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ASYMPTOTIC NUMERICAL METHOD FOR SINGULARLY PERTURBED THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH A DISCONTINUOUS SOURCE TERM

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Abstract. A class of singularly perturbed two point Boundary Value Problems (BVPs) of reaction-diffusion type for third order Ordinary Differential Equations (ODEs) with a small positive parameter (ε) multiplying the highest derivative and a discontinuous source term is considered. The BVP is reduced to a weakly coupled system consisting of one first order ordinary differential equation with a suitable initial condition and one second order singularly perturbed ODE subject to boundary conditions. In order to solve this system, a computational method is suggested. First, in this method, we find the zero order asymptotic expansion approximation of the solution of the weakly coupled system. Then, the system is decoupled by replacing the first component of the solution by its zero order asymptotic expansion approximation of the solution in the second equation. After that the second equation is solved by a finite difference method on Shishkin mesh (a fitted mesh method). Examples are provided to illustrate the method.

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1. Introduction

Singularly perturbed differential equations arise in many branches of science and engineering. The solutions of such equations have boundary and interior layers. That is, there are thin layer(s) where the solution changes rapidly, while away from the layer(s) the solution behaves regularly and changes slowly. So the numerical treatment of singularly perturbed differential equations gives major computational difficulties, and in recent years, a large number of special purpose methods have been developed to provide accurate numerical solutions [1, 2, 6, 9] which cover mostly second order equations. But only a very few authors have developed numerical methods for singularly perturbed higher order

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differential equations [11]. Moreover, most of them have concentrated only on the problems with smooth data. Of course, some authors [3, 4, 5, 7, 10] have recently considered Singular Perturbation Problems (SPPs) for second order ODEs with discontinuous source term and discontinuous convection coefficient. Due to the discontinuity at one or more points in the interior domain, this gives rise to an interior layer(s) in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Therefore, these types of SPPs have to be dealt with separately and carefully. In this paper, an asymptotic numerical method for singularly perturbed reaction-diffusion type third order ODE with a discontinuous source term is developed. The classification of singularly perturbed higher order problems (reaction-diffusion/convection-diffusion) depend on how the order of the original equation is affected if one sets $\varepsilon = 0$. If the order is reduced by one, we say that the problem is of convection-diffusion type, and of reaction-diffusion type if the order is reduced by two.

Motivated by the works of [4, 11], a class of singularly perturbed BVPs for third order ODEs of reaction-diffusion type with discontinuous right-hand side term is considered on the unit interval $\Omega = (0, 1)$. A single discontinuity in the right-hand side is assumed to occur at a point $d \in \Omega$. It is convenient to introduce the notation $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and to denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$. The corresponding class of BVPs is

$$-\varepsilon y'''(x) + b(x)y'(x) + c(x)y(x) = f(x),\ x \in (\Omega^- \cup \Omega^+), \tag{1.1}$$

$$y(0) = p, \quad y'(0) = q, \quad y'(1) = r, \tag{1.2}$$

where ε is a small positive parameter, $b(x), c(x)$ are sufficiently smooth functions on $\bar{\Omega}$ such that

$$b(x) \geq \beta > 0, \tag{1.3}$$

$$0 \geq c(x) \geq -\gamma, \gamma > 0, \tag{1.4}$$

$$\beta - \theta\gamma \geq \eta > 0, \quad \text{for some } \theta > 2 \text{ arbitrarily close to } 2, \text{ for some } \eta. \tag{1.5}$$

It is assumed that f is sufficiently smooth on $\Omega^- \cup \Omega^+$; the left and right limit of f and their derivatives are assumed to exists at $x = d$. The discontinuity in the source term, in general, gives rise to interior layer in the first derivative of the solution. Because f is discontinuous at d, the solution y of (1.1)-(1.2) does not necessarily have a continuous third derivative at the point d, that is, y does not belong to the class of functions $C^3(\Omega)$. Hence the class of functions, where y belongs to it, is taken as $C^1(\bar{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^- \cup \Omega^+)$.

In the following, C is a generic constant independent of the nodes, mesh sizes and the perturbation parameter ε. We use the norm $||w||_D = \sup_{x\in D} |w(x)|$.

2. Preliminaries

In this section, a maximum principle is presented for the following problem, and then, using this principle, a stability result for the same problem is derived. Further, an asymptotic expansion approximation is constructed for the solution and a theorem is presented to establish its accuracy. The singularly perturbed BVP (1.1)-(1.2) can be transformed into an equivalent problem of the form

$$\text{(2.1)}\quad \begin{cases} P_1\bar{y}(x) \equiv y_1'(x) - y_2(x) = 0, \ x \in \Omega \cup \{1\}, \\ P_2\bar{y}(x) \equiv -\varepsilon y_2''(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), \ x \in (\Omega^- \cup \Omega^+), \end{cases}$$

$$\text{(2.2)}\quad \begin{cases} y_1(0) = p, \\ y_2(0) = q, \\ y_2(1) = r, \end{cases}$$

where $\bar{y} = (y_1, y_2)^T$, $y_1 \in C^1(\bar{\Omega})$ and $y_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [4].

