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ON ENDOMORPHISM MONOIDS OF PARTIAL ORDERS AND CENTRAL RELATIONS[1]

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Abstract. In this paper we characterize pairs of Rosenberg relations (ρ, σ) with the property that the endomorphism monoid of one of the relations is properly contained in the endomorphism monoid of the other relation. We focus on the situations where one of the relations is a partial order, or a central relation.

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1. Introduction

In this paper we continue the work from [4, 5, 6, 7] on the description of all pairs (ρ, σ) of Rosenberg relations satisfying $\mathrm{End}\{\rho\} \subset \mathrm{End}\{\sigma\}$. Here, we focus on the situations where one of the relations is a partial order, or a central relation. Results presented here, together with the results of [4, 5, 6, 7] cover all but two cases (see Table 1), bringing us, therefore, closer to the complete characterization.

This line of research was motivated by the paper [1], where the completeness for some special structures (concrete near-rings) was investigated using techniques from clone theory. It appeared that unary parts (traces) of the maximal clones that contain the operation $+$ correspond to the maximal near-rings containing the identity map. Moreover, if for every two distinct unary parts $M_i^{(1)}$ and $M_j^{(1)}$ of such maximal clones we have $M_i^{(1)} \not\subseteq M_j^{(1)}$, then every unary part is a maximal near–ring. It is natural to ask what goes on in the general case, i.e. what the relationship between any two traces of maximal clones on a finite set is. As it was expected, the width of this poset is doubly exponential, but it was rather surprising to find out that its height is equal to the size of the underlying set. Moreover, it turns out that the structure of this poset is quite rich.

2. Preliminaries

Throughout the paper we assume that A is a finite set and $|A| \geqslant 3$. Let $O_A^{(n)}$ denote the set of all n-ary operations on A (so that $O_A^{(1)} = A^A$) and let

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$O_A = \bigcup_{n\geqslant 1} O_A^{(n)}$ denote the set of all finitary operations on A. For $F \subseteq O_A$ let $F^{(n)} = F \cap O_A^{(n)}$ be the set of all n-ary operations in F. A set $C \subseteq O_A$ of finitary operations is a *clone of operations on* A if it contains all projection maps $\pi_i^n : A^n \to A : (x_1, \ldots, x_n) \mapsto x_i$ and is closed with respect to composition of functions in the following sense: whenever $g \in C^{(n)}$ and $f_1, \ldots, f_n \in C^{(m)}$ for some positive integers m and n then $g(f_1, \ldots, f_n) \in C^{(m)}$, where the composition $h = g(f_1, \ldots, f_n)$ is defined by $h(x_1, \ldots, x_m) = g(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$.

For a clone C, the unary part $C^{(1)}$ of C, will be referred to as *the trace of* C.

Let $R_A^{(n)}$ denote the set of all n-ary relations on A (including the empty relation) and let $R_A = \bigcup_{n\geqslant 1} R_A^{(n)}$ denote the set of all finitary relations on A. If ρ is a nonempty relation in $R_A^{(n)}$ we say that its *arity* is n and write $\mathrm{ar}(\rho) = n$. The arity of the empty relation is undefined.

We say that an n-ary operation f preserves an h-ary relation ρ if the following holds:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{h1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{h2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{hn} \end{bmatrix} \in \rho \text{ implies } \begin{bmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ f(a_{h1}, a_{h2}, \ldots, a_{hn}) \end{bmatrix} \in \rho.$$

For a set Q of relations and for a set F of operations let

$$\begin{aligned} \mathrm{Pol}_A\, Q &= \{f \in O_A : f \text{ preserves every } \rho \in Q\} \\ \mathrm{Inv}_A\, F &= \{\rho \in R_A : \text{every } f \in F \text{ preserves } \rho\}. \end{aligned}$$

Let $\mathrm{Pol}_A^{(n)}\, Q = (\mathrm{Pol}_A\, Q) \cap O_A^{(n)}$. For an h-ary relation $\theta \subseteq A^h$ and a unary operation $f \in A^A$ it is convenient to write

$$f(\theta) = \{(f(x_1), \ldots, f(x_h)) : (x_1, \ldots, x_h) \in \theta\}.$$

Then clearly f preserves θ if and only if $f(\theta) \subseteq \theta$. It follows that $\mathrm{Pol}_A^{(1)}\, Q$ is the endomorphism monoid of the relational structure (A, Q). Therefore instead of $\mathrm{Pol}_A^{(1)}\, Q$ we simply write $\mathrm{End}_A\, Q$. We shall omit the subscript A in $\mathrm{Pol}_A\, Q$, $\mathrm{End}_A\, Q$ and $\mathrm{Inv}_A\, F$ and write simply $\mathrm{Pol}\, Q$, $\mathrm{End}\, Q$ and $\mathrm{Inv}\, F$. If, however, $F \subseteq O_B$ and $Q \subseteq R_B$ for some $B \subset A$ we shall keep the subscript and write $\mathrm{Pol}_B\, Q$, $\mathrm{End}_B\, Q$ and $\mathrm{Inv}_B\, F$ to indicate that we restrict our attention to operations and relations on B.

For a relation ρ of arity k, let ρ^{irr} denote the *irreflexive part of* ρ, that is, the set of all $(x_1, \ldots, x_k) \in \rho$ such that $x_i \neq x_j$ for all $i \neq j$. A relation ρ is *irreflexive* if $\rho = \rho^{\mathrm{irr}}$.

Let $B \subseteq A$. We say that $f \in O_A^{(1)}$ *irreflexively preserves* an irreflexive relation $\rho \in R_B^{(k)}$ if whenever $(b_1, \ldots, b_k) \in \rho$ and $(f(b_1), \ldots, f(b_k)) \in (B^k)^{\mathrm{irr}}$ then $(f(b_1), \ldots, f(b_k)) \in \rho$. We say that an irreflexive relation ρ is an *irreflexive* *invariant of* $F \subseteq O_A^{(1)}$ *on* B if every operation in F irreflexively preserves ρ. The set of all irreflexive invariants of F on B will be denoted by $\mathrm{Irr}_B\, F$.

If the underlying set is finite and has at least three elements, then the lattice of clones has cardinality $2^{\aleph_0}$. However, one can show that the lattice of clones on a finite set has a finite number of coatoms, called *maximal clones*, and that every clone distinct from O_A is contained in one of the maximal clones. One of the most influential results in clone theory is the explicite characterization of the maximal clones, obtained by I. G. Rosenberg as the culmination of the work of many mathematicians. It is usually stated in terms of the following six classes of finitary relations on A (the so-called *Rosenberg relations*).

(R1) *Bounded partial orders.* These are partial orders on A with a least and a greatest element.

(R2) *Nontrivial equivalence relations.* These are equivalence relations on A distinct from $\Delta_A := \{(x,x) : x \in A\}$ and A^2.

(R3) *Permutational relations.* These are relations of the form $\{(x,\pi(x)) : x \in A\}$, where π is a fixpoint-free permutation of A with all cycles of the same length p, where p is a prime.

