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ON KINEMATICS OF SEMI-EUCLIDEAN SUBMANIFOLDS ON THE PLANE IN E_1^3

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Abstract. In this study, we obtained an equation of homothetic motion of any smooth semi-Euclidean submanifold M on its tangent plane at the contact points, along pole curves which are trajectories of instantaneous rotation centers at the contact points. Also, we gave some remarks for the homothetic motions that are both sliding and rolling at every moment. We establish a surprising relationship between the curvatures of the moving and fixed pole curves.

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1. Preliminaries

We know that the angular velocity vector has an important role in kinematics of two rigid bodies, especially one rolling on another, [1], [8] and [9]. Mathematicians and physicists have interpreted rigid body motions in various ways. K. Nomizu [9] has studied the 1-parameter motions of orientable surface M on tangent space along the pole curves using parallel vector fields at the contact points and he gave some characterizations of the angular velocity vector of rolling without sliding. H. H. Hacisalihoğlu showed some properties of 1-parameter homothetic motions in Euclidean space [4]. In this study we define the homothetic motion of M on the tangent plane of M and we shall give some results and conditions using any vector field and Frenet frames along smooth pole curves on M and on the tangent plane for a homothetic motion.

The homothetic motion in a 3-dimensional semi-Euclidean space with the index 1 is generated by the transformation

$$(1.1)\qquad \begin{array}{rcll} F: E_1^3 & \longrightarrow & E_1^3 & \\ X & \longrightarrow & Y & = hAX + C \end{array}$$

where $A \in SO_1(3)$, and X and C are 3×1 vectors. The elements of A , C and h are continuously differentiable functions of time-dependent parameter t and the elements of X are coordinates of a point in the body. By differentiating (1.1) we have

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$$Y' = hAX' + (h'A + hA')\,X + C' \tag{1.2}$$

where $(h'A + hA')\,X + C'$ is the sliding velocity of X. We call X a pole point if the sliding velocity of X vanishes and locus of points of X call the pole curve. We take B as hA, so the equation of the moving pole curve is $X = -(B')^{-1}C'$. Substitution X with $X = -(B')^{-1}C'$ in (1.1) we obtain fixed-pole curve $Y = -B(B')^{-1}C' + C$. Now we examine the matrix $B(B')^{-1}$.

$$B(B')^{-1} = hA\left(h'^{-1}A^{-1} + h^{-1}\,(A')^{-1}\right) = \underbrace{hh'^{-1}I_3}_{\varphi} + \underbrace{A'A^{-1}}_{S}$$

where φ and S are respective sliding and rolling parts of (1.1). For $S \neq 0$, there is a uniquely determined vector $W\,(t)$ such that $S(U)$ equal to the cross-product $W(t) \times U$ for every vector U. The vector $W\,(t)$ is called the angular velocity at instant t and the homothetic motion F in (1.1) is called rolling if $W\,(t)$ is tangent to N, and F is rolling if $W\,(t)$ is normal to N at the contact point of M and N at an instant t [11].

2. Introduction

It is well known that in a Lorentzian manifold we can find three types of submanifolds: Space-like (or Riemannian), time-like (Lorentzian) and light-like (degenerate or null), depending on the induced metric in the tangent vector space. Lorentz surfaces has been examined in numerous articles and books. In this article, however, we have examined some characteristics belonging to the surface by making some special choices of coordinate curves on the surface which are on the intersection points of tangent vector spaces. Let IR^3 be endowed with the pseudoscalar product of X and Y is defined by

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 - x_3y_3 \qquad X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3)$$

$(IR^3, \langle , \rangle)$ is called 3-dimensional Lorentzian space denoted by E_1^3. The Lorentzian vector product is defined by

$$X \Lambda Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

A vector X in E_1^3 is called a space-like, light-like, time-like vector if $\langle X, X\rangle > 0$, $\langle X, X\rangle = 0$ or $\langle X, X\rangle < 0$, respectively. For a non-null vector $X \in E_1^3$, the norm of X defined by

$$\|X\| = \sqrt{|\,\langle X, X\rangle\,|}$$

and X is called a unit vector if $\|X\| = 1$[6].

An arbitrary curve (α) in E_1^3 is called a space-like, light-like or time-like if all of its velocity vectors α' space-like, light-like or time-like, respectively. Let T , N and B be tangent, principal normal and binormal vector fields of α, respectively. If α is a space-like curve with a space-like or time-like principal normal N, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = \varepsilon = \pm 1$, $\langle B,B\rangle = -\varepsilon$, $\langle T,N\rangle = \langle T,B\rangle = \langle N,B\rangle = 0$. If α is a space-like curve with a null principal normal N, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = \langle B,B\rangle = 0$, $\langle T,N\rangle = \langle T,B\rangle = 0$, $\langle N,B\rangle = 1$. If α is a time-like curve then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.3}$$

where $\langle T,T\rangle = -1$, $\langle N,N\rangle = \langle B,B\rangle = 1$, $\langle T,N\rangle = \langle T,B\rangle = \langle N,B\rangle = 0$ [3, 7].

