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UNSTEADY FLOW OF A DUSTY FLUID BETWEEN TWO OSCILLATING PLATES UNDER VARYING CONSTANT PRESSURE GRADIENT

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Abstract. The problem of flow of a viscous incompressible embedded fluid with dust particles between two oscillating parallel plates is discussed using differential geometry techniques. The analysis applies to flows with plates oscillating in their own planes and the influence of constant pressure gradient. Initially, the fluid and dust particles are at rest. The expressions for exact velocities of fluid and dust particles are obtained by using Laplace transform methods. The changes in the velocity profiles at different times are shown graphically.

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1. Introduction

The influence of dust particles on viscous flows has great importance in petroleum industry and in the purification of crude oil. Other important applications of dust particles in boundary layer, include soil erosion by natural winds and dust entrainment in a cloud during nuclear explosion. Also, such flows have occur in a wide range of areas of technical importance like fluidization, flow in rocket tubes, combustion, paint spraying, and more recently, blood flows in capillaries.

P.G. Saffman [12] has discussed the stability of the laminar flow of a dusty gas in which the dust particles are uniformly distributed. Liu [9] has studied the flow induced by an oscillating infinite flat plate in a dusty gas. Michael and Miller[10] investigated the motion of dusty gas with uniform distribution of the dust particles placed in the semi-infinite space above a rigid plane boundary. Later, Samba Siva Rao [13] have obtained the analytical solutions for the dusty fluid flow through a circular tube under the influence of constant pressure gradient, using appropriate boundary conditions.

To investigate the kinematical properties of fluid flows in the field of fluid mechanics some researchers like Kanwal [8], Trusdell [14], Indrasena [7], Purushotham [11], Bagewadi, Shantharajappa and Gireesha [1, 2, 3] have applied

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differential geometry techniques. Further, in [2, 3] the authors studied two-dimensional dusty fluid flow in the Frenet frame field system. Recently, in [5, 6] the authors studied the flow of unsteady dusty fluid under varying different pressure gradients like constant, periodic and exponential. The present paper deals with investigation of the laminar flow of an unsteady viscous liquid with uniform distribution of dust particles between two oscillating plates under the influence of constant pressure gradient in anholonomic co-ordinate system. Further, by considering that the fluid and dust particles are at rest initially, the analytical expressions are obtained for velocities of the fluid and dust particles. The changes in the velocity profiles at different times are shown graphically.

2. Equations of Motion

The equations of motion of unsteady viscous incompressible fluid with uniform distribution of dust particles are given by [12]:

For fluid phase

$$\nabla.\overrightarrow{u} = 0 \quad \text{(Continuity)} \tag{2.1}$$

$$\frac{\partial\overrightarrow{u}}{\partial t}+(\overrightarrow{u}.\nabla)\overrightarrow{u}=-\rho^{-1}\nabla p+\upsilon\nabla^2\overrightarrow{u}+\frac{kN}{\rho}(\overrightarrow{v}-\overrightarrow{u}) \quad \text{(Linear Momentum)} \tag{2.2}$$

For dust phase

$$\nabla.\overrightarrow{v} = 0 \quad \text{(Continuity)} \tag{2.3}$$

$$\frac{\partial\overrightarrow{v}}{\partial t}+(\overrightarrow{v}.\nabla)\overrightarrow{v} = \frac{k}{m}(\overrightarrow{u}-\overrightarrow{v}) \quad \text{(Linear Momentum)} \tag{2.4}$$

We have the following nomenclature:

$\overrightarrow{u}-$velocity of the fluid phase, $\overrightarrow{v}-$velocity of dust phase, $\rho-$density of the gas, $p-$pressure of the fluid, $N-$number of density of dust particles, $\upsilon-$kinematic viscosity, $k = 6\pi a\mu-$Stoke's resistance (drag coefficient), $a-$spherical radius of dust particle, $m-$mass of the dust particle, $\mu-$the coefficient of viscosity of fluid particles, $t-$time.

