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HERMITE EXPANSIONS OF ELEMENTS OF GELFAND-SHILOV SPACES IN QUASIANALYTIC AND NON QUASIANALYTIC CASE

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Abstract. We study the Gelfand-Shilov spaces of Roumieu and Beurling type in the quasianalytic and nonquasianalytic case and characterize elements of the spaces in terms of the coefficients of their Fourier-Hermite expansion. The nontriviality conditions we assume on the spaces are new and weaker than the usually considered, and therefore a lot of classical spaces appear to be just examples of the spaces we consider in the paper.

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1. Introduction

In order to study the classes of functionals that are invariant under the Fourier transform, but larger than the classes of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, I.M. Gelfand and G.E Shilov ([5]) introduced spaces $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d)$, $\alpha \geq 1/2$. Their topological duals have been successfully used in differential operators theory, in spectral analysis and more recently, (in non-quasianalytic case) in theory of pseudodifferential operators ([12]). The special cases, when the test spaces are quasianalytic (i.e. when $\alpha \in [1/2, 1]$) are important for applications, see for example [4] and [10], where it was conjectured that the properties of the space of Fourier hyper-functions, which is isomorphic with $\mathcal{S}^1_1$ are well adapted for the use in quantum field theory with a fundamental length.

In the paper we study the Gelfand-Shilov spaces of Roumieu and Beurling type ($\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$) and their duals which generalize all nontrivial Gelfand-Shilov $\mathcal{S}^{\alpha}_{\alpha}$ and Pilipović spaces $\sum^{\alpha}_{\alpha}(\mathbb{R}^d)$ ([11]) in quasianalytic and nonquasianalytic case in a uniform way.

We give characterization of the spaces $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ and their duals in terms of the Hermite coefficients of their elements. Let us emphasize that Langenbruch in [9], using different methods, proved the same characterization for the spaces we consider. During the preparation of the paper we

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were not aware of the paper [9]. Here we give the characterization using analytic methods, instead of mathematical induction - especially in proving rather subtle estimates of the growth of Hermite functions and its derivatives.

In the special case when $\{M_p\}$ is a Gevrey sequence $\{p!^{\alpha}\}, \alpha \geq 1/2$, the characterization was proved independently, and by different methods, by Zhang [18], Kaspiriovskii [6], Avantiagi [1].

Let us note that the nontriviality conditions (M.3)" and (M.3)"' for the Gelfand-Shilov spaces, which we assume in the paper are new in the literature related to the Gelfand-Shilov spaces. They are much weaker than usually used. Therefore, we analyze them thoroughly. Our conditions (M.3)" and (M.3)"' are equivalent with Langenbruch's conditions [9, (1.2)], but are given in a compact form, from which is clear that a lot of classical spaces are just examples of the spaces we consider in the paper.

The examples of the spaces $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ are:

- for $M_p = p^{\alpha p}$, the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is the Gelfand-Shilov space $\mathcal{S}_{\alpha}^{\alpha}$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the Pilipović space $\sum_{\alpha}^{\alpha}$;

- for $M_p = p^p$, the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is isomorphic with the Sato space $\mathcal{F}$, the test space for Fourier hyperfunctions $\mathcal{F}'$, and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the Silva space $\mathcal{G}$, the test space for extended Fourier hyperfunctions $\mathcal{G}'$;

- Braun-Meise-Taylor space $\mathcal{S}_{\{\omega\}}$, $\omega \in \mathcal{W}$, studied in the series of papers by the same authors (see [2] and references thereih), is the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$, where
$$M_p = \sup_{\rho>0} \rho^p e^{-\omega(\rho)}.$$
The sequence satisfies the conditions (M.1), (M.2) and (M.3)', and it is in general different from a Gevrey sequence.

- Beurling-Björk space $\mathcal{S}_{\omega}$, $\omega \in \mathcal{M}_c$, introduced in [3], is equal to the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, where
$$M_p = \sup_{\rho>0} \rho^p e^{-\omega(\rho)}.$$
The sequence satisfies the conditions (M.1) and (M.3)', and it is in general different from a Gevrey sequence. If we assume additionally that $\omega(\rho) \geq C(\log \rho)^2$ for some $C > 0$, then (M.2) is satisfied.

- In [8] Korevaar developed a very general theory of Fourier transforms, based on a set of original and well motivated ideas. In order to obtain a formal class of objects which contain functions of exponential growth and which is closed under Fourier transform he introduced objects called pansions of exponential growth. From characterization theorem [8, Theorem 92.1] and our results it follows that exponential pansions are exactly tempered ultradistributions of Roumieu-Komatsu type, for $M_p = p^{p/2}$.

In the paper, the sequence $\{M_p\}_{p\in\mathbb{N}_0}$, which generates the Denjoy-Carleman classes $C^{\{M_p\}}(\mathbb{R}^d)$ and $C^{(M_p)}(\mathbb{R}^d)$ is a sequence of positive numbers. We suppose that it satisfies the first two standard conditions in ultradistributional theory: the conditions (M.1) - logarithmic convexity and (M.2) - separativity condition. We do not suppose nonquasianaliticity of Denjoy-Carleman classes $C^{\{M_p\}}(\mathbb{R}^d)$ and $C^{(M_p)}(\mathbb{R}^d)$ of functions, which is the standard nontriviality condition in the theory of ultradistributions (the condition (M.3)' in [7]). Instead, we suppose a weaker condition (M.3)" (resp. (M.3)"'), which is minimal nontriviality condition appropriate for the spaces $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$, (resp. $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$). The introduction of conditions (M.3)" (resp. (M.3)"') gives us the possibility to treat the quasianalytic and nonquasinanalytic cases in a unified way. In nonquasianalytic case, the dual space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is the space of tempered ultradistributions and in quasianalytic case the elements of $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ are hyperfunctions.

An example of a class of sequences which satisfy the above conditions is:

$$M_p = p^{sp}(\log p)^{tp}, \quad p \in \mathbb{N}, \quad s \geq 1/2,\ t \geq 0, \tag{1.1}$$

and (only) in Beurling-Komatsu case we assume additionally $s + t > 1/2$.

If the nonquasianalytic condition (M.3)' is also satisfied, the spaces $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ are the proper subspaces of Roumieu-Komatsu and of Beurling-Komatsu ultradistributions (see [7]). If however, the condition (M.3)' is not satisfied, these spaces of ultradistributions are trivial, nevertheless the spaces which are studied in this paper are not.