Remark 2.1. *Hereafter, only the above problem (2.1)-(2.2) is considered subject to conditions (1.3)-(1.5). The condition (1.3) says that the problem (2.1)-(2.2) is a non-turning point problem. The condition (1.4) is imposed to ensure that the system (2.1)-(2.2) is quasi-monotone (Definition (2.1) of [8] and [9]). The condition (1.5) helps to establish the maximum principle for the system (2.1)-(2.2), and, using this principle, we can establish a uniform stability result.*

Theorem 2.2. *The problem (1.1)-(1.2) has a solution $y \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^- \cup \Omega^+)$.*

Proof. Following the procedure adopted in [4], it can be proved that the problem (1.1)-(1.2) has a solution belonging to the class stated in the statement of the theorem. □

2.1. Maximum Principle and Stability Result

Theorem 2.3. **(Maximum Principle)** *Suppose that $\bar{u} = (u_1, u_2)^T$, $u_1 \in C^1(\bar{\Omega})$ and $u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies*

$$u_1(0) \geq 0, \ u_2(0) \geq 0, \ u_2(1) \geq 0,$$

$$P_1\bar{u}(x) \geq 0, \quad \forall \quad x \in \Omega \cup \{1\},$$

$$P_2\bar{u}(x) \geq 0, \quad \forall \quad x \in (\Omega^- \cup \Omega^+),$$

and

$$[u_2'](d) \leq 0. \quad \textit{Then} \quad \bar{u}(x) \geq \bar{0}, \forall x \in \bar{\Omega}.$$

Proof. Define $\bar{s}(x) = (s_1(x), s_2(x))$ as

$$s_1(x) = (1+\delta)x + \delta, \quad x \in \bar{\Omega} \quad \text{and} \quad 0 < \delta \ll 1,$$

$$s_2(x) = \begin{cases} (1/2) - (x/8) + (d/8), \ x \in \Omega^- \cup \{0, d\}, \\ (1/2) - (x/4) + (d/4), \ x \in \Omega^+ \cup \{1\}, \end{cases}$$

where $s_1 \in C^1(\bar{\Omega})$ and $s_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$. Then $\bar{s}(x) > \bar{0}$,

$$P_1\bar{s} = s_1' - s_2 = \begin{cases} (1/2) + \delta + (x/8) - (d/8) > 0, \, x \in \Omega^- \cup \{d\}, \\ (1/2) + \delta + (x/4) - (d/4) > 0, \, x \in \Omega^+ \cup \{1\}, \end{cases}$$

$$\begin{aligned} P_2\bar{s} &= -\varepsilon s_2'' + b(x)s_2 + c(x)s_1 \\ &= \begin{cases} b(x)[(1/2) - (x/8) + (d/8)] + c(x)[(1+\delta)x + \delta], \\ b(x)[(1/2) - (x/4) + (d/4)] + c(x)[(1+\delta)x + \delta], \end{cases} \\ &\geq \begin{cases} (1/8)(\beta - \theta\gamma) \geq (\eta/8) > 0, \, x \in \Omega^-, \\ (1/4)(\beta - \theta\gamma) \geq (\eta/4) > 0, \, x \in \Omega^+. \end{cases} \end{aligned}$$

Assume the theorem is not true. We define

$$\zeta = \max\left\{\max_{x\in\bar{\Omega}}\left(-\frac{u_1}{s_1}\right), \max_{x\in\bar{\Omega}}\left(-\frac{u_2}{s_2}\right)\right\}$$

Then $\zeta > 0$ and there exists a point x_0 such that

$$\left(-\frac{u_1}{s_1}\right)(x_0) = \zeta \quad \text{or} \quad \left(-\frac{u_2}{s_2}\right)(x_0) = \zeta, \quad \text{or both.}$$

Further, $x_0 \in (\Omega^- \cup \Omega^+)$ or $x_0 = d$. Also, $(u_i + \zeta s_i)(x) \geq 0$, $i = 1, 2$, $x \in \bar{\Omega}$.

Case 1: $\left(-\frac{u_1}{s_1}\right)(x_0) = \zeta$ and $x_0 \in \Omega \cup \{1\}$. That is

$$(u_1 + \zeta s_1)(x_0) = 0 \Rightarrow (u_1 + \zeta s_1) \, \text{attains its minimum at} \, x_0.$$

Therefore

$$0 < P_1(\bar{u} + \zeta\bar{s})(x_0) = (u_1 + \zeta s_1)'(x_0) - (u_2 + \zeta s_2)(x_0) \leq 0.$$

It is a contradiction.

Case 2a: $\left(-\frac{u_2}{s_2}\right)(x_0) = \zeta$ and $x_0 \in (\Omega^- \cup \Omega^+)$. That is

$$(u_2 + \zeta s_2)(x_0) = 0 \Rightarrow (u_2 + \zeta s_2) \, \text{attains its minimum at} \, x_0.$$

Therefore

$$\begin{aligned} 0 &< P_2(\bar{u} + \zeta\bar{s})(x_0) \\ &= -\varepsilon(u_2 + \zeta s_2)''(x_0) + b(x_0)(u_2 + \zeta s_2)(x_0) + c(x_0)(u_1 + \zeta s_1)(x_0) \\ &\leq 0. \end{aligned}$$

It is a contradiction.

Case 2b: $\left(-\frac{u_2}{s_2}\right)(x_0) = \zeta$ and $x_0 = d$. That is

$$(u_2 + \zeta s_2)(x_0) = 0 \Rightarrow (u_2 + \zeta s_2) \, \text{attains its minimum at} \, x_0.$$

Therefore

$$\begin{aligned} 0 \le [(u_2+\zeta s_2)'](d) &= [u_2'](d)+\zeta[s_2'](d) \\ &\le 0+\zeta((-1/4)+(1/8)) < 0. \end{aligned}$$

It is a contradiction. Hence the proof of the theorem. □

Theorem 2.4. **(Stability Result)** *If* $y_1 \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^- \cup \Omega^+)$, $y_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ *then*

$$|y_i(x)| \le C \max\{|y_1(0)|, |y_2(0)|, |y_2(1)|, ||P_1\bar{y}||_{\Omega\cup\{1\}}, ||P_2\bar{y}||_{(\Omega^-\cup\Omega^+)}\},$$

for $i = 1, 2$ *and* $x \in \bar{\Omega}$.