Let $\overrightarrow{s}, \overrightarrow{n}, \overrightarrow{b}$ be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruences formed by fluid phase velocity and dusty phase velocity lines respectively. Geometrical relations are given by the Frenet formulae [4]

(2.5)
$$
\begin{aligned}
&i) \quad \frac{\partial \vec{s}}{\partial s} = k_s \vec{n}, \;\; \frac{\partial \vec{n}}{\partial s} = \tau_s \vec{b} - k_s \vec{s}, \;\; \frac{\partial \vec{b}}{\partial s} = -\tau_s \vec{n} \\
&ii) \quad \frac{\partial \vec{n}}{\partial n} = k'_n \vec{s}, \;\; \frac{\partial \vec{b}}{\partial n} = -\sigma'_n \vec{s}, \;\; \frac{\partial \vec{s}}{\partial n} = \sigma'_n \vec{b} - k'_n \vec{n} \\
&iii) \quad \frac{\partial \vec{b}}{\partial b} = k''_b \vec{s}, \;\; \frac{\partial \vec{n}}{\partial b} = -\sigma''_b \vec{s}, \;\; \frac{\partial \vec{s}}{\partial b} = \sigma''_b \vec{n} - k''_b \vec{b} \\
&iv) \quad \nabla.\vec{s} = \theta_{ns} + \theta_{bs}; \;\; \nabla.\vec{n} = \theta_{bn} - k_s; \;\; \nabla.\vec{b} = \theta_{nb}
\end{aligned}
$$

where $\partial/\partial s$, $\partial/\partial n$ and $\partial/\partial b$ are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, tangential, principal normal and binormal. The functions (k_s, k'_n, k''_b) and $(\tau_s, \sigma'_n, \sigma''_b)$ are the curvatures and torsions of the above curves and θ_{ns} and θ_{bs} are normal deformations of these spatial curves along their principal normal and binormal respectively.

3. Formulation and Solution of the Problem

This paper deals with the study of a viscous, incompressible, dusty fluid bounded by two oscillating plates. The flow is due to the influence of oscillation of plates and the constant pressure gradient. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size. The number density of the dust particles is taken as a constant throughout the flow. Under these assumptions the flow will be a parallel flow in which the streamlines are along the tangential direction and the velocities vary along the binormal direction and with time t, since we extended the fluid to infinity in the principal normal direction.

Since we have assumed that a constant pressure gradient is imposed on the system for $t > 0$, we can write

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} = a_o$$

where a_o is a constant.

By virtue of the system of equations (2.5) the intrinsic decomposition of equations (2.2) and (2.4) gives the following forms;

(3.1)
$$\frac{\partial u_s}{\partial t} = \nu\left[\frac{\partial^2 u_s}{\partial b^2} - C_r u_s\right] + \frac{kN}{\rho}(v_s - u_s) + a_o$$

(3.2)
$$2u_s^2 k_s = \nu\left[2\sigma''_b \frac{\partial u_s}{\partial b} - u_s k_s^2\right]$$

$$0 = \nu \left[u_s k_s \tau_s - 2k_b'' \frac{\partial u_s}{\partial b} \right] \tag{3.3}$$

$$\frac{\partial v_s}{\partial t} = \frac{k}{m}(u_s - v_s) \tag{3.4}$$

$$2v_s^2 k_s = 0 \tag{3.5}$$

where $C_r = (\sigma_n'^2 + k_n'^2 + k_b''^2 + \sigma_b''^2)$ is called curvature number [3].

From equation (3.5) we see that $v_s^2 k_s = 0$, which implies either $v_s = 0$ or $k_s = 0$. The choice $v_s = 0$ is impossible, since if it happens then $u_s = 0$, which shows that the flow does not exist. Hence $k_s = 0$, it suggests that the curvature of the streamline along the tangential direction is zero. Thus, no radial flow exists.