In Section 2. we prove the basic identification of the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and its dual space, with the sequence spaces of the Fourier-Hermite coefficients of their elements. First we prove that the test space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ can be identified with the space of sequences of ultrafast falloff, i.e. (in one-dimensional case) the space of sequences of complex numbers $\{a_n\}_{n\in\mathbb{N}_0}$ satisfying for some $\theta > 0$ the following

$$\sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta\sqrt{n}\,)] < \infty.$$

Here, $M(\cdot)$ is the associated function for the sequence $\{M_p\}_{p\in\mathbb{N}_0}$ defined by

$$M(\rho) = \sup_{p\in\mathbb{N}_0}\ \log\frac{\rho^p}{M_p}, \quad \rho > 0. \tag{1.2}$$

In the special case (1.1), one have $M(\rho) = \rho^{\frac{1}{s}}(\log\,\rho)^{-\frac{t}{s}}$, $\rho \gg 0$.

Next we prove that the dual space $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ can be identified with the space of sequences of ultrafast growth, i.e. (in one-dimensional case) the space of sequences $\{b_n\}_{n\in\mathbb{N}_0}$ which, for every $\theta > 0$, satisfy

$$\sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta\sqrt{n})] < \infty.$$

There is an analogy between the Gelfand-Shilov spaces of Roumieu and Beurling type. It would be more appropriate to call the latter type of the spaces the generalized Pilipovic spaces. One can modify the results obtained for one type of spaces to another, but there are differences, about which one should take care. Therefore, in Section 3 we obtain sequential characterization of generalized Pilipović spaces $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ and state the kernel theorem for the spaces of tempered ultradistributions of Beurling-Komatsu type.

In the last section we give the proofs of the two essential lemmas. The first one gives appropriate estimation for the growth of the derivatives of Hermite functions. We need sharper estimations for the derivatives of Hermite functions than the estimations usually given in the literature (see for example [17, p 122]).

1.1. Notations and basic notions

Throughout the paper by C we denote a positive constant, not necessarily the same at each occurrence. Let $\{M_p\}_{p\in\mathbb{N}_0}$ be a sequence of positive numbers, where $M_0 = 1$.

The Denjoy-Carleman class $C^{\{M_p\}}(\mathbb{R}^d)$ is a class of smooth functions φ such that there exist $m > 0$ and $C > 0$ so that

$$||\varphi^{(\alpha)}||_\infty \leq Cm^{|\alpha|}M_{|\alpha|}, \quad |\alpha| \in \mathbb{N}^d, \tag{1.3}$$

where we use the multi-index notation:

$$\varphi^{(\alpha)}(x) = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}\cdots(\partial/\partial x_d)^{\alpha_d}\varphi(x).$$

and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$. The class of functions equipped with a natural topology is the space of ultradifferentiable functions of Roumie-Komatsu type $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ (for the definition see [7]). In the special case, when $\{M_p\}_{p\in\mathbb{N}_0}$ is a Gevrey sequence $\{p^{sp}\}_{p\in\mathbb{N}_0}$, the space is the Gevrey space $\mathcal{G}^{\{s\}}(\mathbb{R}^d)$.

In the paper we define the Gelfand-Shilov space of Roumieu type as subclasses of the Denjoy-Carleman class $C^{\{M_p\}}(\mathbb{R}^d)$ invariant under Fourier transform, closed under the differentiation and multiplication by a polynomial, and equip them with appropriate topologies.

We assume that the sequence $\{M_p\}_{p\in\mathbb{N}_0}$ satisfy

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}, \quad p = 1, 2, \ldots.$
(*logarithmic convexity*)

(M.2) *There exist constants* A, $H > 0$ *such that*
$M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q}, \quad p = 0, 1, \ldots$
(*separativity condition or stability under ultradifferential operators*)

(M.3)" *There exist constants* $C, L > 0$ *such that*
$p^{\frac{p}{2}} \leq C\, L^p M_p, \quad p = 0, 1, \ldots$
(*nontriviality condition for the spaces* $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$)

In Section 4, where we discus generalized Pilipović spaces (Gelfand-Shilov spaces of Beurling type), instead of (M.3)" we assume:

(M.3)"' *For every $L > 0$, there exists $C > 0$ such that*
$p^{\frac{p}{2}} \leq CL^p M_p, \quad p = 1, 2, \ldots.$
(*nontriviality condition for the spaces* $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$).

To discuss our results in the context of Komatsu's ultradistributions, let us state condition :

(M.3)' $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$
(*nonquasianalyticity*)

The condition (M.1) is of a technical nature, which simplifies the work and involves no loss of generality. This is a well-known fact for the Denjoy-Carleman classes of functions.

The condition (M.2) is standard in the ultradistribution theory. It implies that the class $C^{\{M_p\}}(\mathbb{R}^d)$ is closed under the (ultra)differentiation (see [7]), and is important in the characterization of Denjoy-Carleman classes in a multidimensional case.

The nontriviality conditions (M.3)" and (M.3)"' are weaker than the condition (M.3)'. Under the conditions (M.3)" and (M.3)"' all Hermite functions are elements of the spaces $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ respectively. The smallest nontrivial Gelfand-Shilov space is $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$. Condition (M.3)" essentially means that the space $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$ is a subset of $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$. The smallest nontrivial Pilipović space does not exist. Note, $\sum_{1/2}^{1/2} = \{0\}$, but the space $\sum_{\alpha}^{\alpha}(\mathbb{R}^d)$, $\alpha > 1/2$, is nontrivial. Moreover, every nontrivial Pilipović space $\sum_{\alpha}^{\alpha}(\mathbb{R}^d)$, contains as a subspace one generalized Pilipović space, for example, the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, where $M_p = p^{p/2}(\log p)^{pt}$, $t > 0$.

In [9], Langenbruch gives another equivalent condition for nontriviality of the spaces of Roumieu and Beurling type: There is $H > 0$ such that for any $C > 0$ there is $B >$) (there are $C > 0$ and $B > 0$ respectively) such that

$$\alpha^{\alpha/2} M_\beta \leq BC^{|\alpha|} H^{|\alpha+\beta|} M_{\alpha+\beta}, \tag{1.4}$$

for any $\alpha, \beta \in \mathbb{N}_0^n$. It is easy to see that the condition (1.4) is (assuming a technical condition (M.1)) equivalent to (M.3)"' (resp. (M.3)"').