(R4) *Affine relations.* For a binary operation $\oplus$ on A let

$$\lambda_\oplus := \{(x,y,u,v) \in A^4 : u \oplus v = x \oplus y\}.$$

A relation ρ is called *affine* if there is an elementary abelian p-group $(A,\oplus,\ominus,0)$ on A such that $\rho = \lambda_\oplus$.

Suppose now that A is an elementary abelian p-group. Then it is well-known that $f \in \mathrm{Pol}\{\lambda_\oplus\}$ if and only if

$$f(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = f(x_1,\ldots,x_n) \oplus f(y_1,\ldots,y_n) \ominus f(0,\ldots,0)$$

for all $x_i, y_i \in A$. In case f is unary, this condition becomes

$$f(x \oplus y) = f(x) \oplus f(y) \ominus f(0).$$

(R5) *Central relations.* All unary relations are central relations. For central relations ρ of arity $h \geqslant 2$ the definition is as follows: ρ is said to be *totally symmetric* if $(x_1,\ldots,x_h) \in \rho$ implies $(x_{\pi(1)},\ldots,x_{\pi(h)}) \in \rho$ for all permutations π, and it is said to be *totally reflexive* if $(x_1,\ldots,x_h) \in \rho$ whenever there are $i \neq j$ such that $x_i = x_j$. An element $c \in A$ is *central* if $(c,x_2,\ldots,x_h) \in \rho$ for all $x_2,\ldots,x_h \in A$. Finally, $\rho \neq A^h$ is called *central* if it is totally reflexive, totally symmetric and has a central element.

(R6) *h-regular relations.* Let $\Theta = \{\theta_1,\ldots,\theta_m\}$ be a family of equivalence relations on A. We say that Θ is an *h-regular family* if every θ_i has precisely h blocks, and additionally, if B_i is an arbitrary block of θ_i for $i \in \{1,\ldots,m\}$, then $\bigcap_{i=1}^m B_i \neq \emptyset$.

An h-ary relation $\rho \neq A^h$ is *h-regular* if $h \geqslant 3$ and there is an h-regular family Θ such that $(x_1, \dots, x_h) \in \rho$ if and only if for all $\theta \in \Theta$ there are distinct i, j with $x_i \theta x_j$.

Note that regular relations are totally reflexive and totally symmetric.

Theorem 2.1. (Rosenberg [8]) *A clone M of operations on a finite set is maximal if and only if there is a relation ρ from one of the classes (R1)–(R6) such that $M = \mathrm{Pol}\{\rho\}$.*

Every central relation ρ can be written as $C_\rho \cup R_\rho \cup T_\rho$, where C_ρ is *the central part of ρ* which consists of all the tuples of distinct elements containing at least one central element, R_ρ is the *reflexive part of ρ* which consists of all the tuples $(x_1, \dots, x_k)$ such that there are $i \neq j$ with $x_i = x_j$, and T_ρ is *the tail of ρ* and consists of all the tuples $(x_1, \dots, x_k)$ such that $x_1, \dots, x_k$ are distinct non-central elements. The center of ρ will be denoted by $Z(\rho)$. This is the set of all central elements of ρ. For a relation ρ by $\mathrm{dom}(\rho)$ we denote the *domain of ρ*, that is, the set D such that $\rho \subseteq D^{\mathrm{ar}(\rho)}$. For most relations in this paper the domain is obvious. However, for the tail T_ρ of a central relation ρ we set $\mathrm{dom}(T_\rho) = A \setminus Z(\rho)$.

There is another way to look at regular relations. Given a finite set A, $|A| \geqslant 3$, and an h-regular family $\Theta = \{\theta_1, \dots, \theta_m\}$ on A, let

$$R_\Theta = \{(x_1, \dots, x_h) : (\forall \theta \in \Theta)(\exists i \neq j) x_i \theta x_j\}$$

denote the corresponding h-regular relation. We define the *elementary (h, m)-relation* $\Psi_{h,m}$ on $\{1, \dots, h\}^m$ in the following way:

$$\Psi_{h,m} = \left\{ \left(\begin{bmatrix} a_1^1 \\ \vdots \\ a_m^1 \end{bmatrix}, \dots, \begin{bmatrix} a_1^h \\ \vdots \\ a_m^h \end{bmatrix} \right) : (\forall i \in \{1, \dots, m\})(\exists j \neq k) a_i^j = a_i^k \right\}.$$

Note that the elementary (h, m)-relation is the h-regular relation on $\{1, \dots, h\}^m$ defined by the h-regular family $\Theta^* = \{\theta_1^*, \dots, \theta_m^*\}$, where

$$\theta_i^* = \left\{ \left(\begin{bmatrix} b_1^1 \\ \vdots \\ b_m^1 \end{bmatrix}, \begin{bmatrix} b_1^2 \\ \vdots \\ b_m^2 \end{bmatrix} \right) : b_i^1 = b_i^2 \right\}.$$

Then, there exists a surjective mapping $\lambda : A \to \{1, \dots, h\}^m$ such that

$$R_\Theta = \{(x_1, \dots, x_h) : (\lambda(x_1), \dots, \lambda(x_h)) \in \Psi_{h,m}\}.$$

Conversely, for every surjective mapping $\lambda : A \to \{1, \dots, h\}^m$ the relation

$$\{(x_1, \dots, x_h) : (\lambda(x_1), \dots, \lambda(x_h)) \in \Psi_{h,m}\}$$

is an h-regular relation.

The complete characterization of all mappings that preserve regular relations can be found in [3]. We present this result without proof.

Denote by $\overline{x}^{(i)}$ the i-th coordinate of the tuple $\overline{x} \in \{1, \dots, h\}^m$. Let f be an n-ary function on the set A. We define the function $f_i : A^n \to \{1, \dots, h\}$ in the following way:

$$f_i(x_1, \dots, x_n) := \lambda_i(f(x_1, \dots, x_n)),$$

where $\lambda_i(x) = \lambda(x)^{(i)} = \pi_i \circ \lambda(x)$.

Proposition 2.2. ([3]) *An n-ary function f on a set A preserves an h-regular relation R_Θ if and only if for each i either f_i has at most $h-1$ distinct values or there exist a permutation s on $\{1, \dots, h\}$, a $j \in \{1, \dots, n\}$ and a $v \in \{1, \dots, m\}$ such that*

$$f_i(x_1, \dots, x_n) = s(\lambda_v(x_j)).$$

The results we have obtained up to now, including the results obtained in this paper, are summarized in Table 1. The entries in this table are to be interpreted in the following way:

- we write $-$ if $\operatorname{End} \rho \nsubseteq \operatorname{End} \sigma$ for every pair (ρ, σ) of relations of the indicated type;
- we write $+$ if there is a complete characterization of the situation $\operatorname{End} \rho \subseteq \operatorname{End} \sigma$;
- we write ? if there is no definite answer, but the reference below the question mark contains some partial results.

