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HILBERT SPACE VALUED GENERALIZED RANDOM PROCESSES – PART II[1]

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Abstract. A multiplication of Wick type for Hilbert space valued generalized random processes is defined, and applications to some classes of linear and nonlinear stochastic differential equations are presented.

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Introduction

Generalized random processes (GRPs) were classified as GRPs (I) and (II) in Part I of this paper. Now we focus our attention om GRPs (I), i.e. linear continuous mappings from a certain space of test functions to some space of classical or generalized random variables.

The paper is organized in the following manner: In the introductory section (Section 1) we provide some basic terminology and notation. In Section 2 we develop the Wick product for Hilbert space valued GRPs (I), while in Section 3 we define the differentiation of GRPs (I) and provide as an example the stochastic analogue of the Dirac delta distribution. In Section 4 we use the series expansion machinery and the Wick product for GRPs (I) for solving a class of linear and a class of nonlinear evolution stochastic differential equations.

1. Preliminaries

We will make use of the notation introduced in Part I of the paper. Here we recall only some of the most necessary facts. For details refer to Part I of the paper and the references therein.

1.1. Basic notation

Let $I \subset \mathbb{R}$ be an open interval, and let $\mathcal{A}$, $\exp\mathcal{A}$, $\mathcal{A}'$, $\exp\mathcal{A}'$ denote the Zemanian spaces constructed by formally self-adjoint linear differential operator of the form

$$\mathcal{R} = \theta_0 D^{n_1}\theta_1 \cdots D^{n_\nu}\theta_\nu = \bar{\theta}_\nu(-D)^{n_\nu}\cdots(-D)^{n_2}\bar{\theta}_1(-D)^{n_1}\bar{\theta}_0 \tag{1}$$

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where $D = d/dx$, θ_k are smooth complex functions without zero-points in I, and n_k are integers $k = 1, 2, \ldots, \nu$. Let ψ_n, $n \in \mathbb{N}$ be the orthonormal base of $\mathcal{A}$. Let $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$ be the Schwartz spaces of tempered distributions and ξ_n, $n \in \mathbb{N}$, be the family of Hermite functions.

Let the basic probability space $(\Omega, \mathcal{F}, P)$ be $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$, where $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions, $\mathcal{B}$ the sigma-algebra generated by weak topology and μ denotes the white noise measure given by the Bochner-Minlos theorem. Let $(L)^2 = L^2(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$ and H_α, $\alpha \in \mathcal{I}$ be the Fourier–Hermite orthogonal basis of $(L)^2$. Denote by $(S)_1$, $(S)_{-1}$ the spaces of the Kondratiev stochastic test functions and stochastic distributions.

Let H be a separable Hilbert space with orthonormal basis e_n, $n \in \mathbb{N}$. The H–valued Zemanian spaces are denoted by $\mathcal{A}(H)$, $\mathcal{A}'(H)$, and the H–valued Kondratiev spaces are denoted by $S(H)_1$, $S(H)_{-1}$.

1.2. Expansion of GRPs (I)

Here we give a brief overview of the results from Part I of the paper. Hilbert space valued GRPs (I) are elements of the spaces $\mathcal{A}(H)^* = \mathcal{L}(\mathcal{A}, S(H)_{-1})$, ${}^{exp}\mathcal{A}(H)^*_k = \mathcal{L}(\mathcal{A}, \exp S(H)_{-1})$ or $\mathcal{L}(\exp \mathcal{A}, S(H)_{-1})$, $\mathcal{L}(\exp \mathcal{A}, expS(H)_{-1})$. The following expansion theorem holds:

Theorem 1.1. *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in \mathcal{A}(H)^*_k$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j}\, e_i, \qquad f_{ij} \in \mathcal{A}_{-k}, \quad i, j \in \mathbb{N} \tag{2}$$

and there exists $k_0 \in \mathbb{N}_0$ such that for each bounded set $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi \in B} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle f_{ij}, \varphi \rangle|^2 \, (2\mathbb{N})^{-k_0 \alpha^j} < \infty. \tag{3}$$

If Φ can be represented in the form (2) and there exists $k_1 \in \mathbb{N}_0$ such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|f_{ij}\|^2_{-k} (2\mathbb{N})^{-k_1 \alpha^j} < \infty, \tag{4}$$

then $\Phi \in \mathcal{A}(H)^*_k$.

Example 1.1. *(see also [4]) Let $n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(i, j) \mapsto n(i, j)$ be the usual bijection given in the following table:*

	1	2	3	4	5	6	7	$\cdots$	j	$\cdots$
1	1	3	6	10	15	21	28	$\cdots$		
2	2	5	9	14	20	27			$\vdots$	
3	4	8	13	19	26				$\vdots$	
4	7	12	18	25					$\vdots$	
5	11	17	24						$\vdots$	
6	16	23							$\vdots$	
7	22								$\vdots$	
$\vdots$	$\vdots$								$\vdots$	
i	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$n(i,j)$	$\cdots$
$\vdots$									$\vdots$	

Let $\varepsilon_j = (0, 0, \ldots, 1, 0, \ldots)$ be a sequence of zeros with the number 1 as the jth component. Then,

$$\beta_i(t,\omega) = \sum_{j=1}^{\infty} \int_0^t \xi_j(s) ds H_{\varepsilon_{n(i,j)}}(\omega), \quad i \in \mathbb{N}$$

is a sequence of independent (one-dimensional d-parameter $\mathbb{R}$–valued) Brownian motions. Rewrite this as

$$\beta_i(t,\omega) = \sum_{k=1}^{\infty} \theta_{ik}(t) H_{\varepsilon_k}(\omega), \quad \theta_{ik}(t) = \begin{cases} \int_0^t \xi_j(s)ds, & k = n(i,j) \\ 0, & k \neq n(i,j) \end{cases}.$$

The formal sum

$$\mathbf{B}(t,\omega) = \sum_{i=1}^{\infty} \beta_i(t,\omega) e_i = \sum_{k=1}^{\infty} \theta_k(t) H_{\varepsilon_k}(\omega), \quad \theta_k(t) = \delta_{n(i,j),k} \int_0^t \xi_j(s)ds\; e_i$$

is an H–valued Brownian motion. Note, the sum converges in $S(H)_{-0}$ for each $t \geq 0$ fixed.