The condition (M.3)' is necessary and sufficient condition that the classes $C^{\{M_p\}}(\mathbb{R}^d)$ has a nontrivial subclass of functions with compact support, i.e. that $C^{\{M_p\}}(\mathbb{R}^d)$ is non-quasianalytic class of functions.

For example, the sequence (1.1) satisfies conditions (M.1), (M.2), (M.3)", and if $t > 0$ also the condition (M.3)"' but not (M.3)' while for $s > 1$ it satisfies the stronger condition (M.3)'.

2. Gelfand-Shilov spaces of Roumieu type

2.1. Basic spaces

We define the set $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ as a subclass of the Denjoy-Carleman class $C^{\{M_p\}}(\mathbb{R}^d)$ which is invariant under Fourier transform, closed under the differentiation and multiplication by a polynomial. This implies that it is a subset of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions and, therefore, of every $L^q(\mathbb{R}^d)$, $q \in [1,\infty]$. The same set can be characterized in one of the following equivalent ways:

1. The set $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is the set of all smooth functions φ such that there exist $C>0$ and $m>0$ such that

$$|| \exp[M(m\,x)]\varphi||_2 < C \quad \textit{and} \quad || \exp[M(m\,x)]\mathcal{F}\varphi||_2 < C,$$

where $||\cdot||_2$ is the usual norm in $L(\mathbb{R}^d)$, $\mathcal{F}$ is the Fourier transform and the function $M(\cdot)$ is defined by (1.2).

2. The set $\mathcal{S}^{\{M_p\}}$ is the set of all smooth functions φ on $\mathbb{R}^d$, such that for some $C>0$ and $m>0$

$$||(1+x^2)^{\beta/2}\varphi^{(\alpha)}||_\infty \le C\, m^{|\alpha|+|\beta|}M_{|\alpha|}M_{|\beta|}, \text{ for every } \alpha,\beta\in\mathbb{N}_0^d. \tag{2.1}$$

The topology of the Gelfand-Shilov space of Roumieu type is the inductive limit topology of Banach spaces $\mathcal{S}^{M_p,m}$, $m>0$, where $\mathcal{S}^{M_p,m}$ denotes the space of smooth functions φ on $\mathbb{R}^d$, such that for some $C>0$ and $m>0$

$$||\varphi||_{\mathcal{S}^{M_p},m} = \sup_{\alpha,\beta\in\mathbb{N}_0^d}\frac{m^{|\alpha|+|\beta|}}{M_{|\alpha|}M_{|\beta|}}||(1+x^2)^{\beta/2}\varphi^{(\alpha)}(x)||_{L^\infty}<\infty, \tag{2.2}$$

equipped with the norm $||\cdot||_{\mathcal{S}^{M_p},m}$. So, $\mathcal{S}^{\{M_p\}} = ind\,lim_{m\to 0}\mathcal{S}^{M_p,m}$

It is a Frechet space. We will denote by $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ the strong dual of the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$.

The Fourier transform is defined on $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{ix\xi}\varphi(x)dx, \quad \varphi\in\mathcal{S}^{\{M_p\}}(\mathbb{R}^d),$$

and on $\mathcal{S}^{\{M_p\}'}$ by

$$\langle\mathcal{F}f,\varphi\rangle = \langle f,\mathcal{F}\varphi\rangle, \;\; f\in\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d), \varphi\in\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$$

The space is a good space for harmonic analysis since the Fourier transform is an isomorphism of $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ onto itself, and the space of tempered distributions $\mathcal{S}'$ is its subspace.

2.2. Hermite functions

We denote by

$$\mathcal{H}_n(x) = (-1)^n\pi^{-1/4}2^{-n/2}(n!)^{-1/2}e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right), \quad n\in\mathbb{N},$$

the *Hermite functions* (the wave functions of a harmonic oscillator), where $\mathcal{H}_{-k}=0$ for $k=1,2,3....$ The functions arise naturally as eigenfunctions of harmonic oscillator Hamiltonian, and so play a vital role in quantum physics, but they are also eigenfunctions of the Fourier transform. This fact will be used often in the paper.

In the paper we will use the properties of the creation and annihilation operators:

$$L^{+} = \frac{1}{\sqrt{2}}\big(x - \frac{d}{dx}\big), \quad L^{-} = \frac{1}{\sqrt{2}}\big(x + \frac{d}{dx}\big):$$

(1.1) $L^{-}L^{+} - L^{+}L^{-} = 1,$

(1.2) $L^{-}\mathcal{H}_n = \sqrt{n}\,\mathcal{H}_{n-1}, \quad L^{+}\mathcal{H}_n = \sqrt{n+1}\,\mathcal{H}_{n+1},$

(1.3) $L^{+}L^{-}\mathcal{H}_n = n\mathcal{H}_n,$

the fact that the sequence $\{\mathcal{H}_n\}_{n\in\mathbb{N}_0}$ is an orthonormal system in $L^2(\mathbb{R})$, and

$$\mathcal{F}[\mathcal{H}_n] = \sqrt{2\pi} i^n \mathcal{H}_n.$$

The Hermite functions in multidimensional case are defined simply by taking the tensor product of the one-dimensional Hermite functions:

$$\mathcal{H}_n(x) = \mathcal{H}_{n_1}(x_1)\mathcal{H}_{n_2}(x_2)\cdots\mathcal{H}_{n_d}(x_d), \quad x = (x_1, x_2, ...x_d) \in \mathbb{R}^d,$$

where $n = (n_1, n_2, ..., n_d) \in \mathbb{N}^d$. The functions $\mathcal{H}_n$, $n \in \mathbb{N}_0^d$, are elements of the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and of the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$. This is an immediate consequence of Lemma 2.1.

Let φ be a smooth function of the fast falloff ($\varphi \in \mathcal{S}(\mathbb{R}^d)$). The numbers

$$a_n(\varphi) = \int_{\mathbb{R}^d} \varphi(x)\mathcal{H}_n(x)dx, \quad n \in \mathbb{N}_0^d$$

will be called the **Fourier-Hermite coefficients** of φ. The sequence of the Fourier-Hermite coefficients $\{a_n(\varphi)\}_{n\in\mathbb{N}_0^d}$ of φ we call the **Hermite representation of** φ.

We will extensively use the following estimations, which we prove in the last section.

