Novi Sad J. Math.
Vol. 37, No. 2, 2007, 207-220

FUZZY VALUED MEASURE BASED INTEGRAL[1]

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Abstract. In this paper we extend the notion of integration in fuzzy set theory. The main purpose is to introduce and develop the notion of integral with respect to fuzzy valued measure where the basic space for fuzzy sets is Banach separable space. Some general properties of that kind of integral are investigated.

AMS Mathematics Subject Classification (2000): 46G10, 28B20

Key words and phrases: fuzzy set, integration, fuzzy valued measure

1. Introduction

Since L. Zadeh published his now classic paper [21] almost thirty years ago, fuzzy set theory has attention from researches in a wide range of scientific areas, especially in recent years. Theoretical advances and applications have been made in many directions. The theory of fuzzy sets, as its name implies, is a theory of graded concepts, a theory in which everything is a matter of degree. This theory was developed to give techniques for dealing with models for natural phenomena which do not lend themselves to analysis by classical methods based on probability theory and bivalent logic. Applications of this theory can be found in artificial intelligence, computer sciences, expert systems, logic, operations research, pattern recognition, decision theory, robotics and others.

In classical set theory if $A \subseteq \mathcal{X}$, this relation can be described by indicator (or characteristic) function $I_A : \mathcal{X} \to \{0,1\}$, where $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \in \mathcal{X}\backslash A$. One can interprete the function I_A as the degree of membership of x in $\mathcal{X}$. There are only two possibilities: 0 or 1. In fuzzy concept the set A is identified with the membership function $u_A : \mathcal{X} \to [0,1]$ where the interpretation $u_A(x)$ is the degree to which "x is in A", or x is compatible with A. Fuzzy set A of $\mathcal{X}$ we identify with its membership function u_A. The set of all functions $u : \mathcal{X} \to [0,1]$ we denote by $\mathcal{F}(\mathcal{X})$ and we say that $\mathcal{F}(\mathcal{X})$ is the set of all fuzzy sets defined on $\mathcal{X}$.

The concept of fuzzy valued measure is a natural generalization of set valued measure. Fuzzy valued measure has the range in the set of fuzzy sets and it is additive in the cense of addition defined in the set of fuzzy sets. Contributions in this field were made, among others, by Puri,Ralescu [11], Ban [2], [11], Stojaković [13], [17], [15].

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In this paper, using an additive fuzzy valued measure with values in separable Banach space, a notion of integral of single valued function with respect to a fuzzy valued measure is defined and some properties are determined. The study of integrals where integrand or measure has codomen in set of fuzzy sets is usually connected with very complicated structures and procedures. That is the reason why some methods and concepts which simplify the research in that field are introduced. Most often it is the α-level method where a fuzzy valued object is decomposed (when it is possible) into a family of set valued or single valued objects. Also, the notion of support function as usefull tool in treating weak convergence, makes it possible to simplify some investigations especially for fuzzy valued measure. Namely, there are some properties of fuzzy valued measure and integral which have an implicative or equivalent connection with its support function.

At the end of the paper, some examples which illustrate the theory are given.

2. Preliminaries

First, for the convenience of the reader, we give a list of symbols used in this paper:

$\mathbb{R}$, $\mathbb{R}_+$	set of reals, set of non negative reals,
$\mathbb{N}$	set of natural numbers,
$(\Omega, \mathcal{A}, \mu)$	complete measure space where $\mathcal{A}$ is a σ-algebra on Ω and μ is a measure on $\mathcal{A}$,
$(\mathcal{X}, \|\cdot\|)$	real separable Banach space with norm $\|\cdot\|$,
$\mathcal{X}^*$	dual of $\mathcal{X}$ (set of bounded linear functionals $x^* : \mathcal{X} \to \mathbb{R}$),
$\mathcal{P}(\mathcal{X})$	set of nonempty subsets of $\mathcal{X}$,
$\mathcal{P}_{(f)(wk)(k)(c)}(\mathcal{X})$	set of nonempty (closed),(weakly compact), (compact), (convex) subsets of $\mathcal{X}$,
$\mathcal{F}(\mathcal{X})$	set of fuzzy sets defined on $\mathcal{X}$ with nonempty α-levels,
$\mathcal{F}_{(f)(wk)(k)(c)}(\mathcal{X})$	subset of $\mathcal{F}(\mathcal{X})$ with (closed), (weakly compact), (compact), (convex) α-levels,
h	Hausdorff metric on $\mathcal{P}_f(\mathcal{X})$,
$L_\infty(\Omega, \mathcal{X}, \mu)$	set of measurable bounded a.e. functions $f : \Omega \to \mathcal{X}$,
$L_1(\Omega, \mathcal{X}, \mu)$	set of measurable, μ-integrable functions $f : \Omega \to \mathcal{X}$,
$\mathcal{L}_1(\Omega, \mathcal{X}, \mu)$	set of measurable, μ-integrable functions $F : \Omega \to \mathcal{P}(\mathcal{X})$,
$\Lambda(\Omega, \mathcal{X}, \mu)$	set of measurable, μ-integrable functions $X : \Omega \to \mathcal{F}(\mathcal{X})$,
S_F	set of measurable, integrable selectors of $F : \Omega \to \mathcal{P}(\mathcal{X})$,
$\sigma_A(x^*)$	support function of a set $A \subseteq \mathcal{X}$, $\sigma_A(x^*) = \sup_{x \in A}(x, x^*)$,
clA	closure of A, $A \subseteq \mathcal{X}$,
$\bar{co}A$	convex closure of A, $A \subseteq \mathcal{X}$,
I_A	characteristic (indicator) function of A,
u_α	α-level of fuzzy set u,
α	in index always denotes α-level of fuzzy set,
$A - B$,	$A - B = \{a - b,\ a \in A, b \in B\}$, $A, B \in \mathcal{P}(\mathcal{X})$,

Throughout this paper let $\mathcal{X}$ be a real separable Banach space, $\mathcal{X}^*$ be its dual space and $(\Omega, \mathcal{A}, \mu)$ be a complete measure space where $\mathcal{A}$ is a σ-algebra and μ is a measure. Hausdorff metric $h : \mathcal{P}_{fb}(\mathcal{X}) \times \mathcal{P}_{fb}(\mathcal{X}) \to \mathbb{R}$ is defined by

$$h(A,B) = \max\{\sup_{y\in B} \inf_{x\in A} \|x-y\|, \sup_{x\in A} \inf_{y\in B} \|x-y\|\},$$

and for $A \subset \mathcal{X}$, $|A| = h(A,\{0\}) = \sup_{x\in A} \|x\|$. By $\sigma_A(\cdot)$ we denote the support function of a set $A \subset \mathcal{X}$ defined by $\sigma_A(x^*) = \sup_{x\in A}(x^*, x)$, $x^* \in \mathcal{X}^*$.