3. Bounded partial orders

Let $(A, \leqslant)$ be a partially ordered set. Recall that $S \subseteq A$ is a retract of A if there is an idempotent monotonous map $f : A \to A$ such that $f(A) = S$. We shall say that $S \subseteq A$ is a *pseudoretract* of A if there is a (not necessarily idempotent) monotonous map $f : A \to A$ such that $f(A) = S$. Thus, pseudoretracts of A are nothing but substructures of A that are at the same time homomorphic images of A. If $\rho \subseteq S^k$ and $\theta \subseteq A^k$ are arbitrary relations, we say that (S, ρ) is a *pseudoretract* of (A, θ) if there is a monotonous map $f : A \to A$ such that $f(A) = S$ and $f(\theta) = \rho$.

We start with the characterization of the relationship between bounded partial orders and regular relations. In [5] we obtained the following partial result:

Proposition 3.1. ([5] Proposition 4.25) *Let $\leqslant$ be a bounded partial order. If R_Θ is an h-regular relation defined by $\Theta = \{\theta_1, \dots, \theta_m\}$ where $m \geqslant 2$, then* $\operatorname{End}(\leqslant) \nsubseteq \operatorname{End} R_\Theta$.

Therefore, if $\operatorname{End}(\leqslant) \subseteq \operatorname{End} R_\Theta$ then $\Theta = \{\theta\}$. We shall now complete the characterization. Let θ be an equivalence relation on A and let S be a subset of A. We write S/θ to denote $S/(\theta \cap S^2)$, the set of equivalence classes of the restriction of θ to S.

ρ \ σ	Bounded partial order	Equivalence relation	Permutational relation	Affine relation	Unary central relation	k–ary central relation, $k \geqslant 2$	h–regular relation
Bounded partial order	− [5]	− [5]	− [5]	− [5]	− [5]	+ Prop. 3.4	+ Prop. 3.2
Equivalence relation	− [5]	− [5]	− [5]	− [5]	− [5]	− [7]	+ [4]
Permutational relation	− [5]	+ [5]	− [5]	+ [5]	− [5]	− [7]	+ [5]
Affine relation	− [5]	− [5]	− [5]	− [5]	− [5]	− [7]	+ [5]
Unary central relation	− [7]	+ [7]	− [7]	− [7]	− [7]	+ [7]	+ [7]
k–ary central relation, $k \geqslant 2$	− [7]	+ [7]	− [7]	− [7]	− [7]	+ Prop. 4.6	+ Prop. 4.10
h–regular relation	− [5]	− [5]	− [5]	− [5]	− [5]	? [5]	? [4]

Table 1: Summary of some partial results for $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$

Proposition 3.2. *Let $(A, \leqslant)$ be a partially ordered set, let Θ be an h-regular family, $h \geqslant 3$, and R_Θ the regular relation generated by the regular family Θ. Then* $\operatorname{End}(\leqslant) \subseteq \operatorname{End} R_\Theta$ *if and only if $\Theta = \{\theta\}$ and for every pseudoretract S of A, if $|S/\theta| = h$ and (S, ρ) is a pseudoretract of (A, θ) then $\rho = \theta|_S$.*

Proof. ($\Rightarrow$) Let $\operatorname{End}(\leqslant) \subseteq \operatorname{End} R_\Theta$. Then by Proposition 3.1 we have $\Theta = \{\theta\}$ for some equivalence relation θ. Let (S, ρ) be a pseudoretract of (A, θ) and let $|S/\theta| = h$. Then there is a monotonous map $f : A \to A$ such that $f(A) = S$ and $f(\theta) = \rho$. Let $B_1, \ldots, B_h$ be the blocks of θ. Since $f \in \operatorname{End}(\leqslant) \subseteq \operatorname{End} R_\Theta$ and $|f(A)/\theta| = |S/\theta| = h$ it follows from Proposition 2.2 that there exists a permutation $\alpha : \{1, \ldots, h\} \to \{1, \ldots, h\}$ such that $f(B_i) \subseteq B_{\alpha(i)}$ for all i. So, $f \in \operatorname{End} \theta$ and hence $\rho = f(\theta) \subseteq \theta|_S$. Let us show that $f(\theta) = \theta|_S$. Take any $(u, v) \in \theta|_S$. Then $u, v \in B_{\alpha(j)}$ for some j. Since $f(B_i) \subseteq B_{\alpha(i)}$ for all i, from $u, v \in S = f(A)$ it follows that there exist $p, q \in B_j$ such that $(f(p), f(q)) = (u, v)$. Therefore, $\rho = f(\theta) = \theta|_S$.

($\Leftarrow$) Assume that $\Theta = \{\theta\}$ and that for every pseudoretract S of A, if $|S/\theta| = h$ and (S, ρ) is a pseudoretract of (A, θ) then $\rho = \theta|_S$. Take any $f \in \operatorname{End}(\leqslant)$. If $|f(A)/\theta| < h$ then f obviously preserves R_Θ, so assume that $|f(A)/\theta| = h$. Let $S = f(A)$ and $\rho = f(\theta)$. Then (S, ρ) is a pseudoretract of (A, θ) whence $\rho = \theta|_S$. So, f preserves θ and consequently $f \in \operatorname{End} R_\Theta$ since $\Theta = \{\theta\}$. □

As for the relationship between endomorphism monoids of bounded parital orders and central relations, in [7] and [5] we obtained the following:

Proposition 3.3. (Proposition 4.10 [7], Proposition 4.19 [5]) *Let $(A, \leqslant)$ be a partially ordered set. If ρ is a unary central relation then* $\operatorname{End}(\leqslant) \not\subseteq \operatorname{End} \rho$. *If ρ is a binary central relation then* $\operatorname{End}(\leqslant) \subseteq \operatorname{End} \rho$ *if and only if $\leqslant$ is not a chain and $\rho = (\leqslant) \cup (\leqslant)^{-1}$.*

We shall now consider central relations of arity $h \geqslant 3$. For $\emptyset \neq D \subset A$ let $\operatorname{Cen}^h_A(D) = \{(x_1, \ldots, x_h) \in (A^h)^{\mathrm{irr}} : \text{there exists an } i \text{ such that } x_i \in D\}$.

Proposition 3.4. *Let $\leqslant$ be a bounded partial order and σ a central relation of arity $h \geqslant 3$. Then* $\operatorname{End}(\leqslant) \subseteq \operatorname{End} \sigma$ *if and only if*

- $T_\sigma \in \operatorname{Irr}_{A \setminus Z(\sigma)} \operatorname{End}(\leqslant)$, *and*
- *for every pseudoretract S of A, if $|S \setminus Z(\sigma)| \geqslant h$ and (S, D) is a pseudoretract of $(A, Z(\sigma))$ then* $\operatorname{Cen}^h_{S \setminus Z(\sigma)}(D) \subseteq T_\sigma$.