Lemma 2.1. *a) If conditions (M.1), (M.2) and (M.3)" are satisfied, there exist $C > 0$ and $m_0 > 0$ such that for every $m \leq m_0$*

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x))| \leq C\, e^{M(8mH\sqrt{n})}. \tag{2.3}$$

b) If conditions (M.1), (M.2) and (M.3)"' are satisfied, then for every $m > 0$ there exists $C > 0$ such that the estimate (2.3) holds.

We will also need the following lemma:

Lemma 2.2. *a) If $\varphi \in C^\infty$ and $N \in \mathbb{N}$ then*

$$(2.4) \qquad (L^- L^+)^N \varphi(x) = 2^N (1 + x^2 - \frac{d^2}{dx^2})^N \varphi(x) = \sum_{p=0}^{2N} \sum_{q=0}^{2N-p} c_{p,q}^{(N)} x^p \varphi^{(q)}(x),$$

where $c_{p,q}^{(N)}$ are constants which satisfy the inequality

$$(2.5) \qquad |c_{p,q}^{(N)}| \leq 26^N (2N - q)^{N - \frac{p+q}{2}}.$$

b) Moreover, if conditions (M.1), (M.2) and (M.3)"are satisfied for $p, q \in \mathbb{N}$, $p + q \leq 2N$, then it holds:

$$(2.6) \qquad |c_{p,q}^{(N)}| \leq 52^N \frac{M_N^2}{M_p M_q}.$$

2.3. Hermite representation of Gelfand-Shilov space of Roumieu type

The fact that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is isomorphic with sequence space $\mathbf{s}$ of sequences of fast falloff, has a lot of important consequences (see for example [15], [16] and [17]), for example simple proofs of the kernel and structure theorems for the space of tempered distributions [15]. An analogue of that property holds for the Gelfand-Shilov space of Roumieu type. In this section we will prove this fact.

By $\mathbf{s}_{M_p,\theta}$, $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}_+^d$, we denote the set of multisequences $\{a_n\}_{n \in \mathbb{N}_0^d}$ of complex numbers which satisfies

$$\|\{a_n\}\|_\theta = \left(\sum_{n \in \mathbb{N}_0^d} |a_n|^2 \exp \left[2 \sum_{k=1}^d M(\theta_k \sqrt{n_k}) \right] \right)^{1/2} < \infty,$$

equipped with the norm $\|\{a_n\}\|_\theta$.

The space $\mathbf{s}_{\{M_p\}}$ of sequences of ultrafast falloff is the inductive limit of the family of spaces $\{\mathbf{s}_{M_p,\theta}, \theta \in \mathbb{R}_+^d\}$, and it is a nuclear space (see [16]).

Theorem 2.3. *The mapping which assigns to each element of $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ its Hermite representation is a topological isomorphism of the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and the space $\mathbf{s}_{\{M_p\}}$ of sequences of ultrafast falloff.*

The space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is nuclear, since the space $\mathbf{s}_{\{M_p\}}$ is nuclear.

We will prove Theorem 2.3 in one-dimensional case, the proof in multidimensional case is an immediate consequence.

Proof. 1. Let $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R})$, then there exists $\mu > 0$ such that

$$||\varphi||_{M_p,\mu} = \sup_{p,q} \frac{\mu^{p+q}}{M_p M_q} ||(1 + x^2)^{p/2} \varphi^{(q)}(x)||_\infty < \infty.$$

>From the property (1.3) of the creation and annihilation operators it follows that

$$a_n(\varphi) = \int \varphi(x)\mathcal{H}_n(x)dx = n^{-N}\int \varphi(x)(L^+L^-)^N\mathcal{H}_n(x)dx = \tag{2.7}$$

$$= n^{-N}\int \Big((L^-L^+)^N\varphi(x)\Big)\mathcal{H}_n(x)dx =$$

$$= n^{-N}\int (1+x^2)\Big((L^-L^+)^N\varphi(x)\Big)\mathcal{H}_n(x)\frac{dx}{1+x^2}.$$

>From Lemma 2.2 and condition (M.2) it follows that

$$(1+x^2)|(L^-L^+)^N\varphi(x)| \le$$

$$\le 52^N M_N^2 \sum_{p=0}^{2N}\sum_{q=0}^{2N-p}\frac{(1+x^2)|x^p\varphi^{(q)}(x)|}{M_pM_q} \le$$

$$\le C\ 52^N H^{2N} M_N^2 \sum_{p=0}^{2N}\sum_{q=0}^{2N-p}\frac{\mu^{p+q}\|(1+x^2)^{(p+2)/2}\varphi^{(q)}(x)\|_\infty}{M_{p+2}M_q}\mu^{-(p+q)} \le$$

$$\le C\ 52^N H^{2N} M_N^2 ||\varphi||_{M_p,\mu}\mu^{-2}\left(\frac{\mu}{e^k}\right)^{-2N}\sum_{p=0}^{2N}\sum_{q=0}^{2N-p}\left(\frac{\mu e^{-k}}{\mu}\right)^{p+q}\left(\frac{\mu}{e^k}\right)^{2N-(p+q)} \le$$

$$\le C\,\theta^N M_N^2 ||\varphi||_{M_p,\mu},$$

where k is a constant such that $\ln\mu \le k$ and $\theta = \sqrt{52}H\mu e^{-k}$.

Since $||\mathcal{H}_n||_{L^2} = 1$, from the above it follows that for each $N \in \mathbb{N}_0$

$$|a_n(\varphi)|^2 \le C\ n^{-2N}\theta^{2N}M_N^4||\varphi||^2_{M_p,\mu},$$

where $\theta = \sqrt{26}H\cdot 2\cdot(1+\mu)$. Therefore, for $N = \alpha+2$ by (M.2) and (M.1)

$$|a_n(\varphi)|^2 \le C\ n^{-2\alpha}n^{-2}\theta^{2\alpha}M_\alpha^4H^{4\alpha}||\varphi||^2_{M_p,\mu} \le Cn^{-2\alpha}n^{-2}(H^2\theta)^{2\alpha}M_{2\alpha}^2||\varphi||^2_{M_p,\mu},$$

which implies

$$||\{a_n\}||_\theta = \left(\sum_{n=0}^{\infty}|a_n|^2\exp[2M(H^2\theta\sqrt{n})]\right)^{1/2} \le C\ ||\varphi||_{M_p,\mu} \le \infty,$$

for $\theta = \sqrt{52}H\cdot 2\cdot(1+\mu)$.