3. Homothetic Motion Of Submanifolds M On Σ

Let us consider the smooth semi-Euclidean manifold M and the tangent plane Σ of M at contact points $P \in M$ along moving and fixed smooth pole curves $X(t)$ on M and $Y(t)$ on Σ starting at P. We shall take a rectangular coordinate system in E_1^3 and the unit vectors $(1,0,0), (0,1,0)$ and $(0,0,1)$ of E_1^3. Let ξ be a unit normal vector field of M along the curve (X). We wish to move homotheticly M on Σ along the smooth pole curves $X(t)$ and $Y(t)$. We can define homothetic motion M on Σ as

$$\begin{array}{lcll} F : M & \longrightarrow & \Sigma & \\ \quad X & \longrightarrow & Y & = BX + C, \quad B = hA \end{array} \tag{3.1}$$

since $F(M)$ is tangent to Σ at the contact points we have $\xi = e_3$ and $\xi = e_2$ for time-like and space-like manifolds respectively. Suppose that and be orthonormal systems along pole curves (X) and (Y) on M and respectively. Let b_1, b_2 and a_1, a_2 be orthonormal vector fields along (X) and (Y) so that

$b_1 = hB^{-1}a_1$ and $b_2 = hB^{-1}a_2$. Hence $\{b_1, b_2, \xi\}$ and $\{a_1, a_2, e_3\}$ (or$\{a_1, a_2, e_2\}$) will be a moving and fixed system for (X) and (Y) respectively. Since $\xi(t) \in Sp\{N, B\}$ we can write

$$\xi(t) = \lambda N + \mu B \tag{3.2}$$

where λ and μ are smooth functions dependent of the time parameter t. To determine the orthogonal matrix A, we have to construct the frames $\{b_1, b_2, \xi\}$ and $\{a_1, a_2, e_3\}$ (or $\{a_1, a_2, e_2\}$) along the pole curves (X) and (Y) respectively. During this operation we make use of a Darboux frame along (X) and tangent and principal normal vector field of (Y) at the contact points on M and Σ respectively.

i. The case when M is submanifold with space-like normal and (X) is space-like curve:

In this case, from (2.1) we take $B\Lambda N = \varepsilon T$, $T\Lambda B = N$, $T\Lambda N = B$ and we have $\lambda^2 - \mu^2 = \varepsilon$ and $\lambda\lambda' - \mu\mu' = 0$ for (X). From (3.2) we obtain

$$\xi\Lambda T = -\mu N - \lambda B \tag{3.3}$$

So we can find semi-orthogonal matrices $P, Q, R \in S_1O(3)$ between the orthonormal systems $\{T, N, B\}$ and $\{e_1, e_2, e_3\}$, $\{T, \xi\Lambda T, \xi\}$ and $\{T, N, B\}$, $\{b_1, b_2, \xi\}$ and $\{T, \xi\Lambda T, \xi\}$ respectively. Hence, the matrix $A_1 = [P]^{-1}[Q]^{-1}[R]^{-1} \in S_1O(3)$ transforms b_1 to e_1, b_2 to e_2 and ξ to e_3. The tangent spaces $Sp\{b_1, b_2\}$ and $Sp\{T, \xi\Lambda T\}$ are the same spaces and hyperbolic angle between b_1 and T (b_2 and $\xi\Lambda T$ respectively) be θ. We designate the skew-symmetric matrix $\frac{dA_1^{-1}}{dt}A_1$ in semi-Euclidean mean as W_1, then W_1 will be:

(3.4)
$$W_1 = \begin{bmatrix} 0 & \left\{\begin{matrix} \varepsilon\lambda k_1 \cosh\theta \\ -\varepsilon\gamma \sinh\theta \end{matrix}\right\} & \theta' + \varepsilon\mu k_1 \\ -\{\varepsilon\lambda k_1 \cosh\theta - \varepsilon\gamma \sinh\theta\} & 0 & \{\varepsilon\lambda k_1 \sinh\theta - \varepsilon\gamma \cosh\theta\} \\ \theta' + \varepsilon\mu k_1 & \left\{\begin{matrix} \varepsilon\lambda k_1 \sinh\theta \\ -\varepsilon\gamma \cosh\theta \end{matrix}\right\} & 0 \end{bmatrix}$$

where $\gamma = \varepsilon k_2 + \lambda\mu' - \lambda'\mu$. Thus we proved the following theorem.