We shall denote by $\mathcal{F}(\mathcal{X})$ the set of fuzzy sets $u : \mathcal{X} \to [0,1]$ for which the α-level sets u_α of u, defined by $u_\alpha = \{x \in \mathcal{X} : u(x) \geq \alpha\}$, $\quad \alpha \in (0,1]$ are nonempty subset of $\mathcal{X}$ for all $\alpha \in (0,1]$. By $\mathcal{F}_{(f)(wk)(k)(c)}(\mathcal{X})$ we will denote a subset of $\mathcal{F}(\mathcal{X})$ whose α-levels are (closed), (weakly compact), (compact), (convex). Notice that there is no any supposition about the set $u_0 = cl\{x \in \mathcal{X} : u(x) > 0\}$

If $X : \Omega \to \mathcal{F}(\mathcal{X})$ is a fuzzy valued function then the function $X_\alpha : \Omega \to \mathcal{P}(\mathcal{X})$ defined by $X_\alpha(\omega) = (X(\omega))_\alpha$ is a set valued function for every $\alpha \in (0,1]$. More about fuzzy and set valued functions can be found in [1], [5], [4], [6], [8], [9], [12], [15], [19], [20].

The fuzzy valued measure (see [2], [11], [13], [17]) is a natural generalization of the set valued measure (see [3], [5], [8],[9], [19]). Let $(\Omega, \mathcal{A})$ be a measurable space with $\mathcal{A}$ a σ-algebra of measurable subsets of the set Ω. If $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ is a mapping such that for every sequence $\{A_i\}_{i\in N}$ of pairwise disjoint elements of $\mathcal{A}$ the next equality is satisfied

$$\mathcal{M}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathcal{M}(A_i),$$

where

$$\left(\sum_{i=1}^{\infty} \mathcal{M}(A_i)\right)(x) = \sup\{\inf_{i\in\mathbb{N}}\{\mathcal{M}(A_i)(x_i)\} \ : \ x = \sum_{i=1}^{\infty} x_i(\text{unc.conv.})\},$$

and $\mathcal{M}(\emptyset) = I_{\{0\}}$, then $\mathcal{M}$ is a fuzzy valued measure. If for every sequence $\{A_i\}_{i\in N}$ of pairwise disjoint elements of $\mathcal{A}$, $\left(\sum_{i=1}^{\infty} \mathcal{M}(A_i)\right)_\alpha = \sum_{i=1}^{\infty} \mathcal{M}_\alpha(A_i)$, $\alpha \in (0,1]$, then $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is a set valued measure. We shall denote by $S_{\mathcal{M}_\alpha}$ the collection of all (vector valued) measure selectors of the set valued measure $\mathcal{M}_\alpha$. The variation $|\mathcal{M}_\alpha|$ of the set valued measures $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}(\mathcal{X})$, is given by $|\mathcal{M}_\alpha|(A) = \sup_\pi \sum_{A\in\pi} |\mathcal{M}_\alpha(A)|$, where the supremum is taken over all partition π of A into a finite number of pairwise disjoint members of $\mathcal{A}$. A fuzzy valued measure $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ is of bounded variation if $|\mathcal{M}_\alpha|(\Omega) < \infty$ uniformly for $\alpha \in (0,1]$. A fuzzy valued measure $\mathcal{M}$ is μ-continuous if $A \in \mathcal{A}$ with $\mu(A) = 0$ implies $\mathcal{M}(A) = I_{\{0\}}$. The set $A \in \mathcal{A}$ is an atom of the fuzzy valued measure $\mathcal{M}$ if $\mathcal{M}(A) \neq I_{\{0\}}$ and for all $B \subset A$, $\mathcal{M}(B) = I_{\{0\}}$. A fuzzy valued measure $\mathcal{M}$ with no atoms is said to be nonatomic.

The integral of a fuzzy valued function $X : \Omega \to \mathcal{F}(\mathcal{X})$ (with respect to measure μ) is a fuzzy set defined by $(\int_A X d\mu)(x) = \sup\{\alpha \in (0,1] : x \in \int_A X_\alpha d\mu\}$, $A \in \mathcal{A}$, where the integral $\int_A X_\alpha d\mu$ of the measurable set valued function

$X_\alpha : \Omega \to \mathcal{P}(\mathcal{X})$ is defined by $\int_A X_\alpha d\mu = \{\int_A f d\mu : f \in S_{X_\alpha}\}$. The integral $\int_\Omega f(\omega) d\mu(\omega)$ is defined in the sense of Bochner. A measurable set valued function $X_\alpha : \Omega \to \mathcal{P}(\mathcal{X})$ is integrably bounded if there exists integrable function $h : \Omega \to \mathbb{R}_+$ such that $|X_\alpha(\omega)| = \sup_{x \in X_\alpha(\omega)} \|x\| \leq h(\omega)$, $\mu - a.e.$ The integral of a fuzzy valued function is a natural generalization of the integral of a set valued function. It has been studied in connection with problems in probability, statistics, measure theory ([2], [4], [6], [12] [15], [19], [20]).

If $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ is a fuzzy valued measure and $X : \Omega \to \mathcal{F}(\mathcal{X})$ is a measurable fuzzy valued mapping, then X is said to be a Radon Nikodým derivative of $\mathcal{M}$ with respect to μ if $\mathcal{M}(A) = \int_A X(\omega) d\mu(\omega)$ for all $A \in \mathcal{A}$ and we write $d\mathcal{M} = X\, d\mu$. A Banach space $\mathcal{X}$ has a Radon Nikodým property (RNP) if for each finite measure space $(\Omega, \mathcal{A}, \mu)$ and each μ-continuous $\mathcal{X}$-valued measure $m : \mathcal{A} \to \mathcal{X}$ of bounded variation, there exists a Bochner integrable function $f : \Omega \to \mathcal{X}$ such that $m(A) = \int_A f d\mu$ for all $A \in \mathcal{A}$.