2. Let for some $\theta > 0$ the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ satisfy

$$||\{a_n\}||_\theta = \left(\sum_{n=0}^{\infty}|a_n|^2\exp[2M(\theta\sqrt{n})]\right)^{1/2} < \infty.$$

It follows that the sequence is a sequence of fast falloff, so the sum $\sum_{n=0}^{\infty} a_n \mathcal{H}_n(x)$ converges to some φ in $\mathcal{S}$. We will prove that φ also belong to the space $\mathcal{S}^{\{M_p\}}(\mathbb{R})$.

Let m_0 and C be positive constants such that for every $m \leq m_0$ holds:

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x))| \leq C\ \exp[M(8mH\sqrt{n})]. \tag{2.8}$$

the existence of which is determined by Lemma 2.1. By using thw Cauchy-Schwartz inequality and Lemma 2.1 we have:

$$\begin{aligned}
&\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}||(1+x^2)^{\beta/2}\left(\sum_{n=0}^{\infty} a_n \mathcal{H}_n\right)^{(\alpha)}||_\infty \leq \\
&\leq C \sum |a_n|_{n=0}^{\infty} \exp\left[M\left(8mH\sqrt{n}\,\right)\right] \leq \\
&\leq C\left(\sum_{n=0}^{\infty} |a_n|^2 \exp\left[2M\left(\theta\sqrt{n}\,\right)\right]\right)^{1/2} \cdot \\
&\cdot \left(\sum_{n=0}^{\infty} \exp[-2M(\theta\sqrt{n})] \exp\left[M\left(8mH\sqrt{n}\,\right)\right]\right)^{1/2}.
\end{aligned}$$

Since $\exp[-M(\theta\sqrt{n})] \leq C$, it follows that for $m < \theta/(8h)$

$$\begin{aligned}
&||\varphi||_{M_p,m} = \sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta}||(1+x^2)^{\beta/2}\left(\sum_{n=0}^{\infty} a_n \mathcal{H}_n\right)^{(\alpha)}||_\infty \leq \\
&\leq C\left(\sum_{n=0}^{\infty} |a_n|^2 \exp\ \left[2M\left(\theta\sqrt{n}\,\right)\right]\right)^{1/2} = ||\{a_n\}||_\theta.
\end{aligned}$$

This concludes the proof of the second part of the theorem. □

Let f be an element of the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$. The numbers

$$a_n(f) = \langle f, h_n \rangle, \quad n \in \mathbb{N}_0^d.$$

will be called the **Fourier-Hermite coefficients** of f, the sequence $\{a_n(f)\}_{n\in\mathbb{N}_0^d}$, the **Hermit representation** of f, and the formal series

$$\sum_{n\in\mathbb{N}_0^d} a_n(f)\mathcal{H}_n(x)$$

will be called the Hermite series of f.

Let us now characterize the Hermite representation of $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$.

Theorem 2.4. *1. If $f \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ then for every $\theta = (\theta_1, ...\theta_d) \in \mathbb{R}^d_+$ its Hermite representation $\{a_n\}_{n\in\mathbb{N}_0^d}$ satisfies*

$$|b_n(f)| \leq \exp\left[\sum_{k=1}^{d} M(\theta_k\sqrt{n_k})\right], \quad n = (n_1, ...n_d) \tag{2.9}$$

and f has the representation:

$$\langle f, \varphi\rangle = \sum_{n\in\mathbb{N}_0^d} b_n(f)a_n(\varphi), \quad \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d),$$

where the sequence $\{a_n(\varphi)\}_{n\in\mathbb{N}_0^d}$ is the Hermite representative of $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$.

2. Conversely, if a sequence $\{b_n\}_{n\in\mathbb{N}_0^d}$ satisfies that for every $\theta = (\theta_1, ...\theta_d) \in \mathbb{R}^d_+$,

$$|b_n| \leq \exp\left[\sum_{k=1}^{d} M(\theta_k\sqrt{n_k})\right], \tag{2.10}$$

it is the Hermite representation of a unique $f \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ and the **Parseval equation** *holds:*

$$\langle f, \varphi\rangle = \sum_{n\in\mathbb{N}_0^d} b_n(f)a_n(\varphi), \quad \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d),$$

where the sequence $\{a_n(\varphi)\}_{n\in\mathbb{N}_0^d}$ is the Hermite representative of $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$.

Proof. For simplicity we will give the proof in one-dimensional case.
1. Let $f \in \mathcal{S}^{\{M_p\}}(\mathbb{R})$ and let $\theta > 0$. Then for every $\mu > 0$ there exists $C > 0$ such that

$$|\langle f, \varphi\rangle| \leq C||\varphi||_{M_p,\mu},$$

for every $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R})$. From the above and Lemma 2.1 it follows that there exist $m_0 > 0$ and $C > 0$ such that for $m < \min(m_0, \theta/(8m))$

$$|b_n(f)| = |\langle f, \mathcal{H}\rangle| \leq C \sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta}||(1 + x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}||_\infty \leq$$

$$\leq C \exp\left[M(8mH\sqrt{n})\right] \leq C \exp\left[M(\theta\sqrt{n})\right].$$

2. Let the sequence $\{b_n\}$ satisfy condition (2.10) for every $\theta > 0$. We will prove that the series $\sum_{n=0}^{\infty} b_n\mathcal{H}_n$ converges in the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$ to an element of the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$ defined by

$$f : \varphi \mapsto \sum_{n=0}^{\infty} b_n a_n(\varphi), \tag{2.11}$$

where $\{a_n(\varphi)\}$ is the Hermit representation of φ. >From the Schwartz inequality it follows that for every $\theta > 0$

$$\sum_{n=0}^{\infty} |b_n||a_n(\varphi)| \leq$$

$$\leq \Big(\sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta\sqrt{n})]\Big)^{1/2} \cdot \Big(\sum_{n=0}^{\infty} |a_n(\varphi)|^2 \exp[2M(\theta\sqrt{n})]\Big)^{1/2} \leq$$

$$\leq C\Big(\sum_{n=0}^{\infty} |a_n(\varphi)|^2 \exp[2M(\theta\sqrt{n})]\Big)^{1/2} \leq C||\varphi||_\theta,$$

which implies that the mapping f defined by (2.11) is an element from $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$.