Theorem 1. *If $\theta' + \varepsilon k_1\mu = 0$, then b_1 and b_2 vector fields are parallel with the connection of M along space-like curve (X). In this case, b_1 and b_2 have no any component in $T_M(X(t))$.*

Similarly, let $\overline{T}, \overline{N}$ and $\overline{B}$ vector fields be Frenet vectors of (Y). Since (Y) is a planar space-like curve, the binormal vector field of (Y) will be of the same direction with the time-like vector e_2, so we take $\overline{B} = e_2$ and (Y) is a space-like curve with time-like principal normal vector field. We can find again the semi-orthogonal matrices $\overline{P}, \overline{Q}, \overline{R} \in S_1O(3)$ between the orthonormal systems $\{\overline{T}, \overline{N}, e_2\}$ and $\{e_1, e_2, e_3\}$, $\{\overline{T}, e_2\Lambda\overline{T}, e_2\}$ and $\{\overline{T}, \overline{N}, e_2\}$, $\{a_1, a_2, e_2\}$ and $\{\overline{T}, e_2\Lambda\overline{T}, e_2\}$. Thus, the matrix $A_2 = [\overline{P}]^{-1}[\overline{Q}]^{-1}[\overline{R}]^{-1}$ transforms a_1 to e_1, a_2 to e_2 and e_3 to e_3 respectively. The tangent spaces $Sp\{a_1, a_2\}$ and $Sp\{\overline{T}, e_2\Lambda\overline{T}\}$ are the same spaces and hyperbolic angle between a_1 and $\overline{T}$ (a_2 and $e_2\Lambda\overline{T}$, respectively) is $\overline{\theta}$. We designate the skew-symmetric matrix $\frac{dA_2^{-1}}{dt}A_2$ in semi-Euclidean mean as W_2, then W_2 will be:

$$W_2 = \begin{bmatrix} 0 & 0 & \overline{\theta}' - \overline{k}_1 \\ 0 & 0 & 0 \\ \overline{\theta}' - \overline{k}_1 & 0 & 0 \end{bmatrix} \tag{3.5}$$

Hence we proved the following theorem.

Theorem 2. *If $\overline{\theta}' - \overline{k}_1$, then a_1 and a_2 vector fields are parallel with the connection of Σ along the space-like curve (Y). In this case, a_1 and a_2 have no any component in $T_\Sigma(Y(t))$.*

Therefore, we can find the matrix A using A_1 and A_2 as $A = A_2A_1^T$ so that A transforms b_1 to a_1, b_2 to a_2 and ξ to e_3. respectively. The skew-symmetric matrix $S = \frac{dA}{dt}A^{-1}$ is instantaneous rotation matrix and S represents a linear transformation as $S : T_\Sigma(Y(t)) \longrightarrow Sp\{e_2\}$. We can find the matrix S using (3.4) and (3.5) as $S = A_2(-W_2 + W_1)A_2^{-1}$. Consequently, the matrix S determines a vector $W \in Sp\{a_1, a_2, e_2\}$. For $P \in M$, we find

$$W|_P = -\left\{ \begin{array}{l} \varepsilon\lambda k_1 \sinh\theta - \\ \varepsilon\gamma\cosh\theta \end{array} \right\} a_1|_P + \left\{ \begin{array}{l} \varepsilon\lambda k_1 \cosh\theta - \\ \varepsilon\gamma\sinh\theta \end{array} \right\} a_2|_P + \left(\begin{array}{c} \theta' + \varepsilon\mu k_1 - \\ \overline{\theta}' + \overline{k}_1 \end{array} \right) e_2|_P \tag{3.6}$$

Hence, we can give the following theorem and remark.

Theorem 3. *Let b_1, b_2 and a_1, a_2 be any orthogonal vector fields along the space-like pole curves (X) and (Y) respectively. Thus F is a homothetic motion if and only if $\theta' + \varepsilon\mu k_1 - \overline{\theta}' + \overline{k}_1 = 0$.*

Remark 1. *If b_1, b_2 and a_1, a_2 are parallel vector fields along the space-like pole curves (X) and (Y) respectively, then F is a homothetic motion defined as $B(b_1) = ha_1$, $B(b_2) = ha_2$ and $B(\xi) = he_2$. In this case, if the space-like pole curves (X) and (Y) are geodesics on M and Σ respectively, θ and $\overline{\theta}$ are constant along the homothetic motion.*