Proof. ($\Rightarrow$) Let $B = A \setminus Z(\sigma)$. Suppose that $T_\sigma \notin \operatorname{Irr}_B \operatorname{End}(\leqslant)$. Then there exist $f \in \operatorname{End}(\leqslant)$ and $(b_1, \ldots, b_h) \in T_\sigma$ such that $(f(b_1), \ldots, f(b_h)) \in (B^h)^{\mathrm{irr}}$ but $(f(b_1), \ldots, f(b_h)) \notin T_\sigma$. Let us show that in this case f does not preserve σ. It is obvious that $(b_1, \ldots, b_h) \in T_\sigma \subseteq \sigma$. On the other hand, $(f(b_1), \ldots, f(b_h)) \notin \sigma$ since the tuple consists of distinct noncentral elements but it does not belong to T_σ. Therefore, $f \in \operatorname{End}(\leqslant) \setminus \operatorname{End} \sigma$.

Suppose now that there is a pseudoretract (S, D) of $(A, Z(\sigma))$ such that $|S \setminus Z(\sigma)| \geqslant h$ but $\mathrm{Cen}^h_{S\setminus Z(\sigma)}(D) \nsubseteq T_\sigma$. Let f be a monotonous map such that $f(A) = S$ and $f(Z(\sigma)) = D$, and take any $(b_1, \dots, b_h) \in \mathrm{Cen}^h_{S\setminus Z(\sigma)}(D) \setminus T_\sigma$. Note that $(b_1, \dots, b_h) \notin \sigma$ since the tuple consists of distinct noncentral elements but it does not belong to T_σ. Also, there is an i such that $b_i \in D$, say $b_1 \in D$. Since $f(A) = S$ and $f(Z(\sigma)) = D$ there exist $c \in Z(\sigma)$ and $a_2, \dots, a_h \in A$ such that $f(c) = b_1$ and $f(a_i) = b_i$ for all $i \geqslant 2$. Now, $(c, a_2, \dots, a_h) \in \sigma$ but $(f(c), f(a_2), \dots, f(a_h)) = (b_1, b_2, \dots, b_h) \notin \sigma$, so $f \in \mathrm{End}(\leqslant) \setminus \mathrm{End}\,\sigma$.

($\Leftarrow$) Suppose that $\mathrm{End}(\leqslant) \nsubseteq \mathrm{End}\,\sigma$ and take any $f \in \mathrm{End}(\leqslant) \setminus \mathrm{End}\,\sigma$. Then there is a $(a_1, \dots, a_h) \in \sigma$ such that $(b_1, \dots, b_h) \notin \sigma$, where $b_i = f(a_i)$ for all i. Therefore, all the b_i's are distinct noncentral elements and consequently all the a_i's are distinct. If $(a_1, \dots, a_h) \in T_\sigma$ then $T_\sigma \notin \mathrm{Irr}_{A\setminus Z(\sigma)} \mathrm{End}(\leqslant)$ since $f \in \mathrm{End}(\leqslant)$, all the b_i's are distinct noncentral elements but $(b_1, \dots, b_h) \notin T_\sigma$. Assume now that $(a_1, \dots, a_h) \in C_\sigma$. Let $S = f(A)$ and $D = f(Z(\sigma))$. Then clearly (S, D) is a pseudoretract of $(A, Z(\sigma))$, but $(b_1, \dots, b_h) \in \mathrm{Cen}^h_{S\setminus Z(\sigma)}(D) \setminus T_\sigma$. □

If σ is a central relation with $T_\sigma = \emptyset$ we have the following simpler version of the above result:

Proposition 3.5. *Let σ be a central relation of arity $h \geqslant 3$ such that $T_\sigma = \emptyset$. Then $\mathrm{End}(\leqslant) \subseteq \mathrm{End}\,\sigma$ if and only if for every pseudoretract S of A, if $|S \setminus Z(\sigma)| \geqslant h$ and (S, D) is a pseudoretract of $(A, Z(\sigma))$ then $D \subseteq Z(\sigma)$.*

Proof. ($\Rightarrow$) Let $\mathrm{End}(\leqslant) \subseteq \mathrm{End}\,\sigma$, let S be a pseudoretract of A such that $|S \setminus Z(\sigma)| \geqslant h$ and let (S, D) be a pseudoretract of $(A, Z(\sigma))$. Then there is a monotonous map f such that $f(A) = S$ and $f(Z(\sigma)) = D$. Since $f \in \mathrm{End}(\leqslant) \subseteq \mathrm{End}\,\sigma$ and $|f(A) \setminus Z(\sigma)| \geqslant h$ it follows that $f(Z(\sigma)) \subseteq Z(\sigma)$. Therefore, $D \subseteq Z(\sigma)$.

($\Leftarrow$) Take any $f \in \mathrm{End}(\leqslant)$, let $S = f(A)$ and $D = f(Z(\sigma))$. Then (S, D) is a pseudoretract of $(A, Z(\sigma))$. If $|f(A) \setminus Z(\sigma)| < h$ then it is easy to see that f preserves σ. If, however, $|f(A) \setminus Z(\sigma)| \geqslant h$ then according to the assumption $D \subseteq Z(\sigma)$ since (S, D) is a pseudoretract of $(A, Z(\sigma))$. Therefore, $f(Z(\sigma)) \subseteq Z(\sigma)$ and consequently f preserves σ. □

4. Central relations

In [7] we solved the following special case of the problem of comparing endomorphism monoids of central relations:

Proposition 4.1. *Let ρ and σ be distinct central relations.*

(*a*) *If σ is a unary central relation then* $\mathrm{End}\,\rho \nsubseteq \mathrm{End}\,\sigma$ *[7, Lemma 3.2].*

(*b*) *Suppose that* $\mathrm{ar}(\sigma) \geqslant 2$ *and* $T_\rho = \emptyset$. *Then* $\mathrm{End}\,\rho \subseteq \mathrm{End}\,\sigma$ *if and only if* $\mathrm{ar}(\rho) < \mathrm{ar}(\sigma)$, $Z(\rho) = Z(\sigma)$ *and* $T_\sigma = \emptyset$ *[7, Proposition 4.1].*

This proposition handles the cases where at least one of the central relations is unary, or $T_\rho = \emptyset$. In this section we consider the remaining case where $\mathrm{ar}(\rho) \geqslant 2$, $\mathrm{ar}(\sigma) \geqslant 2$ and $T_\rho \neq \emptyset$.

The main problem in the analysis of central relations comes from the fact that the tail of a central relation can be arbitrary (actually, any totally symmetric irreflexive relation). Hence, if ρ and σ are central relations and $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$ then T_σ should be a sort of an invariant of T_ρ, but this does not mean that $T_\sigma \in \operatorname{Inv}\operatorname{End} T_\rho$.

The paper [6] raises an intriguing point in the analysis of the structure of endomorphism monoids of central relations. We are now going to make the approach of [6] explicite and describe the tools in a fashion of Fraïssé-type analysis of first-order structures that will enable us to precisely formulate the feeling that "T_σ is a sort of an invariant of T_ρ".