Proof. Set $M = C \max\{|y_1(0)|, |y_2(0)|, |y_2(1)|, ||P_1\bar{y}||_{\Omega\cup\{1\}}, ||P_2\bar{y}||_{(\Omega^-\cup\Omega^+)}\}$. Define two barrier functions

$$\bar{w}^{\pm}(x) = (w_1^{\pm}(x), w_2^{\pm}(x)) \quad as$$

$$w_1^{\pm}(x) = M[(1+2\delta)x+\delta] \pm y_1(x), \quad 0 < \delta \ll 1$$

$$w_2^{\pm}(x) = M \pm y_2(x).$$

We have

$$\begin{aligned} P_1\bar{w}^{\pm}(x) &= w_1^{\pm'}(x) - w_2^{\pm}(x) \\ &= M2\delta \pm P_1\bar{y} \ge 0, \end{aligned}$$

and

$$\begin{aligned} P_2\bar{w}^{\pm}(x) &= -\varepsilon w_2^{\pm''}(x) + b(x)w_2^{\pm}(x) + c(x)w_1^{\pm}(x) \\ &= b(x)M + c(x)M[(1+2\delta)x+\delta] \pm P_2\bar{y} \ge 0, \end{aligned}$$

by a proper choice of C. Furthermore, we have

$$w_1^{\pm}(0) = M\delta \pm y_1(0) \ge 0, \;\; w_2^{\pm}(0) = M \pm y_2(0) \ge 0,$$

$$w_2^{\pm}(1) = M \pm y_2(1) \ge 0, \text{ and } \quad [w_2^{\pm'}](d) = [y_2'](d) = 0,$$

by a proper choice of C. Applying Theorem 2.3 to the barrier functions $\bar{w}^{\pm}(x)$, we get the desired result. □

Corollary 2.5. *If* (y_1, y_2) *is the solution of the BVP (2.1)-(2.2) with the conditions (1.3)-(1.5), then we have*

$$|y_i(x)| \le C \max\{|p|, |q|, |r|, ||f||_{(\Omega^-\cup\Omega^+)}\}, \quad i = 1, 2, \quad x \in \bar{\Omega}.$$

2.2. Asymptotic Expansion Approximation

Using one of the perturbation methods [6] we can construct an asymptotic expansion for the solution of the BVP (2.1)-(2.2) as follows. Motivated by the perturbation theory for SPPs, let (u_{01}, u_{02}) be the solution of the following problem:

$$\begin{cases} u_{01}'(x) - u_{02}(x) = 0, \\ b(x)u_{02}(x) + c(x)u_{01}(x) = f(x) \\ u_{01}(0) = p, \quad u_{01}(d-) = u_{01}(d+). \end{cases}$$

That is, u_{01} is given by

$$b(x)u_{01}'(x) + c(x)u_{01}(x) = f(x),\ x \in (\Omega^- \cup \Omega^+), \tag{2.3}$$

$$u_{01}(0) = p, \quad u_{01}(d-) = u_{01}(d+). \tag{2.4}$$

Obviously $u_{01} \in C^0(\bar{\Omega}) \cap C^1(\Omega^- \cup \Omega^+)$. Let u_{02} be defined by

$$u_{02}(x) = \frac{f(x) - c(x)u_{01}(x)}{b(x)}, \quad x \in \Omega^- \cup \Omega^+. \tag{2.5}$$

Further, Let $\bar{v}_{L0} = (v_{L01}, v_{L02})$ and $\bar{v}_{R0} = (v_{R01}, v_{R02})$ be the left layer corrections given by

$$v_{L01} = -\sqrt{\frac{\varepsilon}{b(0)}}v_{L02}, \quad v_{L02} = k_1 e^{-x\sqrt{b(0)/\varepsilon}}, \quad x \in \{0\} \cup \Omega^-,$$

$$v_{R01} = -\sqrt{\frac{\varepsilon}{b(d)}}v_{R02}, \quad v_{R02} = k_2 e^{-(x-d)\sqrt{b(d)/\varepsilon}}, \quad x \in \Omega^+ \cup \{1\},$$

and let $\bar{w}_{L0} = (w_{L01}, w_{L02})$ and $\bar{w}_{R0} = (w_{R01}, w_{R02})$ be the right layer corrections given by

$$w_{L01} = \sqrt{\frac{\varepsilon}{b(d)}}w_{L02}, \quad w_{L02} = k_3 e^{-(d-x)\sqrt{b(d)/\varepsilon}}, \quad x \in \{0\} \cup \Omega^-,$$

$$w_{R01} = \sqrt{\frac{\varepsilon}{b(1)}}w_{R02}, \quad w_{R02} = k_4 e^{-(1-x)\sqrt{b(1)/\varepsilon}}, \quad x \in \Omega^+ \cup \{1\},$$

where constants k_1, k_2, k_3 and k_4 will be fixed soon.
Define

$$y_{1,as}^*(x) = \begin{cases} u_{01}(x) + v_{L01}(x) + w_{L01}(x), & x \in \{0\} \cup \Omega^-, \\ u_{01}(x) + v_{R01}(x) + w_{R01}(x), & x \in \Omega^+ \cup \{1\}. \end{cases}$$

and

$$y_{2,as}^*(x) = \begin{cases} u_{02}(x) + v_{L02}(x) + w_{L02}(x), & x \in \{0\} \cup \Omega^-, \\ u_{02}(x) + v_{R02}(x) + w_{R02}(x), & x \in \Omega^+ \cup \{1\}. \end{cases}$$

We now choose the constants k_1, k_2, k_3 and k_4 such that $y^*_{2,as} \in C^0(\bar{\Omega}) \cap C^1(\Omega)$, that is,

$$y^*_{2,as}(0) = y_2(0), \quad y^*_{2,as}(1) = y_2(1),$$
$$y^*_{2,as}(d+) = y^*_{2,as}(d-), \quad y^{*'}_{2,as}(d+) = y^{*'}_{2,as}(d-).$$