Theorem 4. *Let F be a homothetic motion. F is only sliding motion without rolling if the space-like pole curves (X) is passing through the flat points of submanifold M, thus the vector field W will vanish at the flat points.*

Proof. Let S_M be shape operator of the space-like submanifold M, then we have,

$$S_M\left(\frac{dX}{dt}\right) = \frac{d\xi}{dt}$$

At the flat point P on (X) of M we have,

$$S_M\left(\left.\frac{dX}{dt}\right|_P\right) = 0$$

and differentiating of $B\xi = he_3$ with respect to t we obtain

$$S\left(e_2\right) = -A\left.\frac{d\xi}{dt}\right|_P$$

where $-A\frac{d\xi}{dt}$ will be at P as follow.

$$-A\left.\frac{d\xi}{dt}\right|_P = \underbrace{\{\varepsilon\lambda k_1\cosh\theta - \varepsilon\gamma\sinh\theta\}}_{\beta_1|_P}\, a_1|_P - \underbrace{\{\varepsilon\lambda k_1\sinh\theta - \varepsilon\gamma\cosh\theta\}}_{\beta_2|_P}\, a_2|_P$$

Finally, we obtain $\beta_1(P) = 0$ and $\beta_2(P) = 0$ so $S = 0$. Consequently, the rolling part of F will vanish. Hence the homothetic motion F is sliding without rolling at the flat points on (X) of M. □

Remark 2. *Let (X) be a space-like pole curve on a smooth submanifold M which does not pass through a flat point of M. There exists a unique homothetic motion of M on the tangent plane at $P = X(t_o)$ such that $Y(t) = F(X(t))$ is the locus of contact points.*

Remark 3. *If b_1, b_2 and a_1, a_2 are parallel vector fields along the planar space-like pole curves (X) and (Y) respectively and $\lambda = 0$, then the homothetic motion F will be a sliding motion.*

Remark 4. *If (X) is a planar asymptotic space-like pole curve on the submanifold M and θ and $\overline{\theta}$ are constant, then $\frac{\overline{k}_1}{k_1}$ is constant.*

Remark 5. *During the homothetic motion, the Darboux vector field W will be null, time-like and space-like if $\gamma = \mp\lambda k_1$, $\gamma = \sqrt{\lambda^2 k_1^2 - 1}$, and $\gamma = \sqrt{\lambda^2 k_1^2 + 1}$, respectively.*

ii. The case when M is a submanifold with space-like normal and (X) is time-like curve:

In this case, from (2.3) we take $B\Lambda N = T$, $T\Lambda B = -N$, $T\Lambda N = B$ and we have $\lambda^2 + \mu^2 = 1$ and $\lambda\lambda' + \mu\mu' = 0$ for (X). From (3.2) we obtain

$$\xi\Lambda T = \mu N - \lambda B \tag{3.7}$$

So we can find the semi-orthogonal matrices $P, Q, R \in S_1O(3)$ between the orthonormal systems $\{T, N, B\}$ and $\{e_1, e_2, e_3\}$, $\{T, \xi\Lambda T, \xi\}$ and $\{T, N, B\}$, $\{b_1, b_2, \xi\}$ and $\{T, \xi\Lambda T, \xi\}$ respectively. Hence, $A_1 = [P]^{-1} [Q]^{-1} [R]^{-1} \in S_1O(3)$ is the matrix which transforms b_1 to e_1, b_2 to e_2 and ξ to e_3. The tangent spaces $Sp\{b_1, b_2\}$ and $Sp\{T, \xi\Lambda T\}$ are the same spaces and hyperbolic angle between b_1 and T (b_2 and $\xi\Lambda T$ respectively) is θ. We designate the skew-symmetric matrix $\frac{dA_1^{-1}}{dt}A_1$ in semi-Euclidean mean as W_1, then W_1 will be follows.

(3.8)
$$W_1 = \begin{bmatrix} 0 & \{\lambda k_1 \sinh\theta + \delta\cosh\theta\} & \theta' + \mu k_1 \\ -\{\lambda k_1 \sinh\theta + \delta\cosh\theta\} & 0 & \{\lambda k_1 \cosh\theta + \delta\sinh\theta\} \\ \theta' + \mu k_1 & \{\lambda k_1 \cosh\theta + \delta\sinh\theta\} & 0 \end{bmatrix}$$

where $\delta = k_2 + \lambda\mu' - \lambda'\mu$. Thus we proved the following theorem.