Equation (3.1) and (3.4) are to be solved subject to the initial and boundary conditions;

$$\left\{ \begin{array}{ll} \text{Initial condition;} & \text{at } t = 0;\ u_s = 0, v_s = 0 \\ \text{Boundary condition;} & \text{for } t > 0;\ u_s = u_0 \sin t,\ \text{at } b = 0 \text{ and } b = h \end{array} \right\} \tag{3.6}$$

We define the Laplace transformations of u_s and v_s as

$$U = \int_0^\infty e^{-st} u_s dt \quad \text{and} \quad V = \int_0^\infty e^{-st} v_s dt \tag{3.7}$$

Applying the Laplace transform onto equations (3.1), (3.4) and to the boundary conditions, then by using the initial conditions one obtains

$$sU = \nu \left[\frac{\partial^2 U}{\partial b^2} - C_r U \right] + \frac{l}{\tau}(V - U) + \frac{a_o}{s} \tag{3.8}$$

$$sV = \frac{1}{\tau}(U - V) \tag{3.9}$$

$$U = \frac{u_0}{1 + s^2}, \ \text{at } b = 0 \text{ and } b = h \tag{3.10}$$

where $l = \frac{mN}{\rho}$ and $\tau = \frac{m}{k}$. Equation (3.9) implies

$$V = \frac{U}{1 + s\tau} \tag{3.11}$$

Eliminating V from (3.8) and (3.11) we obtain the following equation

$$\frac{d^2 U}{db^2} - Q^2 U = -\frac{a_o}{s\nu} \tag{3.12}$$

where $Q^2 = \left(C_r + \frac{s}{\nu} + \frac{sl}{\nu(1+s\tau)}\right)$.

The velocities of fluid and dust particle are obtained by solving the equation (3.12) under to the boundary conditions (3.10) as follows

$$\begin{aligned} U &= \frac{u_o}{1+s^2}\left\{\frac{\sinh(Qb)-\sinh(Q(b-h))}{\sinh(Qh)}\right\} \\ &+\frac{a_o}{Q^2\nu s}\left[\frac{\sinh(Q(b-h))-\sinh(Qb)}{\sinh(Qh)}+1\right]. \end{aligned}$$

Using U in (3.11) we obtain V as

$$\begin{aligned} V &= \frac{u_o}{(1+s^2)(1+s\tau)}\left[\frac{\sinh(Qb)-\sinh(Q(b-h))}{\sinh(Qh)}\right] \\ &+\frac{a_o}{Q^2\nu s(1+s\tau)}\left[\frac{\sinh(Q(b-h))-\sinh(Qb)}{\sinh(Qh)}+1\right]. \end{aligned}$$

By taking the inverse Laplace transform to U and V, one can obtain (Appendix A)

$$\begin{aligned} u_s &= \frac{u_o}{E^2+F^2}\left((AE-BF)sint+(BE+AF)\cos t\right) \\ &+ \frac{a_o}{C_r\nu}\left(\frac{\sinh(\sqrt{C_r}(b-h))-\sinh(\sqrt{C_r}b)}{\sinh(\sqrt{C_r}h)}+1\right) \\ &+ u_o\pi\nu\frac{2}{h^2}\sum_{n=0}^{\infty}(-1)^n(2n+1)\sin\left(\frac{2n+1}{h}\pi b\right) \\ &\times \left[\frac{(1+x_1\tau)^2e^{x_1t}}{(1+x_1^2)\left((1+x_1\tau)^2+l\right)}+\frac{(1+x_2\tau)^2e^{x_2t}}{(1+x_2^2)\left((1+x_2\tau)^2+l\right)}\right] \\ &- \frac{2a_o}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sin\left(\frac{2n+1}{h}\pi b\right)\left(\frac{(1+x_1\tau)^2e^{x_1t}}{x_1\left((1+x_1\tau)^2+l\right)}+\frac{(1+x_2\tau)^2e^{x_2t}}{x_2\left((1+x_2\tau)^2+l\right)}\right) \end{aligned}$$