The constants are given by

$$\begin{aligned}
k_1 &= [y_2(0) - u_{02}(0)] - k_3 e^{-d\sqrt{b(d)/\varepsilon}},\\
k_2 &= \frac{k_{21} + k_{22}}{k_{23}},\\
k_3 &= \frac{k_{31}\{\sqrt{b(d)} + \sqrt{b(1)}e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\}}{k_{34} + k_{35}}\\
&\quad + \frac{(k_{32} + k_{33})\{1 - e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\}}{k_{34} + k_{35}},\\
k_4 &= [y_2(1) - u_{02}(1)] - k_2 e^{-(1-d)\sqrt{b(d)/\varepsilon}},\\
k_{21} &= [y_2(0) - u_{02}(0)]e^{-d\sqrt{b(0)/\varepsilon}} + k_3\left(1 - e^{-d\sqrt{(b(0)+b(d))/\varepsilon}}\right),\\
k_{22} &= [u_{02}(d+) - u_{02}(d-)] - [y_2(1) - u_{02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}},\\
k_{23} &= \left(1 - e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\right),\\
k_{31} &= \{[y_2(1) - u_{02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}} - [y_2(0) - u_{02}(0)]e^{-d\sqrt{b(0)/\varepsilon}}\}\\
&\quad + [u_{02}(d+) - u_{02}(d-)],\\
k_{32} &= \sqrt{b(1)}[y_2(1) - u_{02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}} + \sqrt{b(0)}[y_2(0) - u_{02}(0)]e^{-d\sqrt{b(0)/\varepsilon}},\\
k_{33} &= \sqrt{\varepsilon}[u'_{02}(d+) - u'_{02}(d-)]\\
k_{34} &= \left(\sqrt{b(d)} + \sqrt{b(1)}e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\right)\left(1 - e^{-d\sqrt{(b(0)+b(d))/\varepsilon}}\right),
\end{aligned}$$

and

$$k_{35} = \left(\sqrt{b(d)} + \sqrt{b(0)}e^{-d\sqrt{(b(0)+b(d))/\varepsilon}}\right)\left(1 - e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\right).$$

We now define

$$y_{1,as}(x) = \begin{cases} y^*_{1,as}(x), & x \in \{0\} \cup \Omega^-,\\ y^*_{1,as}(d-) = y^*_{1,as}(d+), & \text{at} \quad x = d\\ y^*_{1,as}(x), & x \in \Omega^+ \cup \{1\}, \end{cases}$$

and

$$y_{2,as}(x) = \begin{cases} y^*_{2,as}(x), & x \in \{0\} \cup \Omega^-,\\ y^*_{2,as}(d-) = y^*_{2,as}(d+), & \text{at} \quad x = d\\ y^*_{2,as}(x), & x \in \Omega^+ \cup \{1\}. \end{cases}$$

Theorem 2.6. *The zero order asymptotic expansion approximation* $\bar{y}_{as} = (y_{1,as}, y_{2,as})^T$ *of the solution* $\bar{y}(x)$ *of (2.1)-(2.2) satisfies the inequality*

$$|y_i(x) - y_{i,as}(x)| \leq C\sqrt{\varepsilon}, \quad x \in \bar{\Omega}, \quad i = 1, 2,$$

Proof. It is easy to prove that

$$|(y_1 - y_{1,as})(0)| \leq C\sqrt{\varepsilon},$$

$$|(y_2 - y_{2,as})(0)| = 0,\ |(y_2 - y_{2,as})(1)| = 0, [(y_2 - y_{2,as})'](d) = 0,$$

and

$$|P_1(\bar{y} - \bar{y}_{as})| = 0, \quad |P_2(\bar{y} - \bar{y}_{as})| \leq C\sqrt{\varepsilon}, \forall x \in \Omega^- \cup \Omega^+.$$

Then by heorem 2.4 we have the required result. □

3. Some analytical and numerical results for singularly perturbed BVP for second order ODEs with a discontinuous source term

We state some results for the following singularly perturbed BVP which are needed in the rest of the paper. Consider the singularly perturbed BVP

$$-\varepsilon {y_2^*}''(x) + b(x)y_2^*(x) = f(x) - c(x)u_{01}(x),\ x \in (\Omega^- \cup \Omega^+), \tag{3.1}$$

$$y_2^*(0) = q, \quad y_2^*(1) = r, \tag{3.2}$$

where $u_{01}(x)$ is defined in the last section.

Remark 3.1. *The BVP (3.1)-(3.2) has a unique solution* $y_2^* \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ *[4].*

3.1. Analytical Results

Theorem 3.2. *If* (y_1, y_2) *and* y_2^* *are solutions of the BVPs (2.1-2.2) and (3.1)-(3.2), respectively, then*

$$|(y_2 - y_2^*)(x)| \leq C\sqrt{\varepsilon}, \quad x \in \bar{\Omega}.$$

Proof. Since (y_1, y_2) is the solution (2.1)-(2.2), then y_2 satisfies the BVP

$$-\varepsilon y_2''(x) + b(x)y_2(x) = f(x) - c(x)y_1(x), \quad x \in (\Omega^- \cup \Omega^+),$$

$$y_2(0) = q, \quad y_2(1) = r.$$

Further, the function $w = y_2 - y_2^*$ satisfies the BVP

$$-\varepsilon w''(x) + b(x)w(x) = -c(x)[y_1(x) - u_{01}(x)], \quad x \in (\Omega^- \cup \Omega^+),$$

$$w(0) = 0, \quad w(1) = 0, \quad [w'](d) = 0.$$

From Theorem 2.6 and the definition of v_{L01} and w_{L01}, we have

$$\begin{aligned}|(y_1 - u_{01})(x)| &\le |(y_1 - y_{1,as})(x)| + |(y_{1,as} - u_{01})(x)|,\\ &\le C\sqrt{\varepsilon}.\end{aligned}$$