Example 1.2. *(see also [4]) The H–valued (one-dimensional, d-parameter) singular white noise is defined by the formal sum*

$$\mathbf{W}(t,\omega) = \sum_{k=1}^{\infty} \kappa_k(t) H_{\varepsilon_k}(\omega), \quad \kappa_k(t) = \delta_{n(i,j),k} \xi_j(t)\, e_i.$$

It is also an element of $S(H)_{-0}$.

Example 1.3. *Let $\mathcal{R} = -\frac{d^2}{dx^2} + x^2 + 1$ and $I = \mathbb{R}$. Then $\mathcal{A} = \mathcal{S}(\mathbb{R})$, $\mathcal{A}' = \mathcal{S}'(\mathbb{R})$ and $\psi_k(t) = \xi_k(t)$, $k \in \mathbb{N}$, where ξ_k are the Hermite functions.*

(i) In Example 1.2 the H–valued one-dimensional d-parameter singular white noise was defined by the formal sum

$$\mathbf{W}(t,\omega)=\sum_{k=1}^{\infty}\kappa_k(t)H_{\varepsilon_k}(\omega),\quad \kappa_k(t)=\delta_{n(i,j),k}\ \xi_j(t)\ e_i.$$

With the Hermite function ξ_j we associate a generalized function $\tilde{\xi}_j\in\mathcal{S}'(\mathbb{R})$ defined by $\langle\tilde{\xi}_j,\varphi\rangle=\int_{\mathbb{R}}\xi_j(t)\varphi(t)dt$, $\varphi\in\mathcal{S}(\mathbb{R})$.

Define $\tilde{\kappa}_k=\delta_{n(i,j),k}\ \tilde{\xi}_j(t)\ e_i$. Then, the white noise $\tilde{\mathbf{W}}$ as an H–valued GRP (I) has the expansion

$$\tilde{\mathbf{W}}=\sum_{k=1}^{\infty}\tilde{\kappa}_k\otimes H_{\varepsilon_k}\quad\in\mathcal{L}(\mathcal{S}(\mathbb{R}),S(H)_{-1}).$$

Condition (4) is also satisfied, because

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_{n(i,j),k}\ \|\xi_j\|^2_{L^2(\mathbb{R})}(2\mathbb{N})^{-p\varepsilon_j}=\sum_{k=1}^{\infty}(2k)^{-p}<\infty,\quad \textit{for } p>1.$$

Note also that $\tilde{\kappa}_k$ is an element of $\mathcal{S}'(\mathbb{R};H)$.

(ii) Let $l>\frac{5}{12}$ and $t_1,t_2,t_3,\ldots\in\mathbb{R}$ such that $t_1\le t_2\le t_3\le\cdots\to\infty$. It is known that the Dirac delta distributions δ_{t_j}, $j\in\mathbb{N}$, belong to $\mathcal{A}_{-l}=\mathcal{S}_{-l}(\mathbb{R})$. Let

$$\Delta_k=\delta_{n(i,j),k}\ \delta_{t_j}\ e_i,\quad k\in\mathbb{N}$$

(to avoid confusion: the first delta is the Kronecker symbol, the second one is the Dirac distribution). With

$$\sum_{k=1}^{\infty}\Delta_k\otimes H_{\alpha^k}\tag{5}$$

is given a GRP (I). We will check condition (4). Since

$$\delta_{t_j}(x)=\sum_{n=1}^{\infty}\xi_n(t_j)\xi_n(x),\quad j\in\mathbb{N},$$

and $\xi_n=\mathcal{O}(n^{-\frac{1}{12}})$, $n\in\mathbb{N}$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_{n(i,j),k}\|\delta_{t_j}\|^2_{-l}(2\mathbb{N})^{-p\alpha^j}&=\sum_{k=1}^{\infty}\|\delta_{t_k}\|^2_{-l}(2\mathbb{N})^{-p\alpha^k}\\&=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|\xi_n(t_k)|^2(2n)^{-2l}(2\mathbb{N})^{-p\alpha^k}\\&\le C\sum_{k=1}^{\infty}(2\mathbb{N})^{-p\alpha^k}\sum_{n=1}^{\infty}n^{-\frac{1}{6}}(2n)^{-2l}<\infty,\end{aligned}$$

for some constant $C>0$ and $p>1$. Hence, the process given by (5) meets the definition of a GRP (I).

1.3. The Wick product

It is a well-known problem that in general one can not define a pointwise multiplication of generalized functions; thus, it is not clear how to deal with nonlinearities. In the framework of white noise analysis this difficulty is solved by introducing the Wick product. First we recall the definition and some basic properties of the Wick product in the Kondratiev spaces (see [3]).

Definition 1.1. *Let $F, G \in (S)_{-1}$ be given by their chaos expansion $F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha(\omega)$, $G(\omega) = \sum_{\beta\in\mathcal{I}} g_\beta H_\beta(\omega)$, $f_\alpha, g_\beta \in \mathbb{R}$. The Wick product of F and G is the unique element in $(S)_{-1}$ defined by:*

$$F\lozenge G(\omega) = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma(\omega).$$

The Wick product is a commutative, associative operation, distributive with respect to addition. By the same formula we defined in [5] the Wick product for $F, G \in exp(S)_{-1}$. It is known that the spaces $(S)_1$, $(S)_{-1}$, $exp(S)_1$ and $exp(S)_{-1}$ are closed under Wick multiplication.

In the Hilbert space valued case, the Wick product is defined analogously (see [4]), and is denoted by the same symbol.