In a similar way as the integral of the fuzzy valued function with respect to measure μ is introduced, the integral of a measurable function $f : \Omega \to \mathbb{R}$ with respect to fuzzy valued measure $\mathcal{M} : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ will be defined in the next section of this paper. For that definition we need the definition of the related structure in the set valued case. Let $(\Omega, \mathcal{A})$ be a measurable space, $\mathcal{X}$ be a Banach space and $m : \mathcal{A} \to \mathcal{X}$ be a countable additive measure of bounded variation such that $(\Omega, \mathcal{A}, m)$ is a complete measure space. For the function $f : \Omega \to \mathbb{R}$ integrable with respect to $|m|$, the integral of f with respect to m can be defined. We denote it by $\int_\Omega f(\omega) dm(\omega)$. Detailed construction and properties of this kind of integral can be found in [3], [10]. If $M : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is a set valued measure of bounded variation, then $\int_\Omega f(\omega) dM(\omega) \stackrel{\text{def}}{=} \{\int_\Omega f(\omega) dm(\omega),\ m \in S_M\}$, where S_M is the set of measure selectors of M ([8]).

Now we give some theorems which will be used in the next section.

Theorem 2.1. *[14] Let $\mathcal{X}$ be a Banach space and $u_i \in \mathcal{F}_k(\mathcal{X})$. If for every $\alpha \in (0,1]$, $\sum_{i=1}^{\infty} |(u_i)_\alpha| < \infty$, then*

$$\left(\sum_{i=1}^{\infty} u_i \right)_\alpha = \sum_{i=1}^{\infty} (u_i)_\alpha, \quad \textit{for every } \alpha \in (0,1],$$

and $\sum_{i=1}^{\infty} u_i \in \mathcal{F}_k(\mathcal{X})$.

Theorem 2.2. *[16] Let $\mathcal{X}$ be a Banach space and $u_i \in \mathcal{F}_{wk}(\mathcal{X})$. If, $\sum_{i=1}^{\infty} |(u_i)_\alpha| < \infty$, uniformly for $\alpha \in (0,1]$, then*

$$\left(\sum_{i=1}^{\infty} u_i \right)_\alpha = \sum_{i=1}^{\infty} (u_i)_\alpha, \quad \textit{for every } \alpha \in (0,1],$$

and $\sum_{i=1}^{\infty} u_i \in \mathcal{F}_{wk}(\mathcal{X})$.

Lemma 2.1. *[11] Let M be a set and $\{M_\alpha : \alpha \in [0,1]\}$ be a family of subsets of M such that (1) $M_0 = M$, (2) $a < b \Rightarrow M_b \subseteq M_a$, (3) $a_1 < a_2 < \cdots <$*

$a_n < \cdots \to a \Rightarrow \cap_{i=1}^{\infty} M_{a_i} = M_a$. *Then the function* $u : M \to [0,1]$ *defined by* $u(x) = \sup\{a \in [0,1] : x \in M_a\}$ *has the property that* $\{x \in M : u(x) \geq a\} = M_a$ *for every* $a \in [0,1]$.

3. Integration

Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space, $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ be a μ- continuous fuzzy valued measure such that for every $\alpha \in (0,1]$, $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is a set valued measure and $f : \Omega \to \mathbb{R}_+$ a measurable function, $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$. Then for every $A \in \mathcal{A}$ the mapping $\mathcal{I}(A) : \mathcal{X} \to (0,1]$ is defined by

$$\mathcal{I}(A)(x) \stackrel{\text{def}}{=} \sup\{\alpha \in (0,1] : x \in \mathcal{I}_\alpha(A)\},$$

where

$$\mathcal{I}_\alpha(A) = \int_A f(\omega) d\mathcal{M}_\alpha(\omega) = \left\{ \int_A f(\omega) dm(\omega), \quad m \in S_{\mathcal{M}_\alpha} \right\}, \ A \in \mathcal{A},$$

for every $\alpha \in (0,1]$. We shall write

$$\mathcal{I}(A) = \int_A f(\omega) d\mathcal{M}(\omega),$$

and we shall call it - integral with respect to fuzzy valued measure. We can extend our definition of integration to integrable functions $f : \Omega \to \mathbb{R}$ using the decomposition $f = f^+ - f^-$, where $f^+ = \frac{1}{2}(|f| + f) \geq 0$, $f^- = \frac{1}{2}(|f| - f) \geq 0$. Then $\int_A f(\omega) d\mathcal{M}(\omega) = \int_A f^+(\omega) d\mathcal{M}(\omega) - \int_A f^-(\omega) d\mathcal{M}(\omega)$.

In the next five theorems we prove some results concerning the basic property of the integral - the property that the integral $\mathcal{I}$ is a new measure on the measurable space $(\Omega, \mathcal{A})$ on which the basic measure $\mathcal{M}$ is defined. From the definition of the integral $\mathcal{I}$ it is easy to notice that the "fuzzy" integral $\mathcal{I}$ is closely related to the "set" integral $\mathcal{I}_\alpha$. That connection will be used often in the proof of theorems. In the first three theorems the range of the measure $\mathcal{M}$ is a set $\mathcal{F}(\mathcal{X})$ of fuzzy sets with compact or compact convex α-levels in a separable Banach space $\mathcal{X}$, and in the fourth theorem α-levels are (only) closed but the Banach space $\mathcal{X}$ is separable and reflexive. In the last theorem the Banach space is finite dimensional.

Theorem 3.1. *Let* $\mathcal{X}$ *be a real separable Banach space and* $\mathcal{M} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ *be a* μ*-continuous fuzzy valued measure of bounded variation. If for every* $\alpha \in (0,1]$ *there exists a set* $C_\alpha \in \mathcal{P}_{kc}(\mathcal{X})$ *such that* $\mathcal{M}_\alpha(A) \subset |\mathcal{M}_\alpha|(A) C_\alpha$ *for all* $A \in \mathcal{A}$*, then for every* $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$, $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ *is a* μ*-continuous fuzzy valued measure of bounded variation. If* $\mathcal{M}$ *is nonatomic, then* $\mathcal{I}$ *is nonatomic to.*

PROOF: Since fuzzy valued measure $\mathcal{M}$ is of bounded variation, for every sequence $\{A_i\}_{i \in N}$ of pairwise disjoint elements of $\mathcal{A}$, according Prop.15 [3], we have $\sum_{i=1}^{\infty}|\mathcal{M}_\alpha(A_i)| \leq \sum_{i=1}^{\infty}|\mathcal{M}_\alpha|(A_i) < \infty$. By the compactness of $\mathcal{M}_\alpha(A)$ for every $\alpha \in (0,1]$ and every $A \in \mathcal{A}$, applying Th 2.1, we get that $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}_{kc}(\mathcal{X})$ is a set valued measure. Since $\mathcal{M} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation, $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}_{kc}(\mathcal{X})$ is a μ-continuous set valued measure of bounded variation. $\mathcal{M}_\alpha$ satisfies all the conditions of Prop. 4.1 and Cor. 5.3 [5], which implies that there exists a unique integrably bounded set valued function $X_\alpha : \Omega \to \mathcal{P}_{kc}(\mathcal{X})$ which is Radon-Nikodým derivative of $\mathcal{M}_\alpha$. Then

$$\mathcal{M}_\alpha(A) = \int_A X_\alpha(\omega) d\mu(\omega)$$

for all $A \in \mathcal{A}$ and all $\alpha \in (0,1]$, which means that $d\mathcal{M}_\alpha = X_\alpha d\mu$.