From this inequality and the stability result given in [4] we have

$$|w(x)| \le C\sqrt{\varepsilon},$$

that is,

$$|(y_2 - y_2^*)(x)| \ \le C\sqrt{\varepsilon}.$$

□

3.2. Numerical Results

Now, we describe a fitted mesh method for the problem (3.1)-(3.2). On $\Omega^- \cup \Omega^+$, a piecewise uniform mesh of N mesh intervals is constructed as follows. The interval $\bar{\Omega}^-$ is subdivided into the three subintervals.

$$[0, \tau_1], \quad [\tau_1, d - \tau_1] \quad \text{and} \quad [d - \tau_1, d]$$

for some τ_1 that satisfies $0 < \tau_1 \le \frac{d}{4}$. On $[0, \tau_1]$ and $[d - \tau_1, d]$ there is a uniform mesh with $\frac{N}{8}$ mesh intervals is placed, while on $[\tau_1, d - \tau_1]$ there is a uniform mesh with $\frac{N}{4}$ mesh intervals. The subintervals $[d, d+\tau_2]$, $[d+\tau_2, 1-\tau_2]$, $[1-\tau_2, 1]$ of $\bar{\Omega}^+$ are treated analogously for some τ_2 satisfying $0 < \tau_2 \le \frac{1-d}{4}$. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \{x_i : 1 \le i \le \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \le i \le N - 1\}.$$

Clearly, $x_{N/2} = d$ and $\bar{\Omega}_\varepsilon^N = \{x_i\}_0^N$. Note that this mesh is a uniform mesh when $\tau_1 = \frac{d}{4}$ and $\tau_2 = \frac{1-d}{4}$. It is fitted to the singular perturbation problem (3.1)-(3.2) by choosing τ_1 and τ_2 to be the following functions of N and ε

$$\tau_1 = \min\left\{\frac{d}{4}, 2\sqrt{\varepsilon/\beta}\,\ln N\right\} \text{ and } \tau_2 = \min\left\{\frac{1-d}{4}, 2\sqrt{\varepsilon/\beta}\,\ln N\right\}.$$

On the piecewise-uniform mesh $\bar{\Omega}_\varepsilon^N$ a standard centered finite difference operator is used. Then the fitted mesh method for (3.1)-(3.2) is

$$-\varepsilon\delta^2 Y_2^*(x_i) + b(x_i)Y_2^*(x_i) = f(x_i) - c(x_i)u_{01}(x_i),\ \forall x_i \in \Omega_\varepsilon^N \setminus \{d\}, \tag{3.3}$$

$$Y_2^*(x_0) = q, \quad Y_2^*(x_N) = r, \quad D^- Y_2^*(x_{N/2}) = D^+ Y_2^*(x_{N/2}) \tag{3.4}$$

where

$$\delta^2 Z_i = \left(\frac{Z_{i+1} - Z_i}{x_{i+1} - x_i} - \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}\right)\frac{2}{x_{i+1} - x_{i-1}}, \quad D^+ Z_i = \frac{Z_{i+1} - Z_i}{x_{i+1} - x_i}$$

and

$$D^- Z_i = \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}.$$

Theorem 3.3. *The error in using the scheme (3.3)-(3.4) to solve the problem (3.1)-(3.2) at the inner grid points* $\{x_i, i = 1, 2, ..., N-1\}$ *satisfies*

$$||(y_2^* - Y_2^*)||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1} \ln N.$$

Proof. See [4]. □

3.3. Description of the Computational Method

Consider the BVP (2.1)-(2.2). Let $u_{01}(x)$ be the solution of the IVP described in the last section. The first step in the method is to replace y_1 by u_{01} in the second equation of the system (2.1). Hence the system (2.1) gets decoupled. In the second step, we find a numerical solution for y_2 by applying the scheme (3.3)-(3.4) to the BVP (3.1)-(3.2). By this one obtains an approximation to the solution of the BVP (2.1)-(2.2), that is, it gives in turn an approximation for the solution and its first derivative of the BVP (1.1)-(1.2).

4. Error Estimate

Theorem 4.1. *Let* (y_1, y_2) *be the solution of (1.1)-(1.2). Further, let* $Y_2^*(x_i)$ *be its numerical solution obtained by the scheme (3.3)-(3.4). Then*

$$||y_2 - Y_2^*||_{\bar{\Omega}_\varepsilon^N} \leq C[N^{-1} \ln N + \sqrt{\varepsilon}].$$

Proof. The result of the present theorem follows from the inequality

$$|(y_2 - Y_2^*)(x_i)| \leq |(y_2 - y_2^*)(x_i)| + |(y_2^*(x_i) - Y_2^*)(x_i)|$$

and Theorems 3.2 and 3.3. □

Remark 4.2. *If a closed form solution is not availiable for* u_{01}*, one can look for numerical solution for this. Accordingly, the form of the error estimate will change.*