Let ρ be a k-ary relation and let $h > k$. We write $(a_1, \ldots, a_h) \overset{\rho}{\hookrightarrow} (b_1, \ldots, b_h)$ to denote that all $a_1, \ldots, a_h$ are distinct, all $b_1, \ldots, b_h$ are distinct, $\{a_1, \ldots, a_h, b_1, \ldots, b_h\} \subseteq \mathrm{dom}(\rho)$, and whenever $i_1, \ldots, i_k \in \{1, \ldots, h\}$ are k distinct indices such that $(a_{i_1}, \ldots, a_{i_k}) \in \rho$, then $(b_{i_1}, \ldots, b_{i_k}) \in \rho$. We say that σ is a *Fraïssé filter over* ρ if

- $\mathrm{ar}(\sigma) > \mathrm{ar}(\rho)$, and
- if $(a_1, \ldots, a_h) \in \sigma$ and $(a_1, \ldots, a_h) \overset{\rho}{\hookrightarrow} (b_1, \ldots, b_h) \in \mathrm{dom}(\sigma)^h$ then $(b_1, \ldots, b_h) \in \sigma$.

Recall that a structure is *homogeneous* if every isomorphism between two finite substructures extends to an automorphism of the structure. Recently, P. Cameron and J. Nešetřil introduced a notion of homomorphism-homogeneity of structures as a straightforward generalization of homogeneity [2]. Let (A, ρ) be a relational structure. A *local homomorphism of* (A, ρ) is a homomorphism $f : (S, \rho|_S) \to (T, \rho|_T)$, where $S, T \subseteq A$. We say that the structure (A, ρ) is *homomorphism-homogeneous* if every local homorphism extends to an endomorphism, that is, if for every homorphism $f : (S, \rho|_S) \to (T, \rho|_T)$, where $S, T \subseteq A$, there is an endomorphism $g \in \mathrm{End}(A, \rho)$ such that $f = g|_S$. We say that *a relation ρ is homomorphism-homogeneous* if (A, ρ) is homomorphism-homogeneous.

Lemma 4.2. *Let ρ be a homomorphism-homogeneous relation of arity $k \geqslant 2$ and let σ be a totally reflexive relation of arity $h > k$ with* $\mathrm{dom}(\rho) = \mathrm{dom}(\sigma)$. *The following are equivalent*

(1) $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$,

(2) *σ is a Fraïssé filter over ρ,*

(3) $\sigma^{\mathrm{irr}} \in \operatorname{Irr}\operatorname{End}\rho$.

Proof. (1) $\Rightarrow$ (2): Suppose $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$. Take any $(a_1, \ldots, a_h) \in \sigma$ and suppose $(a_1, \ldots, a_h) \overset{\rho}{\hookrightarrow} (b_1, \ldots, b_h)$. Let $S = \{a_1, \ldots, a_h\}$, $T = \{b_1, \ldots, b_h\}$ and $f : S \to T : a_i \mapsto b_i$. Then f is a local homomorphism of (A, ρ), and by homomorphism-homogeneity of ρ it extends to an endomorphism $g \in \operatorname{End}\rho$. So, $g \in \operatorname{End}\sigma$, and consequently $(g(a_1), \dots, g(a_h)) \in \sigma$. But $g(a_i) = f(a_i) = b_i$, for all i. Therefore, $(b_1, \dots, b_h) \in \sigma$.

(2) $\Rightarrow$ (3): Let σ be a Fraïssé filter over ρ and let us show that every $g \in \operatorname{End}\rho$ irreflexively preserves σ^{irr}. Take any $g \in \operatorname{End}\rho$ and any $(a_1, \dots, a_h) \in \sigma^{\text{irr}}$ such that the $g(a_i)$'s are all distinct. Since $g \in \operatorname{End}\rho$ it follows that $(a_1, \dots, a_h) \overset{\rho}{\hookrightarrow} (g(a_1), \dots, g(a_h))$, whence $(g(a_1), \dots, g(a_h)) \in \sigma^{\text{irr}}$ since σ is a Fraïssé filter over ρ.

(3) $\Rightarrow$ (1): Assume that $\sigma^{\text{irr}} \in \operatorname{Irr}\operatorname{End}\rho$, take any $g \in \operatorname{End}\rho$ and let us show that $g \in \operatorname{End}\sigma$. Take any $(a_1, \dots, a_h) \in \sigma$ and let $b_i = g(a_i)$. If there are $i \neq j$ such that $b_i = b_j$ then $(b_1, \dots, b_h) \in \sigma$ since σ is totally reflexive. Otherwise, all the b_i's are distinct. Since $\sigma^{\text{irr}} \in \operatorname{Irr}\operatorname{End}\rho$ and $g \in \operatorname{End}\rho$ we obtain $(b_1, \dots, b_h) \in \sigma^{\text{irr}} \subseteq \sigma$. Therefore, $g \in \operatorname{End}\sigma$. □

Lemma 4.3. *Every central relation is homomorphism-homogeneous.*

Proof. Let ρ be a central relation and $f : (S, \rho|_S) \to (T, \rho|_T)$, where $S, T \subseteq A$, a local homomorphism of (A, ρ). Choose an arbitrary central element c of ρ and define an extension $g : A \to A$ of f by

$$g(x) = \begin{cases} f(x), & x \in S, \\ c, & \text{otherwise.} \end{cases}$$

Then it is easy to see that g preserves ρ. Take any $(a_1, \dots, a_k) \in \rho$. If $f(a_i) = f(a_j)$ for some $i \neq j$ then trivially $(f(a_1), \dots, f(a_k)) \in \rho$. If all the $f(a_i)$'s are distinct, then all the a_i's are also distinct. If there is a j such that $a_j \notin S$, then $f(a_j) = c$ and $(f(a_1), \dots, f(a_k)) \in \rho$. Finally, if $\{a_1, \dots, a_k\} \subseteq S$ then from the fact that f is a local homomorphism of (A, ρ) it follows that $(f(a_1), \dots, f(a_k)) \in \rho$. □

Corollary 4.4. *Let ρ be a central relation of arity $k \geqslant 2$ and let σ be a totally reflexive relation of arity h. Then* $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$ *if and only if $h > k$ and σ is a Fraïssé filter over ρ.*

Proof. In view of the previous two lemmas it suffices to show that $h > k$. But from [4, Proposition 5.3] we know that if ρ and σ are totally reflexive and totally symmetric distinct relations such that σ contains at least one tuple $(b_1, \dots, b_h)$ of distinct elements and $\sigma \neq A^h$, where $h = \operatorname{ar}(\sigma)$, then $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$ implies $\operatorname{ar}(\rho) < \operatorname{ar}(\sigma)$. □

Although rather general, the above statement can be used to derive certain information about central relations.