Since $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$, $X_\alpha : \Omega \to \mathcal{P}_{kc}(\mathcal{X})$ and $|f\, X_\alpha| \leq |f||X_\alpha|$, it follows that $fX_\alpha : \Omega \to \mathcal{P}_{kc}(\mathcal{X})$ defined by $(f\, X_\alpha)(\omega) = f(\omega)X_\alpha(\omega)$ is a μ-integrable, integrable bounded set valued function.

In order to prove that $\int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega) = \int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega)$, we shall show that $\int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega) \subseteq \int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega)$ and $\int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega) \subseteq \int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega)$.

If we suppose that $x \in \int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega)$, then there exists a measure selection m_α of $\mathcal{M}_\alpha$ such that $x = \int_\Omega f(\omega)dm_\alpha(\omega)$. It means that there exists an integrable selection h_α of X_α which is Radon Nikodým derivative of m_α, i.e. $dm_\alpha = h_\alpha d\mu$. Then

$$x = \int_\Omega f(\omega)h_\alpha(\omega)d\mu(\omega) \in \int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega),$$

that is, $\int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega) \subseteq \int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega)$. On the other hand, if $x \in \int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega)$, then there exists a measurable selection $h_\alpha \in S_{X_\alpha}$ such that $x = \int_\Omega f(\omega)h_\alpha(\omega)d\mu(\omega)$. Since h_α is integrably bounded it follows that

$$\int_A h_\alpha(\omega)d\mu(\omega) = m_\alpha(A), \quad A \in \mathcal{A},$$

that is, $h_\alpha\, d\mu = dm_\alpha$. It is obvious that $m_\alpha \in S_{\mathcal{M}_\alpha}$ and

$$x = \int_\Omega f(\omega)h_\alpha(\omega)d\mu(\omega) = \int_\Omega f(\omega)dm_\alpha(\omega) \in \int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega).$$

Having proved that $\int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega) = \int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega)$, it remains to recall that $fX_\alpha = (fX)_\alpha$ and

$$\mathcal{I}_\alpha(\Omega) = \int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega) = \int_\Omega f(\omega)X_\alpha(\omega)d\mu(\omega) = \left(\int_\Omega f(\omega)X(\omega)d\mu(\omega)\right)_\alpha .$$

By Cor. 5.4 [5], $\int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega) \in \mathcal{P}_{kc}(\mathcal{X})$.

In order to prove that the family $\{\mathcal{I}_\alpha(\Omega)\}_{\alpha\in(0,1]} \in \mathcal{P}_{kc}(\mathcal{X})$ defines uniquely the fuzzy set $\mathcal{I}(\Omega)$, we use Lemma 2.1. From the relation $\alpha \geq \beta \Rightarrow \mathcal{M}_\alpha(\Omega) \subseteq \mathcal{M}_\beta(\Omega)$, we get $\mathcal{I}_\alpha(\Omega) \subseteq \mathcal{I}_\beta(\Omega)$. Let us consider the sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots \to \alpha$. Equality $\cap_{i=1}^{\infty}\mathcal{M}_{\alpha_i}(A) = \mathcal{M}_\alpha(A)$, $A \in \mathcal{A}$, implies $\cap_{i=1}^{\infty} S_{\mathcal{M}_{\alpha_i}} = S_{\mathcal{M}_\alpha}$. By Prop.5.2 [9], $S_{\mathcal{M}_\beta} = \{\int g\, d\mu : g \in S_{X_\beta}\}$ for all $\beta \in \{\alpha, \alpha_1, \alpha_2, \cdots, \alpha_i \cdots\}$, where $d\,\mathcal{M}_\beta = X_\beta d\,\mu$. It is obvious that $S_{X_\alpha} \subseteq \cap_{i=1}^{\infty} S_{X_{\alpha_i}}$. Since sets on the both sides of the last inequality are closed, if $S_{X_\alpha} \neq \cap_{i=1}^{\infty} S_{X_{\alpha_i}}$, then there exists a selection $g \in \cap_{i=1}^{\infty} S_{X_{\alpha_i}}$, $g \notin S_{X_\alpha}$. The last assumption leads to the conclusion that there exists $m_g = \int g\, d\mu$, $m_g \in \cap_{i=1}^{\infty} S_{\mathcal{M}_{\alpha_i}}$, $m_g \notin S_{\mathcal{M}_\alpha}$, which contradicts the fact that $\cap_{i=1}^{\infty} S_{\mathcal{M}_{\alpha_i}} = S_{\mathcal{M}_\alpha}$. Knowing that $X_\alpha(\omega) \in \mathcal{P}_{kc}(\mathcal{X})$ for all $\alpha \in (0,1]$, and all $\omega \in \Omega$, we get $X_\alpha(\omega) = \cap_{i=1}^{\infty} X_{\alpha_i}(\omega)$ for all $\omega \in \Omega$. According the statement (IV), page 107 and Cor.5.4 from [5], all integrals $\int_\Omega f(\omega) X_\alpha(\omega) d\mu(\omega)$, $\int_\Omega f(\omega) X_{\alpha_n}(\omega) d\mu(\omega)$ could be considered as a Bochner integrals in a separable real Banach space $\hat{\mathcal{X}}$ in which the space $\mathcal{P}_{kc}(\mathcal{X})$ is embedded as a closed convex cone. Since all X_{α_n} are uniformly integrably bounded by $|X_{\alpha_1}|$, the next implication $X_{\alpha_n}(\omega) \stackrel{n\to\infty}{\longrightarrow} X_\alpha(\omega) \Rightarrow \int_\Omega f(\omega) X_{\alpha_n}(\omega) d\mu(\omega) \stackrel{n\to\infty}{\longrightarrow} \int_\Omega f(\omega) X_\alpha(\omega) d\mu(\omega)$ is true in the space $\hat{\mathcal{X}}$. Now, applying Lemma 2.1, we get that the family of sets $\left\{\int_\Omega f(\omega) d\mathcal{M}_\alpha(\omega)\right\}_{\alpha\in(0,1]} \in \mathcal{P}_{kc}(\mathcal{X})$ generates one and only one fuzzy set $\mathcal{I}(\Omega) : \mathcal{X} \to [0,1]$ defined by

$$\mathcal{I}(\Omega)(x) = \sup\left\{\alpha \in (0,1] : x \in \int_\Omega f(\omega) d\mathcal{M}_\alpha(\omega)\right\}.$$

The preceding arguments are now repeated for any $A \in \mathcal{A}$ instead of Ω.