The equation $f = \sum_{n=0}^{\infty} b_n \mathcal{H}_n$ holds in the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$, since

$$\langle f, \varphi\rangle = \lim_{k\to\infty} \sum_{n=0}^{k} b_n a_n(\varphi) = \lim_{k\to\infty} \sum_{n=0}^{k} b_n \langle \mathcal{H}_n, \varphi\rangle =$$

$$= \lim_{k\to\infty} \langle \sum_{n=0}^{k} b_n \mathcal{H}_n, \varphi\rangle,$$

by virtue of the completeness of the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$ we have that $f = \sum_{n=0}^{\infty} b_n \mathcal{H}_n$ in the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$. □

3. Gelfand-Shilov spaces of Beurling type - Generalized Pilipović space

The definition of the Denjoy-Carleman class $C^{(M_p)}(\mathbb{R}^d)$ differs slightly from the standard one. It is a class of functions φ such that for every $m > 0$ there exists $C > 0$ so that equation (1.3) holds. The class of functions equipped with a natural topology is the space of ultradifferentiable functions of Beurling-Komatsu type $\mathcal{E}^{(M_p)}(\mathbb{R}^d)$ (see [7]). In the special case, when $\{M_p\}_{p\in\mathbb{N}_0}$ is a Gevrey sequence $\{p^{sp}\}_{p\in\mathbb{N}_0}$, the space is the Gevrey space $\mathcal{G}^{(s)}(\mathbb{R}^d)$.

We define the set $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ as a subclass of the Denjoy-Carleman class $C^{(M_p)}(\mathbb{R}^d)$ which is invariant under Fourier transform, and closed under the differentiation and multiplication by a polynomial. Analogously as in Section 2, one can characterize the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ in one of the following equivalent ways:

1. The set $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the set of all smooth functions φ such that for every $m > 0$ there exists $C > 0$ so that

$$|| \exp[M(m\,x)]\varphi||_2 < C \quad \textit{and} \quad || \exp[M(m\,x)]\mathcal{F}\varphi||_2 < C.$$

2. The set $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the set of all smooth functions φ on $\mathbb{R}^d$, such that for every $m > 0$ there exists $C > 0$ so that

$$||(1+x^2)^{\beta/2}\varphi^{(\alpha)}||_\infty \leq C\, m^{|\alpha|+|\beta|} M_{|\alpha|} M_{|\beta|}, \text{ for every } \alpha, \beta \in \mathbb{N}_0^d. \tag{3.1}$$

The topology of the generalized Pilipović space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the projective limit topology of the Banach spaces $\mathcal{S}^{M_p,m}$, $m > 0$, where $\mathcal{S}^{M_p,m}$ is defined as in Section 2. Let us stress out that every nontrivial Pilipović space $\sum_\alpha^\alpha(\mathbb{R}^d)$ contains as a subspace one generalized Pilipović space, for example, the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, where $M_p = p^{p/2}(\log p)^{pt}$.

We will denote by $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ the strong dual of the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$. It contains space of tempered distributions as a proper subspace and the Fourier transform maps it into itself.

Analogously as in Section 2, one can prove the following theorem:

Theorem 3.1. *The mapping which assigns to each element of $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ its Hermite representation is a topological isomorphism of the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ and the space $\mathbf{s}_{(M_p)}$ of sequences of ultrafast falloff, where the space $\mathbf{s}_{(M_p)}$ is the space of sequences of ultrafast falloff and is the projective limit of the family of spaces $\{\mathbf{s}_{M_p,\theta}, \theta \in \mathbb{R}_+^d\}$, defined in Section 2.*

Since the space $\mathbf{s}_{(M_p)}$ is nuclear, from the above theorem follows the nuclearity of the generalized Pilipović space.

By analogous argument as in Section 2 one can prove the theorem which characterizes Hermite representation of the elements of the space $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$.

Theorem 3.2. *1. If $f \in \mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ then for some $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}_0^d$ its Hermite representation $\{a_n\}_{n\in\mathbb{N}_0^d}$ satisfy*

$$|b_n(f)| \leq \exp\left[\sum_{k=1}^{d} M(\theta_k\sqrt{n_k})\right], \quad n = (n_1, ...n_d) \tag{3.2}$$

and f has the representation:

$$\langle f, \varphi\rangle = \sum_{n\in\mathbb{N}_0^d} b_n(f) a_n(\varphi), \quad \varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d),$$

where the sequence $\{a_n(\varphi)\}_{n\in\mathbb{N}_0^d}$ is the Hermite representative of $\varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$.

2. Conversely, if a sequence $\{b_n\}_{n\in\mathbb{N}_0^d}$ satisfies for some $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}_0^d$,

$$|b_n| \leq \exp\left[\sum_{k=1}^{d} M(\theta_k\sqrt{n_k})\right], \tag{3.3}$$

it is the Hermite representation of a unique $f \in \mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ and the Parseval equation holds:

$$\langle f, \varphi\rangle = \sum_{n=0}^{\infty} b_n(f) a_n(\varphi), \quad \varphi \in \mathcal{S}^{(M_p)},$$

where the sequence $\{a_n(\varphi)\}_{n\in\mathbb{N}_0^d}$ *is the Hermite representative of* $\varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$.

4. Proofs of Lemmas

Let us prove Lemmas 2.1 and 2.2.

Lemma 2.1 *a) If conditions (M.1), (M.2) and (M.3)" are satisfied, there exist* $C > 0$ *and* $m_0 > 0$ *such that for every* $m \le m_0$

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x))| \le C\, e^{M(8mH\sqrt{n})}. \tag{4.1}$$

b) If conditions (M.1), (M.2) and (M.3)"' are satisfied, for every $m > 0$ *there exists* $C > 0$ *such that the estimate (2.3) holds.*

Proof.

$$\mathcal{F}[\mathcal{H}_n] = \sqrt{2\pi}\, i^n\, \mathcal{H}_n, \quad \text{and} \quad \frac{d^\alpha}{dx^\alpha}\mathcal{F}[\varphi] = \mathcal{F}[(ix)^\alpha\varphi]$$

(see for example [17]) it follows that

$$\mathcal{H}_n^{(\alpha)} = i^{\alpha-n}\frac{1}{\sqrt{2\pi}}\mathcal{F}[\xi^\alpha\mathcal{H}_n] \quad \text{and} \quad \xi^{2\gamma}\mathcal{F}[\varphi] = \mathcal{F}[(-D^2)^\gamma\varphi].$$