Theorem 5. *If $\theta' + k_1\mu = 0$, then b_1 and b_2 vector fields are parallel with the connection of M along the space-like curve (X). In this case, b_1 and b_2 have no any component in $T_M(X(t))$.*

Similarly, let $\overline{T}, \overline{N}$ and $\overline{B}$ vector fields be Frenet vectors of (Y). Since the (Y) is a planar space-like curve, the binormal vector field of (Y) will be of the same direction as the time-like vector e_2. So we take $\overline{B} = e_2$ and (Y) is a space-like curve with time-like principal normal vector field. We can find again the semi-orthogonal matrices $\overline{P}, \overline{Q}, \overline{R} \in S_1O(3)$ between the orthonormal systems $\{\overline{T}, \overline{N}, e_2\}$ and $\{e_1, e_2, e_3\}$, $\{\overline{T}, e_2\Lambda\overline{T}, e_2\}$ and $\{\overline{T}, \overline{N}, e_2\}$, $\{a_1, a_2, e_2\}$ and $\{\overline{T}, e_2\Lambda\overline{T}, e_2\}$. Thus, the matrix $A_2 = [\overline{P}]^{-1} [\overline{Q}]^{-1} [\overline{R}]^{-1}$ transforms a_1 to e_1, a_2 to e_3 and e_2 to e_2 respectively. The tangent spaces $Sp\{a_1, a_2\}$ and $Sp\{\overline{T}, e_2\Lambda\overline{T}\}$ are the same spaces and the hyperbolic angle between a_1 and $\overline{T}$ (a_2 and $e_2\Lambda\overline{T}$ respectively) will be $\overline{\theta}$. We designate the skew-symmetric matrix $\frac{dA_2^{-1}}{dt}A_2$ in semi-Euclidean mean as W_2, then W_2 will be as follows:

$$W_2 = \begin{bmatrix} 0 & 0 & \overline{\theta'} + \overline{k}_1 \\ 0 & 0 & 0 \\ \overline{\theta'} + \overline{k}_1 & 0 & 0 \end{bmatrix} \tag{3.9}$$

Hence we proved the following theorem.

Theorem 6. *If* $\overline{\theta'} + \overline{k}_1$*, then* a_1 *and* a_2 *vector fields are parallel with the connection of* Σ *along the space-like curve* (Y)*. In this case,* a_1 *and* a_2 *have no any component in* $T_\Sigma(Y(t))$*.*

Therefore we can find the matrix A, using A_1 and A_2 as $A = A_2 A_1^T$ so that A transforms b_1 to a_1, b_2 to a_2 and ξ to e_2, respectively. The skew-symmetric matrix $S = \frac{dA}{dt} A^{-1}$ is an instantaneous rotation matrix and S represents a linear transformation as $S : T_\Sigma(Y(t)) \longrightarrow Sp\{e_2\}$. We can find the matrix S using (3.8) and (3.9) as $S = A_2 (-W_2 + W_1) A_2^{-1}$. Consequently, the matrix S determines a vector $W \in Sp\{a_1, a_2, e_2\}$. For $P \in M$, we find

(3.10)
$$W|_P = \left\{ \begin{array}{l} \lambda k_1 \sinh\theta + \\ \delta\cosh\theta \end{array} \right\} a_1|_P + \left\{ \begin{array}{l} \lambda k_1 \cosh\theta + \\ \delta\sinh\theta \end{array} \right\} a_2|_P + \left(\begin{array}{c} \theta' + \mu k_1 - \\ \overline{\theta'} - \overline{k}_1 \end{array} \right) e_2|_P$$

Thus we can give the following theorems and remarks.

Theorem 7. *Let* b_1*,* b_2 *and* a_1*,* a_2 *be any orthogonal vector fields along the space-like pole curves* (X) *and* (Y) *respectively. Hence* F *is a homothetic motion if and only if* $\theta' + \mu k_1 - \overline{\theta'} - \overline{k}_1 = 0$.

Remark 6. *If* b_1*,* b_2 *and* a_1*,* a_2 *are parallel vector fields along the time-like pole curves* (X) *and* (Y) *respectively, then* F *is a homothetic motion defined as* $B(b_1) = ha_1$*,* $B(b_2) = ha_2$ *and* $B(\xi) = he_2$*. In this case, if the space-like pole curves* (X) *and* (Y) *are geodesics on* M *and* Σ *respectively,* θ *and* $\overline{\theta}$ *are constant along the homothetic motion.*

Theorem 8. *Let* F *be a homothetic motion.* F *is only sliding motion without rolling if the time-like pole curves* (X) *passing through the flat points of the submanifold* M*, thus the vector field* W *will vanish at the flat points.*