5. Adjoint System [9, 8]

Consider the BVP (2.1)-(2.2). Suppose that the condition (1.4) is not met, that is, the system (2.1)-(2.2) is not quasi-monotone. Then we adjoint the following system to the BVP (2.1)-(2.2):

$$\begin{cases} \hat{y_1}'(x) - \hat{y_1} = 0, \\ -\varepsilon \hat{y_2}''(x) + b(x)\hat{y_2}(x) - c^+(x)\hat{y_3}(x) + c^-(x)\hat{y_1}(x) = -f(x), \\ \hat{y_3}'(x) - \hat{y_4} = 0, \\ -\varepsilon \hat{y_4}''(x) + b(x)\hat{y_4}(x) - c^+(x)\hat{y_1}(x) + c^-(x)\hat{y_3}(x) = f(x), \end{cases} \tag{5.1}$$

$$\begin{cases} \hat{y_1}(0) = -p, \quad \hat{y_2}(0) = -q, \quad \hat{y_2}(1) = -r, \\ \hat{y_3}(0) = p, \quad \hat{y_4}(0) = q, \quad \hat{y_4}(1) = r, \end{cases} \tag{5.2}$$

where $x \in (\Omega^- \cup \Omega^+)$, and

$$c^+(x) = \begin{cases} c(x) & \text{if } c(x) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $c^-(x) = c(x) - c^+(x)$ and $\bar{\hat{y}} = (\hat{y_1}, \hat{y_2}, \hat{y_3}, \hat{y_4})$. It is easy to verify that if $\bar{y} = (y_1, y_2)$ is a solution of (2.1)-(2.2) then $\bar{\hat{y}} = (-y_1, -y_2, y_1, y_2)$ is a solution of (5.1)-(5.2). It is obvious to note that all the results derived earlier for the BVP (2.1)-(2.2) are still valid, even if the condition (1.4) is not met.

Remark 5.1. *In the adjoint system, the number of equations is doubled and hence occupies more memory spaces. Still we need this, in order to have the maximum principle and the stability result. To avoid the possibility of c^+ and c^- loosing smoothness and only belonging to $C(\Omega)$, we assume that $c(x)$ is positive throughout the interval.*

6. Numerical Examples

In this section we present numerical examples to illustrate the method presented in Section 3.3.

Example 6.1. *Consider the singularly perturbed BVP with the discontinuous source term:*

$$-\varepsilon y'''(x) + 4y'(x) - y(x) = f(x),\ x \in (\Omega^- \cup \Omega^+),$$
$$y(0) = 1, \quad y'(0) = 0, \quad y'(1) = 0,$$

where

$$f(x) = \begin{cases} 0.7 & 0 \leq x \leq 0.5, \\ -0.6 & 0.5 < x \leq 1. \end{cases}$$

For this problem

$$u_{01}(x) = \begin{cases} -0.7 + 1.7e^{x/4}, \quad x \in \{0\} \cup \Omega^-, \\ 0.6 + 1.7e^{x/4} - 1.3e^{-(0.5-x)/4}, \quad x \in \Omega^+ \cup \{0.5, 1\}. \end{cases}$$

Example 6.2. *Consider the singularly perturbed BVP with the discontinuous source term:*

$$-\varepsilon y'''(x) + (1+x)y'(x) = f(x),\ x \in (\Omega^- \cup \Omega^+),$$
$$y(0) = 1, \quad y'(0) = 1, \quad y'(1) = 0,$$

where

$$f(x) = \begin{cases} x & 0 \le x \le 0.5, \\ (1+x)^2 & 0.5 < x \le 1. \end{cases}$$

For this problem

$$u_{01}(x) = \begin{cases} x - \log(1+x) + 1, & x \in \{0\} \cup \Omega^-, \\ x + \frac{x^2}{2} + 0.875 - \log(1+0.5), & x \in \Omega^+ \cup \{0.5, 1\}. \end{cases}$$

The nodal errors and orders of convergence are estimated using the double mesh principle. Define the double mesh difference to be

$$D_\varepsilon^N = \max_{x_i \in \bar{\Omega}_\varepsilon^N} |(y^N - \bar{y}^{2N})(x_i)|, \text{and} \quad D^N = \max_\varepsilon D_\varepsilon^N$$

where $\bar{y}^{2N}$ is the piecewise linear interpolant of the mesh function Y^{2N} onto $[0,1]$. From these quantities the orders of convergence are computed from

$$p^N = \log_2(\frac{D^N}{D^{2N}}).$$

The corresponding approximate maximum pointwise error is taken to be

$$E_\varepsilon^N = \max_{x_i \in \bar{\Omega}_\varepsilon^N} |(y^N - \bar{y}^{4096})(x_i)|, \text{and} \quad E^N = \max_\varepsilon E_\varepsilon^N.$$

The computed maximum pointwise errors E_ε^N, E^N and the computed orders of convergence p^N for the above BVPs are given in Tables 1 and 2.

ε	Number of mesh points N					
	32	64	128	256	512	1024
2^{0}	5.06849-03	2.48122e-03	1.21279e-03	5.84870e-04	2.72481e-04	1.16679e-04
2^{-2}	5.73510e-03	2.74061e-03	1.32349e-03	6.34411e-04	2.94669e-04	1.25989e-04
2^{-4}	2.30746e-03	1.05560e-03	4.98761e-04	2.36479e-04	1.09240e-04	4.64790e-05
2^{-6}	1.60257e-03	4.07401e-04	1.67867e-04	7.96740e-05	3.68226e-05	1.57036e-05
2^{-8}	5.98678e-03	1.59222e-03	4.04474e-05	1.01254e-05	3.47772e-05	1.47904e-05
2^{-10}	1.35154e-02	5.98529e-03	1.59095e-03	4.03262e-04	1.00056e-04	2.38508e-05
2^{-12}	1.35005e-02	6.41069e-03	2.26165e-03	7.52706e-04	2.29287e-04	6.29280e-05
2^{-14}	1.34984e-02	6.42455e-03	2.25568e-03	7.66612e-04	2.39763e-04	6.94339e-05
2^{-16}	1.34974e-02	6.42452e-03	2.25568e-03	7.66607e-04	2.39766e-04	6.94332e-05
2^{-18}	1.34968e-02	6.42450e-03	2.25568e-03	7.66604e-04	2.39762e-04	6.94321e-05
2^{-20}	1.34965e-02	6.42449e-03	2.25568e-03	7.66603e-04	2.39761e-04	6.94328e-05
2^{-22}	1.34964e-02	6.42450e-03	2.25567e-03	7.66615e-04	2.39764e-04	6.94323e-05
2^{-24}	1.34963e-02	6.42450e-03	2.25567e-03	7.66613e-04	2.39763e-04	6.94316e-05
2^{-26}	1.34963e-02	6.42450e-03	2.25568e-03	7.66613e-04	2.39760e-04	6.94330e-05
2^{-28}	1.34963e-02	6.42448e-03	2.25567e-03	7.66610e-04	2.39767e-04	6.94331e-05
2^{-30}	1.34963e-02	6.42449e-03	2.25568e-03	7.66616e-04	2.39768e-04	6.94319e-05
2^{-32}	1.34963e-02	6.42449e-03	2.25568e-03	7.66610e-04	2.39764e-04	6.94324e-05
2^{-34}	1.34963e-02	6.42450e-03	2.25567e-03	7.66612e-04	2.39764e-04	6.94319e-05
2^{-36}	1.34963e-02	6.42450e-03	2.25568e-03	7.66611e-04	2.39765e-04	6.94336e-05
2^{-38}	1.34963e-02	6.42450e-03	2.25568e-03	7.66614e-04	2.39763e-04	6.94320e-05
2^{-40}	1.34963e-02	6.42449e-03	2.25568e-03	7.66614e-04	2.39763e-04	6.94320e-05
E^N	1.35154e-02	6.42455e-03	2.26165e-03	7.66616e-04	3.22532e-04	1.38027e-04
D^N	4.39456e-03	4.39433e-03	1.36849e-03	6.70967e-04	2.85582e-04	1.16008e-04
p^N	0.00073	1.68305	1.02828	1.23233	1.29967	—