$$
\begin{aligned}
v_s &= \frac{u_o}{(E^2+F^2)(1+\tau^2)}\left((AE-BF)(\sin t-\tau\cos t)+(BE+AF)(\cos t+\tau\sin t)\right) \\
&+ \frac{a_o}{C_r\nu}\left(\frac{\sinh(\sqrt{C_r}(b-h))-\sinh(\sqrt{C_r}b)}{\sinh(\sqrt{C_r}h)}+1\right) \\
&+ u_o\pi\nu\frac{2}{h^2}\sum_{n=0}^{\infty}(-1)^n(2n+1)\sin\left(\frac{2n+1}{h}\pi b\right) \\
&\times \left[\frac{(1+x_1\tau)e^{x_1 t}}{(1+x_1^2)\left((1+x_1\tau)^2+l\right)}+\frac{(1+x_2\tau)e^{x_2 t}}{(1+x_2^2)\left((1+x_2\tau)^2+l\right)}\right] \\
&- \frac{2a_o}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sin\left(\frac{2n+1}{h}\pi b\right)\left(\frac{(1+x_1\tau)e^{x_1 t}}{x_1\left((1+x_1\tau)^2+l\right)}+\frac{(1+x_2\tau)e^{x_2 t}}{x_2\left((1+x_2\tau)^2+l\right)}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= -\frac{1}{2\tau}\left(1+l+\nu C_r\tau+\nu\tau\frac{n^2\pi^2}{h^2}\right) \\
&\quad +\frac{1}{2\tau}\sqrt{\left(1+l+\nu C_r\tau+\nu\tau\frac{n^2\pi^2}{h^2}\right)^2-4\tau\nu\left(C_r+\frac{n^2\pi^2}{h^2}\right)}
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= -\frac{1}{2\tau}\left(1+l+\nu C_r\tau+\nu\tau\frac{n^2\pi^2}{h^2}\right) \\
&\quad -\frac{1}{2\tau}\sqrt{\left(1+l+\nu C_r\tau+\nu\tau\frac{n^2\pi^2}{h^2}\right)^2-4\nu\tau\left(C_r+\frac{n^2\pi^2}{h^2}\right)} \\
y_1 &= -\frac{1}{2\tau}\left(1+l+\nu C_r\tau\right)+\frac{1}{2\tau}\sqrt{\left(1+l+\nu C_r\tau\right)^2-4C_r\nu\tau} \\
y_2 &= -\frac{1}{2\tau}\left(1+l+\nu C_r\tau\right)-\frac{1}{2\tau}\sqrt{\left(1+l+\nu C_r\tau\right)^2-4C_r\nu\tau}
\end{aligned}
$$

$$A = \sinh(\alpha b)\cos(\beta b) - \sinh(\alpha(b-h))\cos(\beta(b-h))$$

$$B = \cosh(\alpha(b-h))\sin(\beta(b-h)) - \cosh(\alpha b)\sin(\beta b)$$

$$E = \sinh(\alpha h)\cos(\beta h), \quad F = \sin(\beta h)\cosh(\alpha h)$$

$$\alpha = \sqrt{\frac{(y_1y_2-1)+\sqrt{(y_1y_2-1)^2+(y_1+y_2)^2}}{2}}$$

$$\beta = \sqrt{\frac{(1-y_1y_2)+\sqrt{(y_1y_2-1)^2+(y_1+y_2)^2}}{2}}$$

Conclusion

The velocity profiles for the fluid and dust particles are drawn in Figure 1 and 2 respectively, which are parabolic. According to the Frenet approximation of a curve in the osculating plane the path of the curve near origin is parabolic. Hence the results obtained here are analogous to the above [4]. It is concluded that the velocity of fluid particles is parallel to the velocity of dust particles. The velocity of both fluid and dust particles, which are nearer to the axis of flow, move with the greater velocity. Further, one can observe that if the dust is very fine, i.e. mass of the dust particles is negligibly small, then the relaxation time of dust particles decreases and ultimately as $\tau \to 0$ the velocities of fluid and dust particles will be the same. Also, we see that as the curvature number increases, the velocity increases too.

Note: Graphs are drawn for the values of $h = 1, \quad r = 1, \quad \nu = 0.5, \quad \tau = 0.5, \quad a_0 = 1, \quad \alpha = 1, \quad u_0 = 1, \quad l = 1$.