Proof. It can be easily proved, similarly as Theorem 4. □

Remark 7. *Let* (X) *be a time-like pole curve on a smooth submanifold* M *which does not pass through a flat point of* M*. There exists a unique homothetic motion of* M *on the tangent plane at* $P = X(t_o)$ *such that* $Y(t) = F(X(t))$ *is the locus of contact points.*

Remark 8. *If* b_1*,* b_2 *and* a_1*,* a_2 *are parallel vector fields along the planar time-like pole curves* (X) *and* (Y) *respectively and* $\lambda = 0$*, then the homothetic motion* F *will be a sliding motion.*

Remark 9. *If* (X) *is a planar asymptotic time-like pole curve on the submanifold* M *and* θ *and* $\overline{\theta}$ *are constant, then* $\frac{\overline{k}_1}{k_1}$ *is constant.*

Remark 10. *During the homothetic motion, the Darboux vector field W will be null, time-like and space-like if $\delta = \mp\lambda k_1$, $\delta = \sqrt{\lambda^2 k_1^2 - 1}$, and $\delta = \sqrt{\lambda^2 k_1^2 + 1}$, respectively.*

iii. The case when M is a submanifold with time-like normal and (X) is space-like curve:

In this case, from (2.1) we take $B\Lambda N = \varepsilon T$, $T\Lambda B = -N$, $T\Lambda N = B$ and we have $\lambda^2 - \mu^2 = -\varepsilon$ and $\lambda\lambda' - \mu\mu' = 0$ for (X). From (3.2) we obtain

$$\xi\Lambda T = -\mu N - \lambda B \tag{3.11}$$

So we can find the semi-orthogonal matrices $P, Q, R \in S_1O(3)$ between the orthonormal systems $\{T, N, B\}$ and $\{e_1, e_2, e_3\}$, $\{T, \xi\Lambda T, \xi\}$ and $\{T, N, B\}$, $\{b_1, b_2, \xi\}$ and $\{T, \xi\Lambda T, \xi\}$ respectively. Hence, $A_1 = [P]^{-1}[Q]^{-1}[R]^{-1} \in S_1O(3)$ is the matrix which transforms b_1 to e_1, b_2 to e_2 and ξ to e_3. The tangent spaces $Sp\{b_1, b_2\}$ and $Sp\{T, \xi\Lambda T\}$ are the same spaces and the angle between b_1 and T (b_2 and $\xi\Lambda T$ respectively) will be θ. We designate the skew-symmetric matrix $\frac{dA_1^{-1}}{dt}A_1$ in semi-Euclidean mean as W_1, then W_1 will be as follows.

(3.12)
$$W_1 = \begin{bmatrix} 0 & \theta' - \varepsilon\mu k_1 & \{-\varepsilon\lambda k_1 \cos\theta + \varepsilon\varphi \sin\theta\} \\ -\theta' + \varepsilon\mu k_1 & 0 & \{\varepsilon\lambda k_1 \sin\theta + \varepsilon\varphi \cos\theta\} \\ \{-\varepsilon\lambda k_1 \cos\theta + \varepsilon\varphi \sin\theta\} & \{\varepsilon\lambda k_1 \sin\theta + \varepsilon\varphi \cos\theta\} & 0 \end{bmatrix}$$

where $\varphi = -\varepsilon k_2 + \lambda\mu' - \lambda'\mu$. Thus we proved the following theorem.

Theorem 9. *If $\theta' - \varepsilon\mu k_1 = 0$, then b_1 and b_2 vector fields are parallel with the connection of M along space-like curve (X). In this case, b_1 and b_2 have no any component in $T_M(X(t))$.*

Similarly, let $\overline{T}, \overline{N}$ and $\overline{B}$ vector fields be Frenet vectors of (Y). Since (Y) is a planar space-like curve, the binormal vector field of (Y) will be of the same direction with the time-like vector e_3. So we take $\overline{B} = e_3$ and (Y) is a space-like curve with time-like principal normal vector field. We can find again a semi-orthogonal matrices $\overline{P}, \overline{Q}, \overline{R} \in S_1O(3)$ between the orthonormal systems $\{\overline{T}, \overline{N}, e_3\}$ and $\{e_1, e_2, e_3\}$, $\{\overline{T}, e_3\Lambda\overline{T}, e_3\}$ and $\{\overline{T}, \overline{N}, e_3\}$, $\{a_1, a_2, e_3\}$ and $\{\overline{T}, e_3\Lambda\overline{T}, e_3\}$. Thus, the matrix $A_2 = [\overline{P}]^{-1}[\overline{Q}]^{-1}[\overline{R}]^{-1}$ transforms a_1 to e_1, a_2 to e_2 and e_3 to e_3 respectively. The tangent spaces $Sp\{a_1, a_2\}$ and $Sp\{\overline{T}, e_2\Lambda\overline{T}\}$ are the same spaces and the angle between a_1 and $\overline{T}$ (a_2 and $e_3\Lambda\overline{T}$ respectively) will be $\overline{\theta}$. We designate the skew-symmetric matrix $\frac{dA_2^{-1}}{dt}A_2$ in semi-Euclidean mean as W_2, then W_2 will be as follows.