Definition 1.2. *Let $F, G \in S(H)_{-1}$ be given by their chaos expansion $F(\omega) = \sum_{i=1}^\infty \sum_{\alpha\in\mathcal{I}} f_{i,\alpha} H_\alpha(\omega) e_i$, $G(\omega) = \sum_{i=1}^\infty \sum_{\beta\in\mathcal{I}} g_{i,\beta} H_\beta(\omega) e_i$, $f_{i,\alpha}, g_{i,\beta} \in \mathbb{R}$. The Wick product of F and G is the unique element in $S(H)_{-1}$ defined by:*

$$\begin{aligned} F\lozenge G(\omega) &= \sum_{i=1}^\infty \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_{i,\alpha} g_{i,\beta} \right) H_\gamma(\omega) e_i \\ &= \sum_{i=1}^\infty (F_i \lozenge G_i(\omega)) e_i, \end{aligned}$$

where $F_i(\omega) = \sum_{\alpha\in\mathcal{I}} f_{i,\alpha} H_\alpha(\omega)$, and $G_i(\omega) = \sum_{\beta\in\mathcal{I}} g_{i,\beta} H_\beta(\omega)$.

This definition is legal, since $S(H)_1$ and $S(H)_{-1}$ are closed under Wick multiplication. In the same manner we can define $F\lozenge G$ for $F, G \in expS(H)_{-1}$ and it is an easy exercise to show (combining the methods in [4] and [5]) that $F\lozenge G \in expS(H)_{-1}$. Also, if $F, G \in expS(H)_1$ then $F\lozenge G \in expS(H)_1$.

Now we recall the definition of a deterministic multiplication of Wick type in $\mathcal{A}'$, which was introduced in [7] and [5]. From now on, when we use Wick products, we will always assume that $\mathcal{A}$ is nuclear, i.e. there exists some $p \geq 0$, such that $M := \sum_{n=1}^\infty \widetilde{\lambda_n}^{-2p} < \infty$.

Definition 1.3. *Let $f, g \in \mathcal{A}'$ be generalized function given by the expansions $f = \sum_{k=1}^\infty a_k \psi_k$, $g = \sum_{k=1}^\infty b_k \psi_k$. Define $f \diamond g$ to be the generalized function* *from $\mathcal{A}'$, given by*

$$f \diamond g = \sum_{n=1}^{\infty} \left(\sum_{\substack{i,j \in \mathbb{N} \\ i+j=n+1}} a_i b_j \right) \psi_n. \tag{6}$$

Proposition 1.1. *If $f = \sum_{i=1}^{\infty} a_i \psi_i \in \mathcal{A}_{-k}$ and $g = \sum_{i=1}^{\infty} b_i \psi_i \in \mathcal{A}_{-l}$, then $f \diamond g \in \mathcal{A}_{-(k+l+p)}$. Moreover,*

$$|\langle f \diamond g, \varphi \rangle|^2 \leq M \|f\|_{-k}^2 \|g\|_{-l}^2 \|\varphi\|_{k+l+p}, \tag{7}$$

for each test function $\varphi \in \mathcal{A}$.

Similarly, one can also define the multiplication of test-functions, under an additional assumption:

Lemma 1.1. *Let there exist a constant $C > 0$, such that*

$$\widetilde{\lambda_{i+j}} \leq C \widetilde{\lambda_i} \widetilde{\lambda_j}, \qquad i, j \in \mathbb{N}.$$

If $f = \sum_{i=1}^{\infty} a_i \psi_i \in \mathcal{A}_k$ and $g = \sum_{i=1}^{\infty} b_i \psi_i \in \mathcal{A}_k$, then $f \diamond g \in \mathcal{A}_{(k-p)}$.

For example, we have $\psi_i \diamond \psi_j = \psi_{i+j-1}$ for arbitrary $i, j \in \mathbb{N}$.

The notion of the Wick product to GRPs (I) was also extended in [5]. Here we summarize the basic results.

Definition 1.4. *Let $\Phi \in \mathcal{A}_k^*$, $\Psi \in \mathcal{A}_l^*$ be two GRPs (I) given by expansions $\Phi = \sum_{i=1}^{\infty} f_i \otimes H_{\alpha^i}$, $f_i \in \mathcal{A}_{-k}$, $i \in \mathbb{N}$, and $\Psi = \sum_{j=1}^{\infty} g_j \otimes H_{\alpha^j}$, $g_j \in \mathcal{A}_{-l}$, $j \in \mathbb{N}$. The Wick product of Φ and Ψ is defined to be*

$$\Phi \blacklozenge \Psi = \sum_{n=1}^{\infty} \left(\sum_{\substack{i,j \in \mathbb{N} \\ \alpha^i + \alpha^j = \alpha^{n+1}}} f_i \diamond g_j \right) \otimes H_{\alpha^n}. \tag{8}$$

Theorem 1.2. *The Wick product $\Phi \blacklozenge \Psi$ from the previous definition is a GRP (I), precisely $\Phi \blacklozenge \Psi \in \mathcal{A}_{k+l+p}^*$.*

The Wick product for GRPs (I) taking values in $exp(S)_{-1}$ can be defined in an analogous way.

Theorem 1.3. *The Wick product $\Phi \blacklozenge \Psi$ defined by the formula (8) of $\Phi \in {}^{exp}\mathcal{A}_k^*$ and $\Psi \in {}^{exp}\mathcal{A}_l^*$ is a GRP (I), precisely $\Phi \blacklozenge \Psi \in {}^{exp}\mathcal{A}_{k+l+p}^*$.*

The Wick product $\diamond$ in $\mathcal{A}'$ can be embedded into the Wick product $\blacklozenge$ in $\mathcal{A}^*$, since each deterministic function can be (trivially) regarded as a stochastic process. Also, the Wick product $\blacklozenge$ acting on $\mathcal{A}^*$ is an extension of the Wick product $\Diamond$ acting on $(S)_{-1}$.

2. The Wick product of H–valued GRPs (I)

The main difference is now that we are not able to define the Wick product $\blacklozenge$ for the whole class of H–valued GRPs (I). This is due to the fact that $S(H)_{-1}$ is not a nuclear space. Therefore, the Wick product will be defined for the class of H–valued GRPs (I) satisfying condition (4).