Corollary 4.5. *Let ρ and σ be central relations, $k = \operatorname{ar}(\rho) \geqslant 2$, $h = \operatorname{ar}(\sigma) \geqslant 2$. If* $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$ *then $h > k$ and $Z(\rho) \subseteq Z(\sigma)$.*

Proof. We already know that $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$ implies $h > k$, [4, Proposition 5.3]. Take any $c \in Z(\rho)$ and $a_2, \dots, a_h \in A$. If $|\{c, a_2, \dots, a_h\}| < h$ then $(c, a_2, \dots, a_h) \in \sigma$ since σ is totally reflexive. If $\{a_2, \dots, a_h\} \cap Z(\rho) \neq \emptyset$ then $(c, a_2, \dots, a_h) \in \sigma$ since the tuple contains a central element of σ. So, assume that $|\{c, a_2, \dots, a_h\}| = h$ and $\{a_2, \dots, a_h\} \cap Z(\rho) = \emptyset$. Take any $d \in Z(\sigma)$. Then all $d, a_2, \dots, a_h$ are distinct and $(d, a_2, \dots, a_h) \in \sigma$. Moreover, $(d, a_2, \dots, a_h) \overset{\rho}{\hookrightarrow} (c, a_2, \dots, a_h)$ since c is a central element of ρ. Now, from $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$ we get by Corollary 4.4 that σ is a Fraïssé filter over ρ. This together with $(d, a_2, \dots, a_h) \in \sigma$ and $(d, a_2, \dots, a_h) \overset{\rho}{\hookrightarrow} (c, a_2, \dots, a_h)$ yields $(c, a_2, \dots, a_h) \in \sigma$. □

Let θ be a k-ary relation and let $h > k$. We say that $d \in \operatorname{dom}(\theta)$ is an *h-central element* for θ if there exists a $B \subseteq \operatorname{dom}(\theta)$ such that $|B| = h - 1$, $d \notin B$ and $\{d\} \times (B^{k-1})^{\mathrm{irr}} \subseteq \theta$. We also say that B *is a set of witnesses for* d.

Proposition 4.6. *Let ρ and σ be central relations, $k = \operatorname{ar}(\rho) \geqslant 2$, $h = \operatorname{ar}(\sigma) \geqslant 2$, and $T_\rho \neq \emptyset$. Then $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$ if and only if*

- $Z(\rho) \subseteq Z(\sigma)$,
- $h > k$,
- *if $d \in \operatorname{dom}(T_\sigma)$ is an h-central element of T_ρ witnessed by $\{b_2, \dots, b_h\} \subseteq \operatorname{dom}(T_\sigma)$ then $(d, b_2, \dots, b_h) \in T_\sigma$, and*
- *$(C_\sigma \cup T_\sigma)|_{\operatorname{dom}(T_\rho)}$ is a Fraïssé filter over T_ρ.*

Proof. ($\Rightarrow$) Suppose $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$. Then $Z(\rho) \subseteq Z(\sigma)$ and $h > k$ according to Corollary 4.5.

Let $d \in \operatorname{dom}(T_\sigma)$ be an h-central element of T_ρ witnessed by $\{b_2, \dots, b_h\} \subseteq \operatorname{dom}(T_\sigma)$. Take any $c \in Z(\rho) \subseteq Z(\sigma)$ and consider

$$f(x) = \begin{cases} d, & x = c \\ x, & x \in \{b_2, \dots, b_h\} \\ c, & \text{otherwise.} \end{cases}$$

Then $f \in \operatorname{End}\rho$. Therefore, $f \in \operatorname{End}\sigma$ which together with $(c, b_2, \dots, b_h) \in \sigma$ yields $(d, b_2, \dots, b_h) \in \sigma$. Since d, b_2, $\dots$, b_h are distinct noncentral elements, we obtain $(d, b_2, \dots, b_h) \in T_\sigma$.

Take any $(a_1, \dots, a_h) \in (C_\sigma \cup T_\sigma)|_{\operatorname{dom}(T_\rho)}$ and let $(a_1, \dots, a_h) \overset{T_\rho}{\hookrightarrow} (b_1, \dots, b_h)$. Take any $c \in Z(\rho) \subseteq Z(\sigma)$ and consider

$$f(x) = \begin{cases} b_i, & x = a_i,\ i \in \{1, \dots, h\} \\ c, & \text{otherwise.} \end{cases}$$

Then $f \in \operatorname{End}\rho$. Therefore, $f \in \operatorname{End}\sigma$ which together with $(a_1, \dots, a_h) \in \sigma$ yields $(b_1, \dots, b_h) \in \sigma$. But $\{b_1, \dots, b_h\} \subseteq \operatorname{dom}(T_\rho)$ so $(b_1, \dots, b_h) \in (C_\sigma \cup T_\sigma)|_{\operatorname{dom}(T_\rho)}$.

($\Leftarrow$) Suppose $\mathrm{End}\,\rho \nsubseteq \mathrm{End}\,\sigma$ and let $h > k$ and $Z(\rho) \subseteq Z(\sigma)$. Take any $f \in \mathrm{End}\,\rho \setminus \mathrm{End}\,\sigma$ and a tuple $(a_1, \ldots, a_h) \in \sigma$ such that $(f(a_1), \ldots, f(a_h)) \notin \sigma$. Let $b_i = f(a_i)$, $i = 1, \ldots, h$. Then $\{b_1, \ldots, b_h\} \cap Z(\sigma) = \emptyset$, all the b_i's are distinct and consequently all the a_i's are distinct.

If there is an i such that $a_i \in Z(\rho)$, then from $f \in \mathrm{End}\,\rho$ it follows that b_i is an h-central element of T_ρ witnessed by $\{b_1, \ldots, b_h\} \setminus \{b_i\}$. Since $\{b_1, \ldots, b_h\} \subseteq A \setminus Z(\sigma)$ we get a contradiction with the third requirement.

If $\{a_1, \ldots, a_h\} \cap Z(\rho) = \emptyset$, then from $f \in \mathrm{End}\,\rho$ it follows that $(a_1, \ldots, a_h) \overset{T_\rho}{\hookrightarrow} (b_1, \ldots, b_h)$. Since $(a_1, \ldots, a_h)$ belongs to $(C_\sigma \cup T_\sigma)|_{\mathrm{dom}(T_\rho)}$ and $(b_1, \ldots, b_h)$ does not, we have that $(C_\sigma \cup T_\sigma)|_{\mathrm{dom}(T_\rho)}$ is not a Fraïssé filter over T_ρ which contradicts the fourth requirement. □

Corollary 4.7. *Let ρ and σ be central relations, $k = \mathrm{ar}(\rho) \geqslant 2$, $h = \mathrm{ar}(\sigma) \geqslant 2$, and $T_\rho \neq \emptyset$. Additionally, assume $Z(\rho) = Z(\sigma)$ and let $B = \mathrm{dom}(T_\rho) = \mathrm{dom}(T_\sigma) = A \setminus Z(\rho)$. Then $\mathrm{End}\,\rho \subseteq \mathrm{End}\,\sigma$ if and only if*

- *$h > k$,*
- *if $d \in B$ is an h-central element of T_ρ witnessed by $\{b_2, \ldots, b_h\} \subseteq B$ then $(d, b_2, \ldots, b_h) \in T_\sigma$, and*
- *T_σ is a Fraïssé filter over T_ρ.*

In case $T_\sigma = \emptyset$ and $Z(\rho) = Z(\sigma)$ we obtain an even simpler characterization:

Proposition 4.8. *Let ρ and σ be central relations, $k = \mathrm{ar}(\rho) \geqslant 2$, $h = \mathrm{ar}(\sigma) \geqslant 2$, and let $T_\sigma = \emptyset$ and $Z(\rho) = Z(\sigma)$. Then $\mathrm{End}\,\rho \subseteq \mathrm{End}\,\sigma$ if and only if $h > k$ and T_ρ has no h-central elements.*

Proof. ($\Leftarrow$) Assume that $\mathrm{End}\,\rho \nsubseteq \mathrm{End}\,\sigma$ and let $h > k$. Take any $f \in \mathrm{End}\,\rho \setminus \mathrm{End}\,\sigma$ and a tuple $(a_1, \ldots, a_h) \in \sigma$ such that $(f(a_1), \ldots, f(a_h)) \notin \sigma$. Let $b_i = f(a_i)$, $i = 1, \ldots, h$. Then $\{b_1, \ldots, b_h\} \cap Z(\sigma) = \emptyset$, all the b_i's are distinct and consequently all the a_i's are distinct. Since $T_\sigma = \emptyset$, from the fact that all the a_i's are distinct it follows that at least one of the a_i's is in $Z(\sigma)$, say, $a_1 \in Z(\sigma)$. But then from $f \in \mathrm{End}\,\rho$ it follows that b_1 is an h-central element of T_ρ witnessed by $\{b_2, \ldots, b_h\}$

($\Rightarrow$) Suppose that $h > k$ and that T_ρ has an h-central element d witnessed by $\{b_2, \ldots, b_h\}$. Then $\{d, b_2, \ldots, b_h\} \subseteq A \setminus Z(\sigma)$ whence follows that $(d, b_2, \ldots, b_h) \notin \sigma$ since $T_\sigma = \emptyset$. Take any $c \in Z(\sigma)$ and consider

$$f(x) = \begin{cases} d, & x = c \\ x, & x \in \{b_2, \ldots, b_h\} \\ c, & \text{otherwise.} \end{cases}$$

Then $f \in \mathrm{End}\,\rho$ since d is an h-central element of T_ρ witnessed by $\{b_2, \ldots, b_h\}$, and $f \notin \mathrm{End}\,\sigma$ since $(c, b_2, \ldots, b_h)$ belongs to σ and $(d, b_2, \ldots, b_h)$ does not. □

We shall say that a central relation ρ is End-*maximal* if $\mathrm{End}\,\rho \nsubseteq \mathrm{End}\,\sigma$ for every central relation $\sigma \neq \rho$.

Proposition 4.9. *A central relation ρ on A is* End-*maximal if and only if* $|A| = \mathrm{ar}(\rho) + |Z(\rho)|$.

Proof. ($\Leftarrow$) Let ρ be a central relation which is not End-maximal. Then there is a central relation σ such that $\operatorname{End}\rho \subset \operatorname{End}\sigma$. This further implies $\mathrm{ar}(\rho) < \mathrm{ar}(\sigma)$ and $Z(\rho) \subseteq Z(\sigma)$. Since σ has at least $\mathrm{ar}(\sigma)$ noncentral elements, we have that $\mathrm{ar}(\sigma) + |Z(\sigma)| \leqslant |A|$, so $\mathrm{ar}(\rho) + |Z(\rho)| < \mathrm{ar}(\sigma) + |Z(\sigma)| \leqslant |A|$.

($\Rightarrow$) Let ρ be a central relation of arity k such that $k + |Z(\rho)| < |A|$ and let $\rho = R_\rho \cup C_\rho \cup T_\rho$ be the decomposition of ρ into its reflexive part, central part and the tail. Let $h = |A| - |Z(\rho)|$ and define R_h, C_h and T_h as follows:

$$\begin{aligned} R_h &= \{(x_1, \dots, x_h) \in A^h : x_i = x_j \text{ for some } i \neq j\}, \\ C_h &= \{(x_1, \dots, x_h) \in (A^h)^{\mathrm{irr}} : x_i \in Z(\rho) \text{ for some } i\}, \\ T_h &= \{(x_1, \dots, x_h) \in (A^h)^{\mathrm{irr}} : \text{ there is an } i \text{ such that} \\ &\qquad (x_i, x_{j_2}, \dots, x_{j_k}) \in T_\rho \text{ whenever } j_2, \dots, j_k \in \\ &\qquad \{1, \dots, h\} \setminus \{i\} \text{ are } k-1 \text{ distinct indices}\}. \end{aligned}$$

Let us show that $\sigma = C_h \cup R_h \cup T_h$ is a central relation such that $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$.

The relation σ is clearly totally reflexive and totally symmetric, and has at least all of $Z(\rho)$ as its central elements. To show that σ is a central relation, we have to show that $\sigma \neq A^h$. Let $A \backslash Z(\rho) = \{b_1, \dots, b_h\}$. Since each b_i is a noncentral element, for every i there exist $d_2, \dots, d_k \in A$ such that $(b_i, d_2, \dots, d_k) \notin \rho$. But then $\{d_2, \dots, d_k\} \subseteq A \setminus Z(\rho)$, so $d_2 = b_{j_2}, \dots, d_k = b_{j_k}$ for some $k-1$ distinct indices $j_2, \dots, j_k \in \{1, \dots, h\} \setminus \{i\}$. Thus, for every i there exist $k-1$ indices $j_2, \dots, j_k \in \{1, \dots, h\} \setminus \{i\}$ such that $(b_i, b_{j_2}, \dots, b_{j_k}) \notin T_\rho$. Therefore, $(b_1, \dots, b_h) \notin T_h$ and hence $(b_1, \dots, b_h) \notin \sigma$.

Clearly, $\mathrm{ar}(\sigma) > \mathrm{ar}(\rho)$ and $Z(\sigma) = Z(\rho)$. Let us show that $\operatorname{End}\rho \subseteq \operatorname{End}\sigma$. Take any $f \in \operatorname{End}\rho$, $(a_1, \dots, a_h) \in \sigma$ and let $d_i = f(a_i)$, $i \in \{1, \dots, h\}$. If there is an i such that $d_i \in Z(\sigma)$ or there are $i \neq j$ such that $d_i = d_j$, then $(d_1, \dots, d_h) \in \sigma$. So, assume now that $d_1, \dots, d_h$ are distinct noncentral elements. Then the a_i's are also all distinct.

If there is an i such that $a_i \in Z(\rho)$ then for every $k-1$ distinct indices $j_2, \dots, j_k \in \{1, \dots, h\} \setminus \{i\}$ we have $(a_i, a_{j_2}, \dots, a_{j_k}) \in \rho$. Since $f \in \operatorname{End}\rho$, we get $(d_i, d_{j_2}, \dots, d_{j_k}) \in \rho$. Moreover, $(d_i, d_{j_2}, \dots, d_{j_k}) \in T_\rho$ since the d_i's are distinct noncentral elements. Therefore, $(d_1, \dots, d_h) \in T_h \subseteq \sigma$.