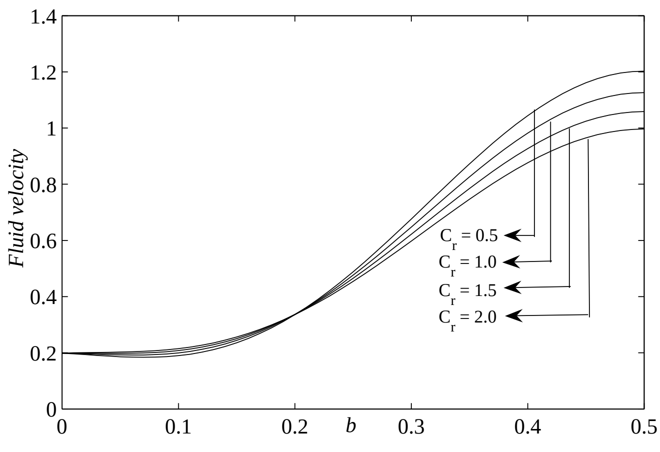

Figure-1: Variation of fluid velocity with b

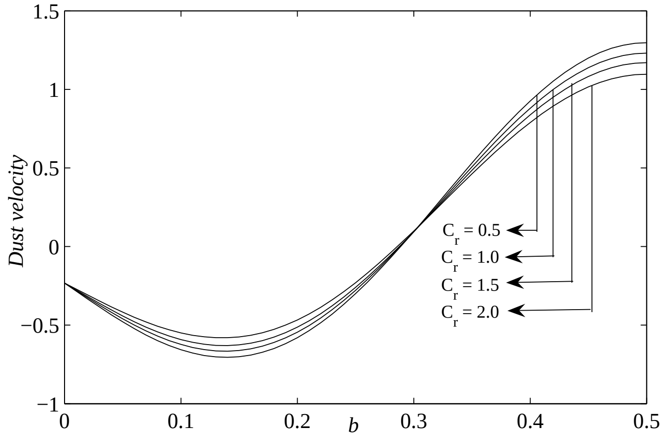

Figure-2: Variation of dust velocity with b

Appendix A

Complex Inversion Formula/Mellin-Fourier integral:

In solving partial differential equations using Laplace transform method, complex variable theory may come in handy for finding inverse transform. The inverse Laplace transform can be expressed as an integral which is known as inverse integral, and this integral can be evaluated by using contour integration methods.

The inverse Laplace Transforms of $U,\ V$ are $u_s,\ v_s$, respectively, and are given by the integrals

$$u_s = \frac{1}{2i\pi}\int_{r-i\infty}^{r+i\infty} e^{xt}U dt \quad \text{and} \quad v_s = \frac{1}{2i\pi}\int_{r-i\infty}^{r+i\infty} e^{xt}V dt$$

which can be evaluated by means of the contour integration. Since there is no branch point, the contour chosen is the closed curve ABC formed by the line $x = r$ and a semi-circle C with the origin at the center and radius R (See Figure 3) so that

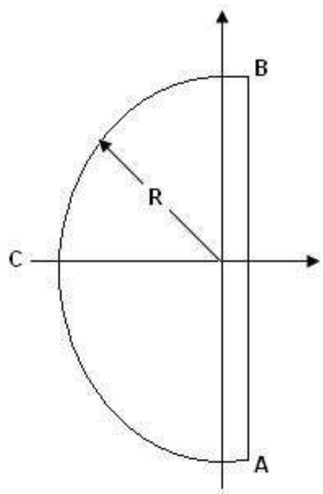

Figure-3: Cantour formed by the line $x = r$ and a semi-circle C with the origin at the center and radius R.

$$\begin{aligned}\int_{r-i\infty}^{r+i\infty} e^{xt}U dt &= \lim_{R\to\infty}\int_{A}^{B} e^{xt}U dt \\ &= \lim_{R\to\infty}\left[\ \oint_{ABC} e^{xt}U dt - \int_{C} e^{xt}U dt\right]\end{aligned}$$

Using Cauchy's theorem of residues and Jordan's lemma, we have

$$u_s = \frac{1}{2i\pi}\int_{r-i\infty}^{r+i\infty} e^{xt}U dt = \text{sum of residues of } \left\{e^{xt}U\right\} \text{ at its poles.}$$

Similarly,

$$v_s = \frac{1}{2i\pi}\int_{r-i\infty}^{r+i\infty} e^{xt}V dt \quad = \text{sum of residues of } \left\{e^{xt}V\right\} \text{ at its poles.}$$

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