In order to prove that $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a fuzzy valued measure, we prove first finite additivity. From Lemma 3 [14] $(\mathcal{I}(A) + \mathcal{I}(B))_\alpha = \mathcal{I}_\alpha(A) + \mathcal{I}_\alpha(B)$ for every $\alpha \in (0,1]$. Since

$$\mathcal{I}_\alpha(A+B) = \int_{A+B} f(\omega) d\mathcal{M}_\alpha(\omega) = \int_{A+B} f(\omega) X_\alpha(\omega) d\mu(\omega) =$$

$$= \int_A f(\omega) X_\alpha(\omega) d\mu(\omega) + \int_B f(\omega) X_\alpha(\omega) d\mu(\omega) = \mathcal{I}_\alpha(A) + \mathcal{I}_\alpha(B),$$

for all $A, B \in \mathcal{A}$, we get the finite additivity for $\mathcal{I}$.

To prove countable additivity of $\mathcal{I}$ we consider first countable additivity of $\mathcal{I}_\alpha$.

If $\{A_n\}_{n\in N}$ is the sequence of pairwise disjoint elements of $\mathcal{A}$, the equality

$$\mathcal{I}_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{k} \mathcal{I}_\alpha(A_n) + \mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right)$$

implies

$$h\left(\mathcal{I}_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right), \sum_{n=1}^{\infty} \mathcal{I}_\alpha(A_n)\right) =$$

$$= h\left(\sum_{n=1}^{k} \mathcal{I}_\alpha(A_n) + \mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right), \sum_{n=1}^{k} \mathcal{I}_\alpha(A_n) + \sum_{n=k+1}^{\infty} \mathcal{I}_\alpha(A_n)\right) \leq$$

$$\leq h\left(\mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right), \sum_{n=k+1}^{\infty} \mathcal{I}_\alpha(A_n)\right) \leq \left|\mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right)\right| + \left|\sum_{n=k+1}^{\infty} \mathcal{I}_\alpha(A_n)\right| \leq$$

$$\left| \int_{\cup_{n=k+1}^{\infty} A_n} f(\omega) d\mathcal{M}_\alpha(\omega) \right| + \sum_{n=k+1}^{\infty} \left| \int_{A_n} f(\omega) d\mathcal{M}_\alpha(\omega) \right| \leq$$

$$\int_{\cup_{n=k+1}^{\infty} A_n} |f(\omega)| \, d \, |\mathcal{M}_\alpha|(\omega) + \sum_{n=k+1}^{\infty} \int_{A_n} |f(\omega)| \, d \, |\mathcal{M}_\alpha|(\omega).$$

Since $\mathcal{M}_\alpha$ is a set valued measure of bounded variation, $|\mathcal{M}_\alpha|$ is a finite positive measure absolutely continuous with respect to μ. Letting $n \to \infty$, we get

$$h\left(\mathcal{I}_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right), \sum_{n=1}^{\infty} \mathcal{I}_\alpha(A_n)\right) \leq$$

$$\leq \int_{\cup_{n=k+1}^{\infty} A_n} |f(\omega)| \, d \, |\mathcal{M}_\alpha|(\omega) + \sum_{n=k+1}^{\infty} \int_{A_n} |f(\omega)| \, d \, |\mathcal{M}_\alpha|(\omega) \overset{k\to\infty}{\longrightarrow} 0.$$

To prove countable additivity of $\mathcal{I}$ we need to establish first that for every sequence $\{A_n\}_{n \in N}$ of pairwise disjoint elements of $\mathcal{A}$

$$\sum_{n=1}^{\infty} |\mathcal{I}_\alpha(A_n)| \leq \sum_{n=1}^{\infty} \int_{A_n} |f(\omega)| \, d \, |\mathcal{M}_\alpha|(\omega) =$$

$$= \sum_{n=1}^{k} \int_{A_n} |f(\omega)| \, d \, |\mathcal{M}_\alpha|(\omega) + \sum_{n=k+1}^{\infty} \int_{A_n} |f(\omega)| \, d \, |\mathcal{M}_\alpha|(\omega) < \infty.$$

Since $\mathcal{I}_\alpha(A)$ are compact for all $\alpha \in (0,1]$ and all $A \in \mathcal{A}$, the last relation allow to apply Th. 2.1,

$$\left(\sum_{i=1}^{\infty} \mathcal{I}(A_i)\right)_\alpha = \sum_{i=1}^{\infty} \mathcal{I}_\alpha(A_i), \quad \text{for every} \ \ \alpha \in (0,1].$$

Further, for every $x \in \mathcal{X}$,

$$\mathcal{I}(\bigcup_{n=1}^{\infty} A_n)(x) = \sup\left\{\alpha \in (0,1] : x \in \mathcal{I}_\alpha(\bigcup_{n=1}^{\infty} A_n)\right\} =$$

$$= \sup\left\{\alpha \in (0,1] : x \in \sum_{n=1}^{\infty} \mathcal{I}_\alpha(A_n)\right\} =$$

$$= \sup\left\{\alpha \in (0,1] : x \in \left(\sum_{n=1}^{\infty} \mathcal{I}(A_n)\right)_\alpha\right\} = \left(\sum_{n=1}^{\infty} \mathcal{I}(A_n)\right)(x)$$

which gives the countable additivity of $\mathcal{I}$.

It is easily seen that $\mathcal{I}$ is absolutely continuous with respect to $\mathcal{M}$, which implies that $\mathcal{I}$ is of bounded variation too. For the same reason $\mathcal{I}(\emptyset) = I_{\{0\}}$. If $\mathcal{M}$ is nonatomic, then all measure selection m of $\mathcal{M}_\alpha$ and all the integrals $\int f(\omega) dm(\omega)$ are nonatomic too. So, $\mathcal{I}$ is also nonatomic.