Definition 2.1. *Let $\Phi \in \mathcal{A}(H)_k^*$, $\Psi \in \mathcal{A}(H)_l^*$ be two H–valued GRPs (I) given by expansions $\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{i,j} \otimes H_{\alpha^j} e_i$, $f_{i,j} \in \mathcal{A}_{-k}$, $i,j \in \mathbb{N}$, and $\Psi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g_{i,j} \otimes H_{\alpha^j}$, $g_{i,j} \in \mathcal{A}_{-l}$, $i,j \in \mathbb{N}$, and let $r_1 \geq 0$, $r_2 \geq 0$ be such that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|f_{i,j}\|_{-k}^2 (2\mathbb{N})^{-r_1\alpha^j} < \infty$ and $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|g_{i,j}\|_{-l}^2 (2\mathbb{N})^{-r_2\alpha^j} < \infty$. The Wick product of Φ and Ψ is defined to be*

$$\Phi \blacklozenge \Psi = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \left(\sum_{\substack{s,r\in\mathbb{N} \\ \alpha^s+\alpha^r=\alpha^{n+1}}} f_{i,s} \diamond g_{i,r} \right) \otimes H_{\alpha^n}\, e_i. \tag{9}$$

We may write (9) also as

$$\Phi \blacklozenge \Psi = \sum_{i=1}^{\infty} (F_i \blacklozenge G_i) e_i,$$

where $F_i = \sum_{j=1}^{\infty} f_{i,j} \otimes H_{\alpha^j} \in \mathcal{A}_k^*$, $G_i = \sum_{j=1}^{\infty} g_{i,j} \otimes H_{\alpha^j} \in \mathcal{A}_l^*$ and $F_i \blacklozenge G_i$ is defined as in Definition 1.4.

Theorem 2.1. *The Wick product $\Phi \blacklozenge \Psi$ from the previous definition is an H–valued GRP (I), precisely $\Phi \blacklozenge \Psi \in \mathcal{A}(H)_{k+l+p}^*$.*

Proof. Due to Lemma 1.1 it follows that $f_{i,s} \diamond g_{i,r} \in \mathcal{A}_{-k-l-p}$ for all $i,s,r \in \mathbb{N}$, and thus, $\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} f_{i,s} \diamond g_{i,r} \in \mathcal{A}_{-k-l-p}$, $n \in \mathbb{N}$. Since $[\Phi \blacklozenge \Psi, \varphi] = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} \langle f_{i,s} \diamond g_{i,r}, \varphi\rangle H_{\alpha^n} e_i$, it remains to prove

$$\sup_{\varphi\in B} \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \left| \sum_{\alpha^s+\alpha^r=\alpha^{n+1}} \langle f_{i,s} \diamond g_{i,r}, \varphi\rangle \right|^2 (2\mathbb{N})^{-r_3\alpha^n} < \infty,$$

for any bounded set $B \subseteq \mathcal{A}$ and some $r_3 \geq 0$. Put $r_3 = r_1 + r_2 + q$, where $q > 1$. Then, according to (7)

$$\begin{aligned}
&\sup_{\varphi\in B}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left|\sum_{\alpha^s+\alpha^r=\alpha^{n+1}}\langle f_{i,s}\diamond g_{i,r},\varphi\rangle\right|^2(2\mathbb{N})^{-r_3\alpha^n}\\
&\leq \sup_{\varphi\in B}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left|\sum_{\alpha^s+\alpha^r=\alpha^{n+1}}\|f_{i,s}\|_{-k}\|g_{i,r}\|_{-l}\|\varphi\|_{k+l+p}\sqrt{M}\right|^2(2\mathbb{N})^{-r_3\alpha^n}\\
&\leq M\sum_{n=1}^{\infty}(2\mathbb{N})^{-q\alpha^n}\sum_{i=1}^{\infty}\left|\sum_{\alpha^s+\alpha^r=\alpha^{n+1}}\|f_{i,s}\|_{-k}(2\mathbb{N})^{-\frac{r_1\alpha^s}{2}}\|g_{i,r}\|_{-l}(2\mathbb{N})^{-\frac{r_2\alpha^r}{2}}\sup_{\varphi\in B}\|\varphi\|_{k+l+p}\right|^2\\
&\leq M\sum_{n=1}^{\infty}(2\mathbb{N})^{-q\alpha^n}\sum_{i=1}^{\infty}\left(\sum_{s=1}^{\infty}\|f_{i,s}\|^2_{-k}(2\mathbb{N})^{-r_1\alpha^s}\sum_{r=1}^{\infty}\|g_{i,r}\|^2_{-l}(2\mathbb{N})^{-r_2\alpha^r}\sup_{\varphi\in B}\|\varphi\|^2_{k+l+p}\right)\\
&= M\sum_{n=1}^{\infty}(2\mathbb{N})^{-q\alpha^n}\sum_{i=1}^{\infty}\sum_{s=1}^{\infty}\|f_{i,s}\|^2_{-k}(2\mathbb{N})^{-r_1\alpha^s}\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\|g_{i,r}\|^2_{-l}(2\mathbb{N})^{-r_2\alpha^r}\sup_{\varphi\in B}\|\varphi\|^2_{k+l+p}\\
&< \infty,
\end{aligned}$$

where we used the property $(2\mathbb{N})^{\alpha^s+\alpha^r} = (2\mathbb{N})^{\alpha^s}(2\mathbb{N})^{\alpha^r}$. □

Note that the Wick product $\lozenge$ acting on $S(H)_{-1}$ can be embedded into $\blacklozenge$ acting on $\mathcal{A}(H)^*$.

Also, the following theorem holds, similarly as in the finite dimensional case.