Table 1: *Maximum pointwise errors E_ε^N for the first derivative y' of Example 6.1.*

7. Summary and Conclusions

We presented a computational method to solve third order singularly perturbed BVPs for ODEs with discontinuous source term, subject to a particular type of boundary conditions. The boundary conditions not only help us to reduce the given third order differential equation into an IVP and one second order BVP, subject to a suitable boundary conditions, but also to establish the maximum principle, a uniform stability result and other necessary estimates. As mentioned in the introduction, the second order singularly perturbed differential equations have been extensively studied. Further, no such results are reported for higher equations and in particular for higher order equations with discontinuous source term. The idea of an adjoint system presented in Section 5 is a new approach for solving a weakly coupled system of differential equations. The nonlinear BVPs of the following form can be solved by linearizing them by Newton's method of quasilinearization, as was done in [1]:

$$\varepsilon y'''(x) = F(x, y'),\ x \in (\Omega^- \cup \Omega^+),\ y(0) = p,\ y'(0) = q,\ y'(1) = r,$$

ε	Number of mesh points N					
	32	64	128	256	512	1024
2^{0}	3.51696-03	1.70900e-03	8.32200e-04	4.00567e-04	1.86439e-04	7.97973e-05
2^{-2}	5.84677e-03	2.76377e-03	1.32691e-03	6.34130e-04	2.94086e-04	1.25643e-04
2^{-4}	1.15754e-03	2.81455e-04	6.68483e-05	1.52140e-05	6.90098e-06	3.49375e-06
2^{-6}	5.02092e-03	2.52739e-03	1.24939e-03	6.05551e-04	2.82781e-04	1.21228e-04
2^{-8}	1.42377e-02	3.88827e-03	1.37845e-03	6.62628e-04	3.08110e-04	1.31797e-04
2^{-10}	5.02944e-02	1.40444e-02	3.83111e-03	9.63111e-04	3.16451e-04	1.35090e-04
2^{-12}	8.20611e-02	4.99647e-02	1.39407e-02	3.79367e-03	9.45190e-04	2.25830e-04
2^{-14}	8.16435e-02	5.23794e-02	2.06983e-02	7.06221e-03	2.22200e-03	5.47942e-04
2^{-16}	8.14671e-02	5.23799e-02	2.06593e-02	6.97759e-03	2.24251e-03	6.51767e-04
2^{-18}	8.13801e-02	5.23385e-02	2.06425e-02	6.97153e-03	2.24058e-03	6.51206e-04
2^{-20}	8.13367e-02	5.23179e-02	2.06341e-02	6.96849e-03	2.23962e-03	6.50928e-04
2^{-22}	8.13150e-02	5.23075e-02	2.06299e-02	6.96695e-03	2.23914e-03	6.50785e-04
2^{-24}	8.13042e-02	5.23024e-02	2.06278e-02	6.96620e-03	2.23892e-03	6.50715e-04
2^{-26}	8.12988e-02	5.22998e-02	2.06267e-02	6.96583e-03	2.23879e-03	6.50679e-04
2^{-28}	8.12961e-02	5.22985e-02	2.06262e-02	6.96564e-03	2.23873e-03	6.50664e-04
2^{-30}	8.12947e-02	5.22978e-02	2.06260e-02	6.96553e-03	2.23869e-03	6.50655e-04
2^{-32}	8.12941e-02	5.22975e-02	2.06258e-02	6.96549e-03	2.23869e-03	6.50651e-04
2^{-34}	8.12937e-02	5.22974e-02	2.06258e-02	6.96547e-03	2.23868e-03	6.50647e-04
E^N	8.20611e-02	5.23799e-02	2.06983e-02	7.06221e-03	2.24251e-03	6.51767e-04
D^N	3.63398e-02	3.61172e-02	1.03419e-02	4.46174e-03	2.027527e-03	9.11242e-04
p^N	0.00886	1.80418	1.21282	1.13788	1.15381	—

Table 2: *Maximum pointwise errors E_ε^N for the first derivative y' of Example 6.2.*