So, we conclude that $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation. $\square$

Theorem 3.2. *Let $\mathcal{X}$ be a real separable Banach space with Radon Nikodým property and $\mathcal{M} : \mathcal{A} \to \mathcal{F}_k(\mathcal{X})$ be a μ-continuous nonatomic fuzzy valued measure of bounded variation. If for every $\alpha \in (0,1]$ there exists a set $C_\alpha \in \mathcal{P}_{kc}(\mathcal{X})$ such that $\mathcal{M}_\alpha(A) \subset |\mathcal{M}_\alpha|(A)C_\alpha$ for all $A \in \mathcal{A}$, then for every $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$, $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous nonatomic fuzzy valued measure of bounded variation.*

Proof: Since Banach space $\mathcal{X}$ has the Radon-Nikodým property, we can apply Th.1.2 [5] which provides convexity of the set $\mathcal{M}_\alpha(A)$. The rest of the proof is the same as the proof of Th. 3.1.

Theorem 3.3. *Let $\mathcal{X}$ be a real separable Banach space, $\mathcal{M} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ be a μ-continuous fuzzy valued measure of bounded variation and $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$. If for every $\alpha \in (0,1]$ and every $A \in \mathcal{A}$, $0 < \mu(A) < \infty$, there exists $B_\alpha \subset A$, $\mu(B_\alpha) > 0$ and compact set $C_\alpha \in \mathcal{X}$ such that $\mathcal{M}_\alpha(D_\alpha)/\mu(D_\alpha) \subset C_\alpha$ for all $D_\alpha \subset B_\alpha$, $\mu(D_\alpha) > 0$, then $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation.*

Proof: Since all the conditions of Th.5.2. [5] are satisfied, there exists a unique Radon Nikodým derivative $X_\alpha : \Omega \to \mathcal{P}_{kc}(\mathcal{X})$, $d\mathcal{M}_\alpha = X_\alpha \, d\mu$. The rest of the proof is the same as in the Th.3.1. □

Theorem 3.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a nonatomic measure space, $\mathcal{X}$ be a real reflexive separable Banach space and $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$. If $\mathcal{M} : \mathcal{A} \to \mathcal{F}_f(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation, then for every $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$, $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation. If $\mathcal{M}$ is nonatomic, then $\mathcal{I}$ is nonatomic to.*

Proof: By Th. 2.2, $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}_f(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation, meaning that $\mathcal{M}_\alpha(A)$ is a closed bounded set for every $\alpha \in (0,1]$ and for every $A \in \mathcal{A}$. Separable reflexive Banach space has the Radon-Nikodým property, so we can apply Th.1.2 [5] which provides convexity of the set $\mathcal{M}_\alpha(A)$. By the consequence of Banach-Saks theorem, since $\mathcal{M}_\alpha(A) \subset \mathcal{X}$ is convex, closure and weak closure of the set $\mathcal{M}_\alpha(A)$ are equal. Now, since the Banach space is reflexive if and only if the unit ball is weakly compact, we can establish that $\mathcal{M}_\alpha(A)$ is weakly compact. Also, separability of reflexive space $\mathcal{X}$ implies separability of $\mathcal{X}^*$. Now, since $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}_{wkc}(\mathcal{X})$, from Cor. I [9], $\mathcal{M}_\alpha$ has a unique Radon-Nikodym derivative $X_\alpha \in \mathcal{L}(\Omega, \mathcal{X}, \mu)$, $X_\alpha : \Omega \to \mathcal{P}_{fc}(\mathcal{X})$. From convexity of $X_\alpha(\omega)$, closure and weak closure of that set are equal, implying $X_\alpha : \Omega \to \mathcal{P}_{wkc}(\mathcal{X})$. As $\mathcal{M}_\alpha$ is a set valued measure of bounded variation, by Prop. 4.1 [5], X_α is integrably bounded set valued function. Now applying the Corollary of Prop.3.1.[8], we get $\mathcal{I}_\alpha(A) \in \mathcal{P}_{wkc}(\mathcal{X})$.

Since $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$, $X_\alpha : \Omega \to \mathcal{P}_{wkc}(\mathcal{X})$ and $|f \, X_\alpha| \leq |f||X_\alpha|$, it follows that $fX_\alpha : \Omega \to \mathcal{P}_{wkc}(\mathcal{X})$, defined by $(f \, X_\alpha)(\omega) = f(\omega)X_\alpha(\omega)$, is a μ-integrable, integrable bounded set valued function.

To prove that the family $\{\mathcal{I}_\alpha(\Omega)\}_{\alpha \in (0,1]}$ defines a fuzzy set, we use Lemma 2.1. From the relation $\alpha \geq \beta \Rightarrow \mathcal{M}_\alpha(\Omega) \subseteq \mathcal{M}_\beta(\Omega)$, we get $\mathcal{I}_\alpha(\Omega) \subseteq \mathcal{I}_\beta(\Omega)$.

Further, let $\{\alpha_n\} \subset (0,1]$ be an increasing sequence converging to $\alpha \in (0,1]$, i.e. $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots \to \alpha \leq 1$. We shell show that $\mathcal{I}_{\alpha_n}(\Omega) \xrightarrow{h} \mathcal{I}_\alpha(\Omega)$. By definition

$$h(\mathcal{I}_{\alpha_n}(\Omega), \mathcal{I}_\alpha(\Omega)) = h\left(\int_\Omega f(\omega) d\mathcal{M}_{\alpha_n}(\omega), \int_\Omega f(\omega) d\mathcal{M}_\alpha(\omega)\right) =$$

$$= h\left(\int_\Omega f(\omega) X_{\alpha_n}(\omega) d\mu(\omega), \int_\Omega f(\omega) X_\alpha(\omega) d\mu(\omega)\right).$$