Theorem 2.2. *The Wick product $\Phi\blacklozenge\Psi$ defined by the formula (9) of $\Phi \in {}^{exp}\mathcal{A}(H)^*_k$ and $\Psi \in {}^{exp}\mathcal{A}(H)^*_l$ is an H–valued GRP (I), precisely $\Phi\blacklozenge\Psi \in {}^{exp}\mathcal{A}(H)^*_{k+l+p}$, provided there exist $r_1 \geq 0$, $r_2 \geq 0$, such that* $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\|f_{i,j}\|^2_{-k}e^{-r_1(2\mathbb{N})^{\alpha^j}} < \infty$, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\|g_{i,j}\|^2_{-l}e^{-r_2(2\mathbb{N})^{\alpha^j}} < \infty$.

3. Differentiation of GRPs (I)

Definition 3.1. *Let $F \in \mathcal{A}(H)^*$. The distributional derivative of F, denoted by $\frac{\partial}{\partial t}F$ is defined by $\left[\frac{\partial}{\partial t}F,\varphi\right] = -\left[F,\frac{\partial}{\partial t}\varphi\right]$, for all $\varphi \in \mathcal{A}$.*

Lemma 3.1. *Let $F = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij}\otimes H_{\alpha^j}e_i \in \mathcal{A}(H)^*$. Then,*

$$\frac{\partial}{\partial t}F = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\partial}{\partial t}f_{ij}\otimes H_{\alpha^j}e_i, \tag{10}$$

where $\frac{\partial}{\partial t}f_{ij}$ is the distributional derivative of f_{ij} in $\mathcal{A}'$.

Proof. The assertion follows from the fact that

$$\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\partial}{\partial t}f_{ij}\otimes H_{\alpha^j}e_i,\varphi\right] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle\frac{\partial}{\partial t}f_{ij},\varphi\rangle H_{\alpha^j}e_i$$
$$= -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_{ij},\frac{\partial}{\partial t}\varphi\rangle H_{\alpha^j}e_i = -\left[F,\frac{\partial}{\partial t}\varphi\right]$$

for all $\varphi\in\mathcal{A}$. Obviously, the condition (3) is satisfied. □

Example 3.1. *Denote by $\delta_y\in\mathcal{S}'(\mathbb{R};H)$ be the H–valued Dirac delta distribution at $y\in\mathbb{R}$. Let $t_j\in\mathbb{R}$, $j\in\mathbb{N}$, such that $t_1\le t_2\le t_3\le\cdots\to\infty$. Then,*

$$\sum_{j=1}^{\infty}\delta_{t_j}\otimes H_{\alpha^j}$$

defines an H–valued GRP (I). Indeed, since $\{\xi_n e_i : n,i\in\mathbb{N}\}$ is an orthogonal basis of $\mathcal{S}'(\mathbb{R};H)$, we can write $\delta_y=\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}d_{in}(y)\xi_n e_i$, $\|\delta_y\|^2_{-k;H}=\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}|d_{in}(y)|^2(2n)^{-k}<\infty$, and, moreover, $\|\delta_y\|^2_{-k;H}$ does not depend on y. Thus,
$\sum_{j=1}^{\infty}\|\delta_{t_j}\|^2_{-k;H}(2\mathbb{N})^{-p\alpha^j}=Const\sum_{j=1}^{\infty}(2\mathbb{N})^{-p\alpha^j}<\infty$, for $p>1$.

It is a well-known fact in the (deterministic) generalized functions theory that if a function $f:\mathbb{R}\to\mathbb{R}$ is differentiable on $\mathbb{R}\setminus\{x_0\}$ and has a jump in $x_0\in\mathbb{R}$, then $Df=f'(x)+C\delta_{x_0}$, where Df is the distributional derivative in $\mathcal{S}'(\mathbb{R})$, $f'(x)$ is the classical derivative, $\delta_{x_0}\in\mathcal{S}'(\mathbb{R})$ is the Dirac delta distribution, and $C=f(x_0^+)-f(x_0^-)$. In this light, the GRP defined in Example 3.1 is a stochastic analogue of the Dirac delta distribution.

Example 3.2. *Let $F=\sum_{j=1}^{\infty}f_jH_{\alpha^j}$, $f_j:\mathbb{R}\to\mathbb{R}$ be a stochastic process in sense of [3], i.e. for each $t\in\mathbb{R}$ fixed $\sum_{j=1}^{\infty}|f_j(t)|^2(2\mathbb{N})^{-p\alpha^j}<\infty$ for some $p>0$. Denote by $\delta_y\in\mathcal{S}'(\mathbb{R})$ the Dirac delta distribution at $y\in\mathbb{R}$. Assume that for each $j\in\mathbb{N}$, the function f_j is differentiable on $\mathbb{R}\setminus\{t_j\}$, has one jump in $t_j\in\mathbb{R}$, and $t_1\le t_2\le t_3\le\cdots\to\infty$. Assume that $\sum_{j=1}^{\infty}|f_j'(t)|^2(2\mathbb{N})^{-p\alpha^j}<\infty$ for each fixed $t\in\mathbb{R}\setminus\{t_1,t_2,\ldots\}$. Let $c_j=f(t_j^+)-f(t_j^-)$, $j\in\mathbb{N}$, be the jump heights. Assume that there exists $C>0$ such that $|c_j|\le C$, $j\in\mathbb{N}$ (i.e. the jump heights are bounded). Then,*

$$\frac{\partial}{\partial t}F=\sum_{j=1}^{\infty}\frac{\partial}{\partial t}f_j\otimes H_{\alpha^j}=\sum_{j=1}^{\infty}\left(f_j'+c_j\delta_{t_j}\right)\otimes H_{\alpha^j}.$$

Since $\sum_{j=1}^{\infty}c_j^2\|\delta_{t_j}\|_{-k}(2j)^{-p}\le C^2\sum_{j=1}^{\infty}\|\delta_{t_j}\|_{-k}(2j)^{-p}<\infty$, the process above is well-defined. Note $\sum_{j=1}^{\infty}f_j'(t)H_{\alpha^j}$ is an element of $(S)_{-1}$ for each fixed $t\in\mathbb{R}$, and $\sum_{j=1}^{\infty}\delta_{t_j}\otimes H_{\alpha^j}$ is the GRP (I) defined in Example 3.1.