$$W_2 = \begin{bmatrix} 0 & \overline{\theta}' - \overline{k}_1 & 0 \\ -\overline{\theta}' + \overline{k}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.13}$$

Hence we proved the following theorem.

Theorem 10. *If* $\overline{\theta'} - \overline{k}_1$*, then* a_1 *and* a_2 *vector fields are parallel with the connection of* Σ *along the space-like curve* (Y)*. In this case,* a_1 *and* a_2 *have no any component in* $T_\Sigma(Y(t))$*.*

Therefore we can find the matrix A using A_1 and A_2 as $A = A_2 A_1^T$ so that A transforms b_1 to a_1, b_2 to a_2 and ξ to e_3, respectively. The skew-symmetric matrix $S = \frac{dA}{dt}A^{-1}$ is instantaneous rotation matrix and S represents a linear transformation $S : T_\Sigma(Y(t)) \longrightarrow Sp\{e_3\}$. We can find the matrix S using (3.12) and (3.13) as $S = A_2\left(-W_2 + W_1\right)A_2^{-1}$. Consequently, the matrix S determines a vector $W \in Sp\{a_1, a_2, e_3\}$. For $P \in M$, we find

(3.14)
$$W|_P = -\left\{\begin{array}{l}\varepsilon\lambda k_1\sin\theta + \\ \varepsilon\varphi\cos\theta\end{array}\right\} a_1|_P + \left\{\begin{array}{l}-\varepsilon\lambda k_1\cos\theta + \\ \varepsilon\varphi\sin\theta\end{array}\right\} a_2|_P + \left(\begin{array}{c}\theta' - \varepsilon\mu k_1 - \\ \overline{\theta'} + \overline{k}_1\end{array}\right) e_3|_P$$

Thus we can give the following theorems and remarks.

Theorem 11. *Let* b_1*,* b_2 *and* a_1*,* a_2 *be any orthogonal vector fields along the space-like pole curves* (X) *and* (Y) *respectively. Hence* F *is a homothetic motion if and only if* $\theta' - \varepsilon\mu k_1 - \overline{\theta'} + \overline{k}_1 = 0$*.*

Remark 11. *If* b_1*,* b_2 *and* a_1*,* a_2 *are parallel vector fields along the space-like pole curves* (X) *and* (Y) *respectively, then* F *is a homothetic motion defined as* $B(b_1) = ha_1$*,* $B(b_2) = ha_2$ *and* $B(\xi) = he_3$*. In this case, if the space-like pole curves* (X) *and* (Y) *are geodesics on* M *and* Σ *respectively,* θ *and* $\overline{\theta}$ *are constant along the homothetic motion.*

Theorem 12. *Let* F *be a homothetic motion.* F *is only sliding motion without rolling if the space-like pole curves* (X) *is passing through the flat points of the submanifold* M*, thus the vector field* W *will vanish at the flat points.*

Proof. It can be easily proved, similarly Theorem 4. □

Remark 12. *Let* (X) *be a space-like pole curve on a smooth submanifold* M *which does not pass through a flat point of* M*. There exists a unique homothetic motion of* M *on the tangent plane at* $P = X(t_o)$ *such that* $Y(t) = F(X(t))$ *is the locus of points of contact.*

Remark 13. *If* b_1*,* b_2 *and* a_1*,* a_2 *are parallel vector fields along the planar space-like pole curves* (X) *and* (Y) *respectively and* $\lambda = 0$*, then the homothetic motion* F *will be sliding motion.*

Remark 14. *If* (X) *is a planar asymptotic space-like pole curve on the submanifold* M, *and* θ *and* $\overline{\theta}$ *are constant then* $\frac{\overline{k}_1}{k_1}$ *is constant.*

Remark 15. *If* $\lambda = k_2 = 0$ *(or* $k_1 = 0$ *and* $\varphi = 0$*) and* $\varphi = \sqrt{1 - \lambda^2 k_1^2}$ *satisfy during the homothetic motion then the Darboux vector field* W *will be null and space-like respectively. Thus* W *will never be time-like during homothetic motion* F.