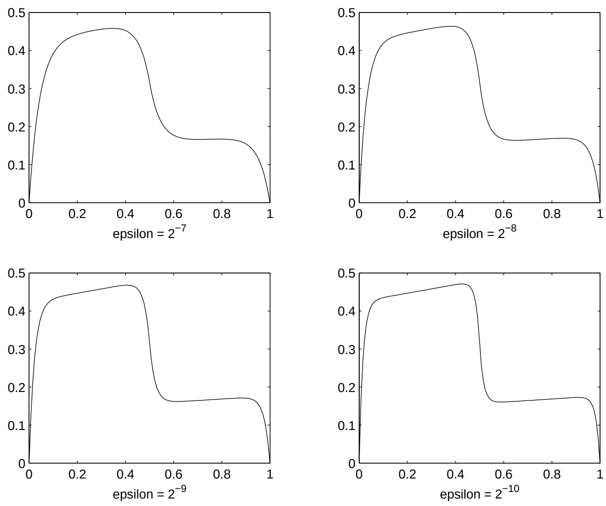

Figure 1: *Graphs of the numerical solution for the first derivative of Example 6.1 for various values of ε with $N = 128$.*

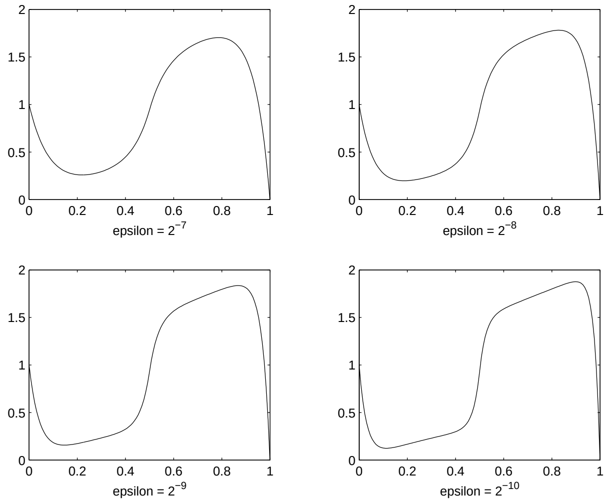

Figure 2: *Graphs of the numerical solution for the first derivative of Example 6.2 for various values of* ε *with* $N = 128$.

where

$$F_{y'}(x, y, y') \geq \beta > 0, x \in (\Omega^- \cup \Omega^+),\, y \in R,$$

$$0 \geq F_{y'}(x, y, y') \geq -\gamma, \quad \gamma > 0, \quad \beta - \theta\gamma > \eta > 0,$$

$\theta > 2$ is arbitrary close to 2.
The main advantage of the present method is that the system (2.1)-(2.2) gets decoupled and hence one can solve first for y_2, that is y', independently of y_1 (this saves memory space). Further, this method exploits the techniques available in the literature for solving SPPs for second order ODEs. It may be noted that an approximation for y_1 (that is y) is taken as u_{01}.

The present work can be extended to the case when $b(x)$ has also a single discontinuity at $x = d$ with $[b](d) \neq 0$ and $b(x) \geq \beta > 0, \quad x \in (\Omega^- \cup \Omega^+)$. In this case there is not much change in this paper except for the calculation of $y_{1,as}, y_{2,as}$. To calculate this, one has to carry out the same procedure as done in Section 2.2, but for the unknowns v_{R02}, w_{L02} and constants k_1 to k_4 which are given below.

$$v_{R02} = k_2 e^{-(x-d)\sqrt{b(d+)/\varepsilon}}, \quad w_{L02} = k_3 e^{-(d-x)\sqrt{b(d-)/\varepsilon}},$$

and

$$k_1 = [y_2(0) - u_{02}(0)] - k_3 e^{-d\sqrt{b(d-)/\varepsilon}},$$

$$k_3 = \frac{k_{31}\{\sqrt{b(d+)} + \sqrt{b(1)}e^{-(1-d)\sqrt{(b(d+)+b(1))/\varepsilon}}\}}{k_{34}+k_{35}} + \\ + \frac{(k_{32}+k_{33})\{1 - e^{-(1-d)\sqrt{(b(d+)+b(1))/\varepsilon}}\}}{k_{34}+k_{35}},$$

$$k_4 = [y_2(1) - u_{02}(1)] - k_2 e^{-(1-d)\sqrt{b(d+)/\varepsilon}},$$

$$k_{21} = [y_2(0) - u_{02}(0)]e^{-d\sqrt{b(0)/\varepsilon}} + k_3\left(1 - e^{-d\sqrt{(b(0)+b(d-))/\varepsilon}}\right),$$

$$k_{22} = [u_{02}(d+) - u_{02}(d-)] - [y_2(1) - u_{02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}},$$

$$k_{23} = \left(1 - e^{-(1-d)\sqrt{(b(d+)+b(1))/\varepsilon}}\right),$$

$$k_{31} = \{[y_2(1) - u_{02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}} - [y_2(0) - u_{02}(0)]e^{-d\sqrt{b(0)/\varepsilon}}\} + \\ + [u_{02}(d+) - u_{02}(d-)],$$

$$k_{32} = \sqrt{b(1)}[y_2(1) - u_{02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}} + \sqrt{b(0)}[y_2(0) - u_{02}(0)]e^{-d\sqrt{b(0)/\varepsilon}},$$

$$k_{33} = \sqrt{\varepsilon}[u'_{02}(d+) - u'_{02}(d-)]$$

$$k_{34} = \left(\sqrt{b(d+)} + \sqrt{b(1)}e^{-(1-d)\sqrt{(b(d+)+b(1))/\varepsilon}}\right)\left(1 - e^{-d\sqrt{(b(0)+b(d-))/\varepsilon}}\right),$$

and

$$k_{35} = \left(\sqrt{b(d-)} + \sqrt{b(0)}e^{-d\sqrt{(b(0)+b(d-))/\varepsilon}}\right)\left(1 - e^{-(1-d)\sqrt{(b(d+)+b(1))/\varepsilon}}\right).$$

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