4. Applications to some classes of SDEs

4.1. A class of linear SDEs

Let $\mathcal{R}$ be of the form (1) and $P(t) = p_n t^n + p_{n-1}t^{n-1} + \cdots + p_1 t + p_0$, $t \in I$, be a polynomial with real coefficients. We give two examples of stochastic differential equations using GRPs (I), and a differential operator $P(\mathcal{R})$ defined as $P(\mathcal{R}) = p_n\mathcal{R}^n + p_{n-1}\mathcal{R}^{n-1} + \cdots + p_1\mathcal{R} + p_0 I$. Note, if ψ_k is an eigenfunction of $\mathcal{R}$, then

$$\begin{aligned} P(\mathcal{R})\psi_k &= (p_k\mathcal{R}^n + p_{n-1}\mathcal{R}^{n-1} + \cdots + p_1\mathcal{R} + p_0 I)\psi_k \\ &= p_n(\widetilde{\lambda_k})^n\psi_k + p_{n-1}(\widetilde{\lambda_k})^{n-1}\psi_k + \cdots + p_1\widetilde{\lambda_k}\psi_k + p_0\psi_k = P(\widetilde{\lambda_k})\psi_k. \end{aligned}$$

Consider an SDE of the form

$$P(\mathcal{R})u = g \tag{11}$$

where $g \in \mathcal{A}(H)^*_r$ is a GRP (I).

Let u and g be given by the series expansions

$$u(t,\omega) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_{ij}(t) \otimes H_{\alpha^j}(\omega)e_i,$$

$$g(t,\omega) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g_{ij}(t) \otimes H_{\alpha^j}(\omega)e_i,\ t \in I, \omega \in \mathcal{S}'(\mathbb{R}),$$

respectively, where u_{ij}, $g_{ij} \in \mathcal{A}'$, $i,j \in \mathbb{N}$.

Let $u_{ij} = \sum_{k=1}^{\infty} a_{ij}^k\psi_k$, $g_{ij} = \sum_{k=1}^{\infty} b_{ij}^k\psi_k$, $i,j \in \mathbb{N}$. Then we have

$$\begin{aligned} P(\mathcal{R})u &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(\mathcal{R})u_{ij} \otimes H_{\alpha^j}e_i \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(P(\mathcal{R})\sum_{k=1}^{\infty} a_{ij}^k\psi_k\right) \otimes H_{\alpha^j}e_i = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{ij}^k P(\widetilde{\lambda_k})\psi_k \otimes H_{\alpha^j}e_i. \end{aligned} \tag{12}$$

In order to solve (11) we may use the method of undetermined coefficients. From (11) and (12) we have

$$\sum_{k=1}^{\infty} a_{ij}^k P(\widetilde{\lambda_k})\psi_k = g_{ij}, \quad i,j \in \mathbb{N}.$$

Finally, we obtain the system

$$a_{ij}^k P(\widetilde{\lambda_k}) = b_{ij}^k, \quad i,j,k \in \mathbb{N}.$$

First case: If $P(\widetilde{\lambda_k}) \neq 0$ for all $k \in \mathbb{N}$, then $a_{ij}^k = \frac{b_{ij}^k}{P(\widetilde{\lambda_k})}$, $i, j, k \in \mathbb{N}$, and the solution of the equation is given by

$$u = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{b_{ij}^k}{P(\widetilde{\lambda_k})} \psi_k \right) \otimes H_{\alpha^j} e_i.$$

In this case the solution exists and it is unique. Note that there exists a constant $C > 0$ such that $P(\widetilde{\lambda_k}) \geq C$, for all $k \in \mathbb{N}$. Thus,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\| \sum_{k=1}^{\infty} \frac{b_{ij}^k}{P(\widetilde{\lambda_k})} \psi_k \right\|_{-r}^2 (2\mathbb{N})^{-p\alpha^j} \leq \frac{1}{C^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\| \sum_{k=1}^{\infty} b_{ij}^k \psi_k \right\|_{-r}^2 (2\mathbb{N})^{-p\alpha^j} < \infty,$$

for some $p > 1$, because $g \in \mathcal{A}(H)_r^*$. Thus, $u \in \mathcal{A}(H)_r^*$.

Second case: Let $P(\widetilde{\lambda_k}) = 0$ for $k = k_1, k_2, \dots k_m$. Then a solution exists if and only if $b_{ij}^{k_1} = b_{ij}^{k_2} = \dots = b_{ij}^{k_m} = 0$, $i, j \in \mathbb{N}$. The solution in this case is not unique, and the coefficients of the solution u are given by

$$u_{ij} = \sum_{\substack{k=1 \\ P(\widetilde{\lambda_k}) \neq 0}}^{\infty} \frac{b_{ij}^k}{P(\widetilde{\lambda_k})} \psi_k + \sum_{s=1}^{m} c_{ij}^s \psi_{k_s}, \qquad i, j \in \mathbb{N},$$

where c_{ij}^s, $s = 1, 2, \dots m$, $i, j \in \mathbb{N}$, are arbitrary real numbers.

4.2. A class of nonlinear SDEs

Now we will consider a class of nonlinear SDEs with Wick products involving H–valued GRPs (I).

Let $P(\mathcal{R})$ be the differential operator defined as in (11). Assume that $P(\widetilde{\lambda_k}) - 1 \neq 0$ and $P(\widetilde{\lambda_k}) \neq 0$ for all $k \in \mathbb{N}$. Consider a nonlinear SDE of the form

$$P(\mathcal{R})X = X \blacklozenge \tilde{W} + g, \tag{13}$$

where $\tilde{W}$ is the singular white noise in $\mathcal{A}(H)_0^*$ given by the expansion

$$\tilde{W}(t, \omega) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} w_{i,n}(t) \otimes H_{\varepsilon_n}(\omega) e_i, \qquad t \in I,\ \omega \in \Omega,$$

and

$$w_{i,n} = \begin{cases} \psi_j(t), & n = m(i,j) \\ 0, & \text{else} \end{cases}$$

and $m(i, j)$ is defined as in Example 1.1.