If the a_i's are distinct noncentral elements, then $(a_1, \dots, a_h) \in T_h$ whence follows that there is an i such that $(a_i, a_{j_2}, \dots, a_{j_k}) \in T_\rho$ whenever $j_2, \dots, j_k \in \{1, \dots, h\} \setminus \{i\}$ are $k-1$ distinct indices. Since $f \in \operatorname{End}\rho$, we get $(d_i, d_{j_2}, \dots, d_{j_k}) \in \rho$. Moreover, $(d_i, d_{j_2}, \dots, d_{j_k}) \in T_\rho$ since the d_i's are distinct noncentral elements. Therefore, $(d_1, \dots, d_h) \in T_h \subseteq \sigma$. $\square$

There exist pairs (ρ, R_Θ) such that ρ is a central relation of arity $k \geqslant 2$, R_Θ is an h-regular relation and $\operatorname{End}\rho \subseteq \operatorname{End}R_\Theta$. First of all, if $\Theta = \{\Delta_A\}$ where $\Delta_A = \{(x,x) : x \in A\}$ then $\operatorname{End}R_\Theta = A^A$ and clearly $\operatorname{End}\rho \subseteq \operatorname{End}R_\Theta$. In case $\Theta \neq \{\Delta_A\}$, we have shown in [7, Propositions 4.6 and 4.7] that $\operatorname{End}\rho \subseteq \operatorname{End}R_\Theta$

implies $k < h$, $\Theta = \{\theta\}$, θ has a single nontrivial block B and $Z(\rho) \subseteq B$. Moreover, if $T_\rho = \emptyset$, we were able to provide a complete characterization [7, Proposition 4.8]: if $T_\rho = \emptyset$ and $\Theta \neq \{\Delta_A\}$ then $\operatorname{End}\rho \subseteq \operatorname{End} R_\Theta$ if and only if $k < h$, $\Theta = \{\theta\}$, θ has a single nontrivial block B and $Z(\rho) = B$. Here we shall deal with the remaining case of $T_\rho \neq \emptyset$ and thus finish the characterization.

Proposition 4.10. *Let ρ be a central relation of arity $k \geqslant 2$ such that $T_\rho \neq \emptyset$ and let R_Θ be an h-regular relation, where $\Theta \neq \{\Delta_A\}$. Then* $\operatorname{End}\rho \subseteq \operatorname{End} R_\Theta$ *if and only if:*

(1) $k < h$;

(2) $\Theta = \{\theta\}$, where $A/\theta = \big\{B, \{b_2\}, \ldots, \{b_h\}\big\}$ and $|B| \geqslant 2$, i.e. B is a single nontrivial block of θ;

(3) $Z(\rho) \subseteq B$;

(4) if b_i is an $(h-1)$-central element of T_ρ for some $i \in \{2, \ldots, h\}$, then $\{b_2, \ldots, b_h\} \setminus \{b_i\}$ cannot be a set of witnesses; and

(5) for all $a_2, \ldots, a_h \in \operatorname{dom}(T_\rho)$, if $(a_2, \ldots, a_h) \overset{T_\rho}{\hookrightarrow} (b_2, \ldots, b_h)$ then $(a_2, \ldots, a_h)$ is a permutation of $(b_2, \ldots, b_h)$.

Proof. ($\Rightarrow$) Suppose $\operatorname{End}\rho \subseteq \operatorname{End} R_\Theta$. Then we know from [7, Propositions 4.6 and 4.7] that $k < h$, $\Theta = \{\theta\}$, θ has a single nontrivial block B and $Z(\rho) \subseteq B$.

Suppose now that (4) does not hold. Then there is an $i \in \{2, \ldots, h\}$ such that b_i is an $(h-1)$-central element of T_ρ with the set of witnesses $\{b_2, \ldots, b_h\} \setminus \{b_i\}$. For the sake of simplicity, let b_2 be an $(h-1)$-central element of T_ρ with the set of witnesses $\{b_3, \ldots, b_h\}$. Take any $c \in Z(\rho)$, $b_1 \in B \setminus \{c\}$ and consider

$$f(x) = \begin{cases} c, & x = b_1 \\ b_2, & x = c \\ x, & x \in \{b_3, \ldots, b_h\} \\ c, & \text{otherwise.} \end{cases}$$

Then $f \in \operatorname{End}\rho \setminus \operatorname{End} R_\Theta$ which contradicts $\operatorname{End}\rho \subseteq \operatorname{End} R_\Theta$.

Finally, assume that (5) does not hold. Then there exist $a_2, \ldots, a_h \in \operatorname{dom}(T_\rho)$ such that $(a_2, \ldots, a_h) \overset{T_\rho}{\hookrightarrow} (b_2, \ldots, b_h)$ and $(a_2, \ldots, a_h)$ is not a permutation of $(b_2, \ldots, b_h)$. Then $\{a_2, \ldots, a_h\} \cap B \neq \emptyset$ since all the a_i's are distinct. If $|\{a_2, \ldots, a_h\} \cap B| \geqslant 2$ take an arbitrary $a_1 \in A \setminus \{a_2, \ldots, a_h\}$. If, however, $|\{a_2, \ldots, a_h\} \cap B| = 1$ take an arbitrary $a_1 \in B \setminus \{a_2, \ldots, a_h\}$. Take any $c \in Z(\rho)$ and consider

$$f(x) = \begin{cases} c, & x = a_1 \\ b_i, & x = a_i,\ i \in \{2, \ldots, h\} \\ c, & \text{otherwise.} \end{cases}$$

Then $f \in \operatorname{End}\rho \setminus \operatorname{End} R_\Theta$ which contradicts $\operatorname{End}\rho \subseteq \operatorname{End} R_\Theta$.

($\Leftarrow$) Suppose $\operatorname{End}\rho \not\subseteq \operatorname{End} R_\Theta$, but (1), (2) and (3) are true. Take any $f \in \operatorname{End}\rho \setminus \operatorname{End} R_\Theta$ and any $(a_1, \ldots, a_h) \in R_\Theta$ such that $(f(a_1), \ldots, f(a_h)) \notin R_\Theta$. Then there is a $b_1 \in B$ such that $(f(a_1), \ldots, f(a_h))$ is a permutation of

$(b_1, \ldots, b_h)$. Since relations we are working with are totally symmetric, without loss of generality we can assume that $b_i = f(a_i)$, $i \in \{1, \ldots, h\}$. If there is an $i \geqslant 2$ such that $a_i \in Z(\rho)$ then b_i is an $(h-1)$-central element for T_ρ witnessed by $\{b_2, \ldots, b_h\} \setminus \{b_i\}$. If, however, $\{a_2, \ldots, a_h\} \cap Z(\rho) = \emptyset$ then from $(a_1, \ldots, a_h) \in R_\Theta$ it follows that $\{a_2, \ldots, a_h\} \cap B \neq \emptyset$, while $f \in \operatorname{End} \rho$ implies $(a_2, \ldots, a_h) \stackrel{T_\rho}{\hookrightarrow} (b_2, \ldots, b_h)$. □

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