>From Hörmander's formula and properties of the support function

$$h\left(\int_\Omega f(\omega) X_{\alpha_n}(\omega) d\mu(\omega), \int_\Omega f(\omega) X_\alpha(\omega) d\mu(\omega)\right) =$$

$$= \sup_{\|x^*\| \leq 1} \left| \sigma_{\int_\Omega f(\omega) X_{\alpha_n}(\omega) d\mu(\omega)}(x^*) - \sigma_{\int_\Omega f(\omega) X_\alpha(\omega) d\mu(\omega)}(x^*) \right| =$$

$$= \sup_{\|x^*\| \leq 1} \left| \int_\Omega (\sigma_{f(\omega) X_{\alpha_n}(\omega)}(x^*) - \sigma_{f(\omega) X_\alpha(\omega)}(x^*)) d\mu(\omega) \right| \leq$$

$$\leq \int_\Omega \sup_{\|x^*\| \leq 1} \left| (\sigma_{f(\omega) X_{\alpha_n}(\omega)}(x^*) - \sigma_{f(\omega) X_\alpha(\omega)}(x^*)) d\mu(\omega) \right| =$$

$$= \int_\Omega h(f(\omega) X_{\alpha_n}(\omega), f(\omega) X_\alpha(\omega)) d\mu(\omega) = \int_\Omega f(\omega) h(X_{\alpha_n}(\omega), X_\alpha(\omega)) d\mu(\omega).$$

If $x_n \in X_{\alpha_n}(\omega)$ and $x \in X_\alpha(\omega)$, we get

$$\inf_{x \in X_\alpha(\omega)} \|x_n - x\| \leq \|x_n - x\| \;\Rightarrow$$

$$\sup_{x_n \in X_{\alpha_n}(\omega)} \inf_{x \in X_\alpha(\omega)} \|x_n - x\| \leq \sup_{x_n \in X_{\alpha_n}(\omega),\, x \in X_\alpha(\omega)} \|x_n - x\| \;\Rightarrow$$

$$\Rightarrow \sup_{x_n \in X_{\alpha_n}(\omega)} \inf_{x \in X_\alpha(\omega)} \|x_n - x\| \leq h(X_{\alpha_n}(\omega) - X_\alpha(\omega), 0),$$

and by the same way

$$\sup_{x \in X_\alpha(\omega)} \inf_{x_n \in X_{\alpha_n}(\omega)} \|x_n - x\| \leq h(X_{\alpha_n}(\omega) - X_\alpha(\omega), 0).$$

The last two inequalities imply relation

$$h(X_{\alpha_n}(\omega), X_\alpha(\omega)) \leq h(X_{\alpha_n}(\omega) - X_\alpha(\omega)), 0) = |(X_{\alpha_n}(\omega) - X_\alpha(\omega)|.$$

Therefore, we finely conclude

$$h(\mathcal{I}_{\alpha_n}(\Omega), \mathcal{I}_\alpha(\Omega)) \leq \int_\Omega f(\omega) h((X_{\alpha_n}(\omega) - X_\alpha(\omega), 0) d\mu(\omega) =$$

$$= \int_\Omega f(\omega) |(X_{\alpha_n}(\omega) - X_\alpha(\omega))| d\mu(\omega).$$

Since $X_{\alpha_1}(\omega)$ is integrable bounded, there exists $\phi(\omega) \in L(\Omega, \mathbb{R})$, $|X_{\alpha_1}(\omega)| < \phi(\omega)$ $\mu-$a.e. Further, the relation $X_\alpha(\omega) \subseteq \cdots \subseteq X_{\alpha_n}(\omega) \subseteq \cdots \subseteq X_{\alpha_2}(\omega) \subseteq X_{\alpha_1}(\omega)$ implies $|X_\alpha(\omega)| < \phi(\omega)$, $|X_{\alpha_n}(\omega)| < \phi(\omega)$, $\mu-$a.e., for all $n \in \mathbb{N}$. Now, the next inequality

$$\int_\Omega |f(\omega)(X_{\alpha_n}(\omega) - X_\alpha(\omega))| d\mu(\omega) = \int_\Omega |f(\omega)||X_{\alpha_n}(\omega) - X_\alpha(\omega)| d\mu(\omega) \leq$$

$$\leq \int_\Omega |f(\omega)|(|X_{\alpha_n}(\omega)| + |X_\alpha(\omega)|) d\mu(\omega) \leq \int_\Omega |f(\omega)| 2\phi(\omega) d\mu(\omega) < \infty,$$

enables the application of Lebesgue's dominated convergence theorem

$$\lim_{n\to\infty} h(\mathcal{I}_{\alpha_n}(\Omega), \mathcal{I}_\alpha(\Omega)) = \lim_{n\to\infty} h\left(\int_\Omega f(\omega) d\mathcal{M}_{\alpha_n}(\omega), \int_\Omega f(\omega) d\mathcal{M}_\alpha(\omega)\right) \leq$$

$$\leq \int_\Omega \lim_{n\to\infty} h(f(\omega)X_{\alpha_n}(\omega), f(\omega)X_\alpha(\omega)) d\mu(\omega) =$$

$$= \int_\Omega f(\omega) \lim_{n\to\infty} h(X_{\alpha_n}(\omega), X_\alpha(\omega)) d\mu(\omega).$$

In the same manner as in the proof of Th.3.1, we can see that $X_\alpha(\omega) = \cap_{i=1}^\infty X_{\alpha_i}(\omega)$ for all $\omega \in \Omega$. Since $X_\alpha(\omega) \in \mathcal{P}_{wkc}(\mathcal{X})$, the last equality yields to $h(X_{\alpha_n}(\omega), X_\alpha(\omega)) \stackrel{n\to\infty}{\longrightarrow} 0$

Now, we can conclude that

$$\lim_{n\to\infty} h(\mathcal{I}_{\alpha_n}(\Omega), \mathcal{I}_\alpha(\Omega)) = \int_\Omega f(\omega) \lim_{n\to\infty} h(X_{\alpha_n}(\omega), X_\alpha(\omega)) d\mu(\omega) = 0.$$

To prove that for every $A \in \mathcal{A}$, the family $\{\mathcal{I}_\alpha(A)\}_{\alpha\in(0,1]}$ defines a fuzzy set, we repeat the preceding proof for any $A \in \mathcal{A}$ instead of Ω.

So, we have proved that $\mathcal{I}(A) \in \mathcal{F}_{wkc}(\mathcal{X})$, for every $A \in \mathcal{A}$.

The rest of the proof is the same as in Th. 3.1. □

The next theorem is a special case of the last theorem when $\mathcal{X}$ is finite dimensional.