Let $g \in \mathcal{A}(H)_r^*$ be of the form

$$g(t, \omega) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} g_{i,n}(t) \otimes H_{\varepsilon_n}(\omega) e_i, \qquad t \in I,\ \omega \in \Omega,$$

such that $g_{i,n} \in \mathcal{A}_{-r}$, $i,n \in \mathbb{N}$; and assume that there exist $q \geq 0$, such that

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\|g_{i,n}\|_{-r}^{2}(2\mathbb{N})^{-q\varepsilon_n} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\|g_{i,n}\|_{-r}^{2}(2n)^{-q} < \infty. \tag{14}$$

Expanding each $g_{i,n}$ in $\mathcal{A}_{-r}$ we get

$$g_{i,n}(t) = \sum_{k=1}^{\infty} g_{i,n,k}\psi_k(t), \qquad g_{i,n,k} \in \mathbb{R},\ t \in I,\ i,n \in \mathbb{N},$$

which yields

$$\sum_{k=1}^{\infty} |g_{i,n,k}|^2 \widetilde{\lambda_k}^{-2r} < \infty. \tag{15}$$

We will look for the solution X of the equation in the form

$$X(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a_{i,n}(t) \otimes H_{\varepsilon_n}(\omega)e_i, \qquad t \in I,\ \omega \in \Omega,$$

where $a_{i,n} \in \mathcal{A}'$, $i,n \in \mathbb{N}$, are the coefficients to be determined. Let $a_{i,n}(t)$, $t \in I$, be given by the expansion

$$a_{i,n}(t) = \sum_{k=1}^{\infty} a_{i,n,k}\psi_k(t), \qquad a_{i,n,k} \in \mathbb{R},\ t \in I,\ n \in \mathbb{N}.$$

Then,

$$P(\mathcal{R})X(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{i,n,k}P(\widetilde{\lambda_k})\psi_k(t)\right) \otimes H_{\varepsilon_n}(\omega)e_i, \quad t \in I,\ \omega \in \Omega, \tag{16}$$

and due to the definition of Wick product:

$$\begin{aligned} &X(t,\omega)\blacklozenge\tilde{W}(t,\omega) + g(t,\omega) = \\ &\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{r+s=n+1} a_{i,s}(t) \diamond w_{i,r}(t) + g_n(t)\right) \otimes H_{\varepsilon_n}(\omega)e_i, \quad t \in I,\ \omega \in \Omega. \end{aligned} \tag{17}$$

Since

$$\begin{aligned} a_{i,s}(t) \diamond w_{i,r}(t) &= \sum_{k=1}^{\infty} a_{i,s,k}\psi_k(t) \diamond w_{i,r}(t) = \sum_{k=1}^{\infty} a_{i,s,k}\psi_k(t) \diamond \psi_j(t) \\ &= \sum_{k=1}^{\infty} a_{i,s,k}\psi_{k+j-1}(t), \end{aligned}$$

for $j \in \mathbb{N}$ such that $r = m(i,j)$, relation (17) becomes

$$X(t,\omega)\blacklozenge\tilde{W}(t,\omega) + g(t,\omega) =$$

(18)
$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{r+s=n+1}\sum_{k=1}^{\infty} a_{i,s,k}\psi_{k+j-1}(t) + g_n(t)\right)\otimes H_{\varepsilon_n}(\omega)e_i,\ t \in I,\ \omega\in\Omega.$$

From (16) and (18) we obtain

(19)
$$\sum_{\substack{s,r\in\mathbb{N}\\ s+r=n+1, r=m(i,j)}}\sum_{k=1}^{\infty} a_{i,s,k}\psi_{k+j-1}(t)+\sum_{k=1}^{\infty} g_{i,n,k}\psi_k(t) = \sum_{k=1}^{\infty} a_{i,n,k}P(\widetilde{\lambda_k})\psi_k(t),\ i,n\in\mathbb{N}.$$

From the system of equations (19) one can recursively determine the coefficients $a_{i,n,k}$, $i,n,k \in \mathbb{N}$.

For $i = 1$, $n = 1$ we have only one possibility how to get $r + s = 2$ ($s = 1, r = 1$) and for $r = 1$ the corresponding j is $j = 1$. Thus, (19) gives

$$\sum_{k=1}^{\infty} a_{1,1,k}\psi_k(t) + \sum_{k=1}^{\infty} g_{1,1,k}\psi_k(t) = \sum_{k=1}^{\infty} a_{1,1,k}P(\widetilde{\lambda_k})\psi_k(t), \quad t\in I,$$

which implies

$$a_{1,1,k} = \frac{g_{1,1,k}}{P(\widetilde{\lambda_k})-1}, \qquad k = 1,2,\ldots \tag{20}$$

For $i = 1$, $n = 2$ we have two possibilities how to get $r + s = 3$ ($s = 1, r = 2$ and $s = 2, r = 1$). For $r = 1$ we have $j = 1$, and for $r = 2$ we also get $j = 1$. Thus, from (19) we obtain

$$\sum_{k=1}^{\infty} a_{1,1,k}\psi_k(t) + \sum_{k=1}^{\infty} g_{1,2,k}\psi_k(t) = \sum_{k=1}^{\infty} a_{1,2,k}P(\widetilde{\lambda_k})\psi_k(t), \quad t\in I,$$

and since $a_{1,1,k}$ are known from the previous step, now we get

$$a_{1,2,k} = \frac{a_{1,1,k}+g_{1,2,k}}{P(\widetilde{\lambda_k})}, \qquad k = 1,2,\ldots$$

For $i = 1$, $n = 3$, and, consequently, $r + s = 4$, we get following triples: $s = 1, r = 3, j = 2$; $s = 3, r = 1, j = 1$ and $s = 2, r = 2, j = 1$. Thus,

$$\sum_{k=1}^{\infty} a_{1,1,k}\psi_{k+1} + \sum_{k=1}^{\infty} a_{1,3,k}\psi_k + \sum_{k=1}^{\infty} a_{1,2,k}\psi_k + \sum_{k=1}^{\infty} g_{1,3,k}\psi_k = \sum_{k=1}^{\infty} a_{1,3,k}P(\widetilde{\lambda_k})\psi_k.$$

