then S and T have a unique common fixed point.

Proof. Take any $x_0 \in X$. For each $n \in N$, we define

$$x_n = \begin{cases} Sx_{n-1} & \text{if } n \text{ is even;} \\ Tx_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Then, by inequality in the above condition (iii) we get

$$\begin{aligned} \delta(x_1, x_2) &\leq \delta(Tx_0, Sx_1) \\ &\leq \alpha(\delta(x_0, x_1), \delta(x_0, Tx_0), \delta(x_1, Sx_1)) \\ &= \alpha(\delta(x_0, x_1), \delta(x_0, x_1), \delta(x_1, x_2)) \leq k\delta(x_0, x_1) \end{aligned}$$

for some $k \in [0, 1)$ as $\alpha \in A$. In general, for any $n \in N$ we get (as in the proof of the previous theorem) that $\delta(x_n, x_{n+1}) \leq k^n \delta(x_0, x_1)$ for some $k \in [0, 1)$. This, by condition (iii), gives

$$d(x_n, x_{n+1}) \leq \delta(x_n, x_{n+1}) \leq k^n \delta(x_0, x_1)$$

for all $n \in N$ with $k \in [0, 1)$. So, x_n is a Cauchy sequence in X with respect to d and hence by condition (ii), $d(x_n, x') \to 0$ for some $x' \in X$.
Since T is given to be continuous with the respect to d we have

$$0 = \lim_{n \to \infty} d(x_{2n-1}, x') = \lim_{n \to \infty} d(Tx_{2n}, x') = d(Tx', x')$$

So that $Tx' = x'$.

Now, by condition (iii)

$$\begin{aligned} \delta(x', Sx') &= \delta(Tx', Sx') \\ &\leq \alpha(\delta(x', x'), \delta(x', Tx'), \delta(x', Sx')) \\ &\leq \alpha(0, 0, \delta(x', Sx')) \\ &\leq 0 \end{aligned}$$

since $\alpha \in A$. Hence $x' = Sx'$. Thus x' is a common fixed point of S and T.

For the uniqueness, let y be any common fixed point of S and T in X. Then by condition (iii),

$$\delta(x', y) = \delta(Tx', Sy) \leq \alpha(\delta(x', y), \delta(x', Tx'), \delta(y, Sy)) \leq \alpha(\delta(x', y), 0, 0) \leq 0,$$

so that $x = y$. □

References

[1] Ahmad, B., Rehman, F. U., Some fixed point theorems in complete metric spaces. Math. Japonica 36 (2) (1991), 239-243.

[2] Bianchini, R., Su un problema di S. Reich riguardante la teori dei punti fissi. Boll. Un. Math. Ital. 5 (1972), 103-108.

[3] Chuanyi, Z., The generalized set-valued contraction and completeness of the metric space. Math. Japonica 35 (1) (1990), 111-118.

[4] Kannan, T., Some results on fixed points. Bull. Culcutta Math Soc. 60 (1968), 71-76.

[5] Khan, M. S., Kubiaczyk, I., Fixed point theorems for point to set maps. Math. Japonica 33 (3) (1988), 409-415.

[6] Khan, M. S., On fixed point theorems. Math. Japonica 23 (2) (1978/79), 201-204.

[7] Reich, S., Kannan, T., Fixed point theorem. Boll. Un. Math. Ital. 4 (1971), 1-11.

[8] Shioji, N., Suzuki, T., Takahashi, W., Contractive mappings, Kannan mappings and metric completeness. Proc. Amer. Math. Soc. 10 (1998), 3117-3124.

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