Theorem 3.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a nonatomic measure space, $\mathcal{X}$ be a finite dimensional Banach space, $\mathcal{M} : \mathcal{A} \to \mathcal{F}_f(\mathcal{X})$ be a μ-continuous fuzzy valued measure of bounded variation and $f \in L_\infty(\Omega, \mathbb{R}_+, \mu)$. Then $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous nonatomic fuzzy valued measure of bounded variation.*

Proof: Since for every $\alpha \in (0,1]$ and every $A \in \mathcal{A}$, $\mathcal{M}_\alpha(A)$ is closed bounded set in finite Banach space, the set $\mathcal{M}_\alpha(A)$ is compact. Applying Th. 2.1, it is easily seen that $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}_k(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation. Now, by the same reasoning as in the proof of Th. 3.4, we get that $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}_{wkc}(\mathcal{X})$ and $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{wkc}(\mathcal{X})$. Since the dimension of Banach space $\mathcal{X}$ is finite, weak and strong topologies coincide, meaning that every weakly compact set is compact. Now, from Th. 2.1 and by the remark noted above, we conclude that $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous nonatomic fuzzy valued measure of bounded variation. □

Next we list some properties of the integral with respect to fuzzy valued measure. The proofs for the next three Lemmas are simple, so they are omitted. Let the fuzzy valued measure $\mathcal{M}$ be such that all integrals in the related lemma exist.

Lemma 3.1. *If $f, g \in L_\infty(\Omega, \mathbb{R}, \mu)$, $\mathcal{M}$ is μ-continuous fuzzy valued measure of bounded variation and $\lambda \in \mathbb{R}$, then*

$$\int_\Omega (f+g)(\omega) d\mathcal{M}(\omega) = \int_\Omega f(\omega) d\mathcal{M}(\omega) + \int_\Omega g(\omega) d\mathcal{M}(\omega)$$

and

$$\int_\Omega \lambda f(\omega) d\mathcal{M}(\omega) = \lambda \int_\Omega f(\omega) d\mathcal{M}(\omega).$$

Lemma 3.2. *Let $\mathcal{M}'$ and $\mathcal{M}''$ be μ-continuous fuzzy valued measures of bounded variation such that $\mathcal{M}'(A)(x) \leq \mathcal{M}''(A)(x)$ for all $A \in \mathcal{A}$ and all $x \in \mathcal{X}$. Then*

$$\left(\int_\Omega f d\mathcal{M}'\right)(x) \leq \left(\int_\Omega f d\mathcal{M}''\right)(x) \quad \text{for all } x \in \mathcal{X}.$$

Lemma 3.3. *Let $(\Omega, \mathcal{A}_1)$ and $(\Omega, \mathcal{A}_2)$ be two measurable spaces, $S : \Omega \to \Omega$ be an $\mathcal{A}_1$-measurable function and $f : \Omega \to R$ a bounded function. If $\mathcal{M} : \mathcal{A}_2 \to \mathcal{F}$ is a μ-continuous fuzzy valued measure of bounded variation, then for every $A \in \mathcal{A}_2$*

$$\int_{S^{-1}(A)} (f \cdot S) d\mathcal{M} = \int_A f d(\mathcal{M} \cdot S).$$

4. Examples

Example 1 Let $(\Omega, \mathcal{A}, P)$ be a probability measure space and let $\mathcal{M} : \mathcal{A} \to \mathcal{F}_f(\mathbb{R})$ be the fuzzy valued measure such that for all $A \in \mathcal{A}$, $\mathcal{M}(A)(P(A)) = 1$, $\mathcal{M}(A)(x) = 0$, for all $x \notin [0, P(A)]$. The fuzzy valued measure with that properties we shall cal **fuzzy valued probability**.

One simple example of that kind of measure is defined by

$$\mathcal{M}(A)(x) = \begin{cases} \frac{x}{P(A)}, & x \in [0, P(A)] \\ 0, & x \notin [0, P(A)] \end{cases}$$

if $P(A) \neq 0$, and $\mathcal{M}(A)(x) = I_{\{0\}}(x)$ if $P(A) = 0$.

If $X : \Omega \to \mathbb{R}$ is a random variable, then

$$\mathcal{I}(A) = \int_A X(\omega) d\mathcal{M}(\omega),$$

$A \in \mathcal{A}$, is a new fuzzy valued probability and $\mathcal{I}(\Omega)$ is fuzzy expectation of X with respect to fuzzy valued probability. $\square$

Example 2 This example illustrates some difficulties we could have in definition and properties of the integral with respect to fuzzy valued measure with noncompact α-levels, if the Banach space is not reflexive. If $\mathcal{X}$ is not reflexive, then the integral defined by $\mathcal{I}(A) = \int_A f(\omega)d\mathcal{M}(\omega)$, need not to have closed α-levels. So, in this case it would be more appropriate to change the basic definition of $\mathcal{I}_\alpha$ putting $\mathcal{I}_\alpha(A) = cl \int_A f(\omega)d\mathcal{M}_\alpha(\omega)$.

On the other hand, since $\mathcal{X}$ is not reflexive, according to [7], there exists two bounded sets $S, T \in \mathcal{P}_{fc}(\mathcal{X})$, $S \cap T = \emptyset$, which can not be separated. It means (as it was shown in [7]) that the set $S - T$ is not closed set. Now, we define the fuzzy valued measure $\mathcal{M} : \mathcal{A} \to \mathcal{F}_f(\mathcal{X})$ on the Lebesque measure space $([0,2), \mathcal{A}, \mu)$ by $\mathcal{M}(A) = \mu(A \cap [0,1))s - \mu(B \cap [1,2))t$, $A \in \mathcal{A}$, where s and t are fuzzy sets such that for some $\beta \in (0,1]$, $s_\beta = S$, $t_\beta = T$. Then, putting $f(x) = 1$ for all $x \in [0,2)$,

$$\mathcal{I}_\beta([0,2)) = cl \int_{[0,2)} f d\mathcal{M}_\beta = cl \left(\int_{[0,1)} S d\mu - \int_{[1,2)} T d\mu \right) = cl(S - T) \neq$$

$$\neq S - T = cl\, S - cl\, T = cl \int_{[0,1)} S d\mu - cl \int_{[1,2)} T d\mu = \mathcal{I}_\beta([0,1)) + \mathcal{I}_\beta([1,2)),$$

which leads to the conclusion that $\mathcal{I}$ is not a fuzzy valued measure. $\square$

5. Conclusion

In this paper the definition and the basic properties of the integral with respect to fuzzy valued measure are given. In all applications which involve measure, when measurement or data are imprecise, the structure defined in this paper can be applied.

The directions of further investigation are numerous: specific properties of integral, application on random case - expectation, conditional expectation, martingales and similar structures, application in economy.

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Received by the editors October 18, 2007