After reordering the indeces in the first sum we get

$$a_{1,3,1} = \frac{a_{1,2,1} + g_{1,3,1}}{P(\widetilde{\lambda_1}) - 1},$$
$$a_{1,3,k} = \frac{a_{1,1,k-1} + a_{1,2,k} + g_{1,3,k}}{P(\widetilde{\lambda_k}) - 1}, \quad k = 2, 3, \dots$$

We follow this schedule for $n = 4, 5, \dots$. Then we fix $i = 2$ and obtain the coefficients for $n = 1, 2, 3, \dots$ given by following recursion formulae:

$$a_{2,1,k} = \frac{g_{2,1,k}}{P(\widetilde{\lambda_k}) - 1}, \qquad k = 1, 2, \dots$$
$$a_{2,2,k} = \frac{a_{2,1,k} + g_{2,2,k}}{P(\widetilde{\lambda_k})}, \qquad k = 1, 2, \dots$$
$$a_{2,3,1} = \frac{a_{2,2,1} + g_{2,3,1}}{P(\widetilde{\lambda_1}) - 1},$$
$$a_{2,3,k} = \frac{a_{2,1,k-1} + a_{2,2,k} + g_{2,3,k}}{P(\widetilde{\lambda_k}) - 1}, \quad k = 2, 3, \dots$$

Then we fix $i = 3$, and so on....

Since for each $r \in \mathbb{N}$ its corresponding $j \in \mathbb{N}$ from $r = m(i, j)$ is always $j < r$, we get a general formula

$$a_{i,n,k} = \frac{g_{i,n,k} + L}{P(\widetilde{\lambda_k}) - 1}, \qquad i \in \mathbb{N},\ n \in \mathbb{N};\ k = n, n+1, \dots \tag{21}$$

where L is a linear combination of a_{i,n_1,k_1} for $n_1 < n$, $k_1 < k$.

Since $P(\widetilde{\lambda_k}) - 1 \neq 0$, $k \in \mathbb{N}$, there exists a constant $K > 0$ such that $|P(\widetilde{\lambda_k}) - 1| \geq K$, $k \in \mathbb{N}$. Relation (15) yields that there exists $C(i, n) > 0$, such that

$$|g_{i,n,k}| \leq C(i, n)\widetilde{\lambda_k}^r, \qquad k \in \mathbb{N}.$$

Similarly, according to (14), there exists $D > 0$, such that

$$\|g_{i,n}\|_{-r}^2 = \sum_{k=1}^{\infty} |g_{i,n,k}|^2 \widetilde{\lambda_k}^{-2r} \leq D(2n)^q, \qquad i, n \in \mathbb{N}.$$

Hence, for each $i, n \in \mathbb{N}$ we have $C(i, n) \leq D(2n)^q$ and consequently

$$|g_{i,n,k}| \leq D(2n)^q \widetilde{\lambda_k}^r, \qquad i, n, k \in \mathbb{N}. \tag{22}$$

We prove now the estimate

$$|a_{i,n,k}| \leq D(2n)^q \widetilde{\lambda_k}^r Q_n(\frac{1}{K}), \quad i, n \in \mathbb{N},\ k \geq n, \tag{23}$$

where Q_n is a polynomial of order n. The proof can be done by induction. For $i = n = 1$ we get from (20) and (22) that $|a_{1,1,k}| \le \frac{1}{K} D 2^q \widetilde{\lambda_k}^r$, $k \in \mathbb{N}$. Assume now (23) holds. Then, from (21) and (22) we get

$$|a_{i+1,n+1,k}| \le \frac{1}{K}\left(|L| + |g_{i+1,n+1,k}|\right) \le \frac{1}{K}\left(|L| + D(2(n+1))^q \widetilde{\lambda_{k+1}}^r\right).$$

Since L is a linear combination of n coefficients a_{i,n_1,k_1}, using the induction hypothesis we get $|L| \le D(2n)^q \widetilde{\lambda_k}^r Q_n(\frac{1}{K})$. Thus, since $\widetilde{\lambda_k}^r \le \widetilde{\lambda}_{k+1}^r$, and $(2n)^q \le (2(n+1))^q$, we obtain

$$|a_{i+1,n+1,k}| \le D(2(n+1))^q \widetilde{\lambda_{k+1}}^r \frac{1}{K}\left(1 + Q_n(\frac{1}{K})\right) = D(2(n+1))^q \widetilde{\lambda_{k+1}}^r R_{n+1}(\frac{1}{K}),$$

where R_{n+1} is some polynomial of order $n+1$. This proves (23).

Let p be such that $\sum_{k=1}^{\infty} \widetilde{\lambda_k}^{-2p} < \infty$ (such p exists, since $\mathcal{A}$ is nuclear). Then, due to (23)

$$\sum_{k=1}^{\infty} |a_{i,n,k}|^2 \widetilde{\lambda_k}^{-2r-2p} \le D^2 (2n)^{2q} Q_n^2(\frac{1}{K}) \sum_{k=1}^{\infty} \widetilde{\lambda_k}^{-2p} < \infty,$$

and thus,

$$\|a_{i,n}\|_{-(r+p)}^2 < \infty, \qquad i, n \in \mathbb{N},$$

which yields that all coefficients $a_{i,n}$, $i, n \in \mathbb{N}$, belong to $\mathcal{A}_{-(r+p)}$.

Also, it holds that,

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \|a_{i,n}\|_{-(r+p)}^2 e^{-s(2\mathbb{N})^{\varepsilon_n}} \le D^2 M \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} (2n)^{2q} Q_n^2(\frac{1}{K}) e^{-2sn}. \tag{24}$$

The series on the right-hand side of (24) can be made convergent if we choose s large enough. Thus, there exists a unique solution X of equation (13) in the space ${}^{exp}\mathcal{A}(H)_{r+p}^*$.

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