Remark 16. *Homothetic motion can not be defined if one of the submanifold* M, (X) *and* (Y) *curves is light-like. In this case the matrix* A *is not an orthogonal (in semi-Euclidean sense). Furthermore, in this case, homothetic motions is not regular motions.*

Example 1. *Let the submanifold* M *be cylinder with the time-like principal normal which has the equation* $x_1^2 - (1 - x_3)^2 = -1$ *and*

$$X(t) = \left(\sinh\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}}, 1 - \cosh\left(\frac{t}{\sqrt{2}}\right)\right)$$

be regular space-like curve with time-like principal normal on M. *We obtain*

$$T = \tfrac{1}{\sqrt{2}}\left(\cosh\left(\tfrac{t}{\sqrt{2}}\right), 1, -\sinh\left(\tfrac{t}{\sqrt{2}}\right)\right), \quad N = \left(\sinh\left(\tfrac{t}{\sqrt{2}}\right), 0, -\cosh\left(\tfrac{t}{\sqrt{2}}\right)\right)$$
$$B = \tfrac{1}{\sqrt{2}}\left(-\cosh\left(\tfrac{t}{\sqrt{2}}\right), 1, \sinh\left(\tfrac{t}{\sqrt{2}}\right)\right), \quad \xi = \left(\sinh\left(\tfrac{t}{\sqrt{2}}\right), 0, -\cosh\left(\tfrac{t}{\sqrt{2}}\right)\right)$$
$$k_1 = \tfrac{1}{2},\ k_2 = \tfrac{-1}{2},\ \psi = \pi,\ \lambda = 1,\ \mu = 0$$

for (X) *and let* $Y(t) = \left(\frac{t^2}{2},\ 0,\ 0\right)$ *be planar space-like curve with space-like principal normal on* Σ. *We find*

$$\overline{T} = \tfrac{1}{\sqrt{2}}(1, 1, 0), \quad \overline{N} = \tfrac{1}{\sqrt{2}}(1, -1, 0), \quad \overline{B} = (0, 0, 1),$$
$$\overline{k}_1 = 0,\ \overline{k}_2 = 0,\ \overline{\psi} = \tfrac{\pi}{2},\ \overline{\lambda} = 0,\ \overline{\mu} = 1$$

for (Y) *curve. Since* $\|dY/dt\| = h$ *we find* $h = t$ *and using* $\frac{dY}{dt} = B\frac{dX}{dt}$ *we obtain* $\overline{\theta}(t) = \theta = \frac{\pi}{2}$ *so the motion will be as follows.*

$$Y(t) = \begin{bmatrix} \frac{t}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right) & \frac{t}{\sqrt{2}} & -\frac{t}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right) \\ -\frac{t}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right) & \frac{t}{\sqrt{2}} & \frac{t}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right) \\ -t\sinh\left(\frac{t}{\sqrt{2}}\right) & 0 & t\cosh\left(\frac{t}{\sqrt{2}}\right) \end{bmatrix} X(t) + \begin{bmatrix} -\frac{t}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right) \\ -\frac{t^2}{\sqrt{2}} + \frac{t}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right) \\ -t + t\cosh\left(\frac{t}{\sqrt{2}}\right) \end{bmatrix}$$

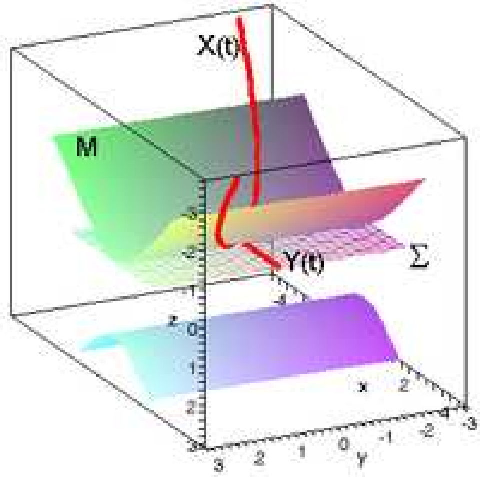
After calculations we obtain the skew-symmetric matrix $S = \frac{dA}{dt}A^{-1}$ *and* W *Darboux vector field from (3.14)*

$$S = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

and

$$W|_P = \left(-\frac{1}{2}, -\frac{1}{2}, 0\right)$$

respectively and the condition $\theta' - \varepsilon\mu k_1 - \overline{\theta'} + \overline{k}_1 = 0 = 0$ *is satisfied. So, the motion* $Y(t) = BX(t) + C$ *is homothetic motion.*



The space-like cylinder rolling its space-like tangent plane at the contact points, along the pole curves X(t) and Y(t), respectively

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