This implies that for an even number $\beta \in \mathbb{N}$ it holds:

$$\begin{aligned}(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x) &= \frac{i^{\alpha-n}}{\sqrt{2\pi}}\mathcal{F}\Big[\Big(1-\frac{d^2}{d\xi^2}\Big)^{\beta/2}(\xi^\alpha\mathcal{H}_n(\xi))\Big] = \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+\xi^2)\Big(1-\frac{d^2}{d\xi^2}\Big)^{\beta/2}(\xi^\alpha\mathcal{H}_n(\xi))\frac{e^{ix\xi}}{1+\xi^2}d\xi.\end{aligned} \tag{4.2}$$

From

$$\xi^\alpha\varphi = 2^{-\frac{\alpha}{2}}(L^- + L^+)^\alpha\varphi$$

and

$$\left(1-\frac{d^2}{dx^2}\right)^\gamma = \left(1-\frac{1}{2}\left(L^- - L^+\right)^2\right)^\gamma$$

we obtain that

(4.3)

$$\begin{aligned}&(1+\xi^2)\Big(1-\frac{d^2}{d\xi^2}\Big)^{\beta/2}[\xi^\alpha\mathcal{H}_n(\xi)] = \\ &= 2^{-\frac{\alpha}{2}}\left(1+2^{-\frac{1}{2}}(L^-+L^+)^2\right)\left(1-2^{-1}(L^- - L^+)^2\right)^{\beta/2}(L^-+L^+)^\alpha\mathcal{H}_n(\xi) = \\ &= 2^{-\frac{\alpha}{2}}\sum_{\gamma=0}^{\beta/2}\binom{\beta/2}{\gamma}\Big(-\frac{1}{2}\Big)^\gamma(L^- - L^+)^{2\gamma}(L^-+L^+)^\alpha\mathcal{H}_n(\xi) +\end{aligned}$$

$$+2^{-\frac{\alpha+1}{2}}\sum_{\gamma=0}^{\beta/2}\binom{\beta/2}{\gamma}\left(-\frac{1}{2}\right)^{\gamma}(L^-+L^+)^2(L^- - L^+)^{2\gamma}(L^-+L^+)^{\alpha}\mathcal{H}_n(\xi)=$$

$$=2^{-\alpha/2}\sum_{\gamma=0}^{\beta/2}\binom{\beta/2}{\gamma}\left(-\frac{1}{2}\right)^{\gamma}\Big[(L^- - L^+)^{2\gamma}(L^-+L^+)^{\alpha}\mathcal{H}_n(\xi)+$$

$$+2^{-\frac{1}{2}}(L^-L^+)^2(L^- - L^+)^{2\gamma}(L^-+L^+)^{\alpha}\mathcal{H}_n(\xi)\Big].$$

The term

$$(L^- - L^+)^{2\gamma}(L^-+L^+)^{\alpha}\mathcal{H}_n(\xi).$$

which appears in the sum on the right-hand side of the above equality is a sum of $2^{\alpha+2\gamma}$ terms of the form

$$L^{\sharp_1}L^{\sharp_2}\cdots L^{\sharp_{2\gamma}}L^{\sharp_{2\gamma+1}}\cdots L^{\sharp_{\alpha+2\gamma}}\mathcal{H}_n(\xi)$$

where $\sharp_j$ stands for $+$ or $-$.

In $\binom{\alpha+2\gamma}{j}$ of them L^+ appears exactly j times, $j\in\{0,1,2,...,\alpha+2\gamma\}$, and in the case

$$L^{\sharp_1}\cdots L^{\sharp_{\alpha+2\gamma}}\mathcal{H}_n(\xi)=c_{\sharp_1\sharp_2\cdots\sharp_{\alpha+2\gamma}}\mathcal{H}_{n+2j-(\alpha+2\gamma)}(\xi), \tag{4.4}$$

where $\mathcal{H}_{-k}:=0$ for $k=1,2,...$ and $C_{\sharp_1\sharp_2\cdots\sharp_{\alpha+2\gamma}}$ is a constant. From $L^-\mathcal{H}_n=\sqrt{n}\,\mathcal{H}_{n-1}$, and $L^+\mathcal{H}_n=\sqrt{n+1}\,\mathcal{H}_{n+1}$, it follows that

$$\begin{aligned}&C_{\sharp_1\sharp_2\cdots\sharp_{\alpha+2\gamma}}\le C_{--\ldots-++\ldots+}=\\&=\Big(\frac{(n+j)!}{n!}\Big)^{1/2}\Big(\frac{(n+j)!}{(n+j-(\alpha+2\gamma-j))!}\Big)^{1/2}\le(n+j)^{(\alpha+2\gamma)/2}.\end{aligned}\tag{4.5}$$

Since $||\mathcal{H}_n||_{L^2}=1$ we have that

$$||(L^- - L^+)^{2\gamma}(L^-+L^+)^{\alpha}\mathcal{H}_n(\xi)||_{L^2}=\sum_{j=0}^{\alpha+2\gamma}\binom{\alpha+2\gamma}{j}(n+j)^{(\alpha+2\gamma)/2}\le \tag{4.6}$$

$$\le(n+\alpha+2\gamma)^{(\alpha+2\gamma)/2}\cdot 2^{\alpha+2\gamma}.$$

Analogously, one can obtain

$$||(L^-+L^+)^2(L^- - L^+)^{2\gamma}(L^-+L^+)^{\alpha}\mathcal{H}_n(\xi)||_{L^2}= \tag{4.7}$$

$$=\sum_{j=0}^{\alpha+2\gamma+2}\binom{\alpha+2\gamma+2}{j}(n+j)^{(\alpha+2\gamma+2)/2}\le(n+\alpha+2\gamma+2)^{(\alpha+2\gamma+2)/2}\cdot 2^{\alpha+2\gamma+2}$$

>From above it follows that for $\beta\in\mathbb{N}$ even:

(4.8)

$$\Big|(1{+}x^2)^{\beta/2}\mathcal{H}^{(\alpha)}(x)\Big| =$$

$$= \frac{1}{\sqrt{2\pi}}\int\Big|(1+\xi^2)\Big(1-\frac{d^2}{d\xi^2}\Big)^{\beta/2}[\xi^\alpha \mathcal{H}_n(\xi)]\Big|\frac{1}{1+\xi^2}d\xi \le$$

$$\le \frac{1}{\sqrt{2\pi}}\,2^{-\frac{\alpha}{2}}\sum_{\gamma=0}^{\beta/2}\binom{\frac{\beta}{2}}{\gamma}\Big(-\frac{1}{2}\Big)^\gamma\Big[\sum_{j=0}^{\alpha+2\gamma}\binom{\alpha+2\gamma}{j}(n+j)^{(\alpha+2\gamma)/2}+$$

$$+\,2^{-1/2}\sum_{j\ge 0}^{\alpha+2\gamma+2}\binom{\alpha+2\gamma+2}{j}(n+j)^{(\alpha+2\gamma+2)/2}\Big]||\mathcal{H}_n||_{L^2}\Big(\int_{\mathbb{R}}\frac{d\xi}{1+\xi^2}\Big)^{1/2} \le$$

from (4.6) and (4.7)

$$\le 2^{-\frac{\alpha}{2}-\frac{1}{2}}\sum_{\gamma=0}^{\beta/2}\binom{\frac{\beta}{2}}{\gamma}\Big((n+\alpha+2\gamma)^{(\alpha+2\gamma)/2}\cdot 2^{\alpha+2\gamma}+$$

$$+\,2^{-\frac{1}{2}}(n+\alpha+2\gamma+2)^{(\alpha+2\gamma+2)/2}\cdot 2^{\alpha+2\gamma+2}\Big) \le$$

$$\le 2^{-\frac{\alpha}{2}-\frac{1}{2}}\sum_{\gamma=0}^{\beta/2}\binom{\frac{\beta}{2}}{\gamma}(n+\alpha+2\gamma+2)^{(\alpha+2\gamma+2)/2}\cdot 2^{\alpha+2\gamma+2}(1+2^{-\frac{1}{2}}) \le$$

$$\le 2^{-\frac{\alpha}{2}-\frac{1}{2}}(n+\alpha+\beta+2)^{(\alpha+\beta+2)/2}\cdot 2^{\alpha+\beta+2}(1+2^{-\frac{1}{2}})\cdot 2^{\beta/2} \le$$

$$\le 2^{\frac{\alpha}{2}+\frac{3}{2}\beta+2}(1+\sqrt{2})(n+\alpha+\beta+2)^{(\alpha+\beta+2)/2} \le$$

$$\le 3\cdot 2^{\frac{\alpha}{2}+\frac{3}{2}\beta+2}\cdot 2^{(\alpha+\beta+2)/2}(\max(n,\alpha+\beta+2))^{(\alpha+\beta+2)/2} \le$$

$$\le C\,2^{\alpha+2\beta}\max(n^{(\alpha+\beta+2)/2},(\alpha+\beta+2)^{(\alpha+\beta+2)/2}).$$

For β odd, we have

$$|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x)| \le |(1+x^2)^{(\beta+1)/2}\mathcal{H}_n^{(\alpha)}(x)| \le$$

$$\le C\,2^{\alpha+2\beta}\max\Big\{\sqrt{n}^{(\alpha+\beta+3)/2},(\alpha+\beta+3)^{(\alpha+\beta+3)/2}\Big\}.$$

Therefore, for every $\alpha,\beta,n\in\mathbb{N}$

$$|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x)| \le C\,4^{\alpha+\beta}\max\Big\{n^{(\alpha+\beta+3)/2},(\alpha+\beta+3)^{(\alpha+\beta+3)/2}\Big\}.$$

This and (M.2) imply that for every $\alpha,\beta\in\mathbb{N}_0$

$$\Big|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x)\Big|$$

$$\le C\frac{M_\alpha M_\beta}{m^{\alpha+\beta}}\frac{(8mH)^{\alpha+\beta+3}\max\big\{n^{(\alpha+\beta+3)/2},(\alpha+\beta+3)^{(\alpha+\beta+3)/2}\big\}}{M_{\alpha+\beta+3}} \le$$

$$\leq C\frac{M_\alpha M_\beta}{m^{\alpha+\beta}} \max\left\{\exp[M(8mH\sqrt{n})], \sup_k \frac{(8mH)^k k^{k/2}}{M_k}\right\}.$$

The above estimation and (M.3)" imply that (4.1) holds for all $m \leq m_0 = (8HL)^{-1}$.

If moreover (M.3)"' holds, then for every $m > 0$ there exists C so that (4.1) holds, which implies the second part of the theorem. □

Lemma 2.2 a) If $\varphi \in C^\infty$ and $N \in \mathbb{N}$ then

$$(4.9) \qquad (L^-L^+)^N\varphi(x) = 2^N(1+x^2-\frac{d^2}{dx^2})^N\varphi(x) = \sum_{p=0}^{2N}\sum_{q=0}^{2N-p} c_{p,q}^{(N)} x^p \varphi^{(q)}(x),$$

where $c_{p,q}^{(N)}$ are constants which satisfy inequality

$$(4.10) \qquad |c_{p,q}^{(N)}| \leq 26^N(2N-q)^{N-\frac{p+q}{2}}.$$

b) Moreover, if conditions (M.1), (M.2) and $k^{k/2} \subset M_k$ are satisfied for $p, q \in \mathbb{N}$, $p+q \leq 2N$, then it holds:

$$(4.11) \qquad |c_{p,q}^{(N)}| \leq\ 52^N \frac{M_N^2}{M_p M_q}.$$

Proof. Let us first prove inequality (4.10) by induction. For $N = 1$ the estimation is obvious. Let us suppose that (4.9) and (4.10) hold for some $N \in \mathbb{N}$. Then

$$2^{N+1}(1+x^2-\frac{d^2}{dx^2})^{N+1}\varphi(x) = \sum_{p=0}^{2N+2}\sum_{q=0}^{2N+2-p} c_{p,q}^{(N+1)} x^p \varphi^{(q)}(x),$$

where

$$c_{p,q}^{(N+1)} = 2\left(c_{p,q}^{(N)} + c_{p-2,q}^{(N)} - c_{p,q-2}^{(N)} - (p+2)(p+1)c_{p+2,q}^{(N)} - 2(p+1)c_{p+1,q-1}^{(N)}\right),$$

for $p, q \in \mathbb{N}$, $p, q \leq 2(N+1)$. The constants $c_{k,l}^{(N)}$ are equal to zero if $k+l > 2N$ or k or l are negative. Therefore,

(4.12)

$$\begin{aligned}
|c_{p,q}^{(N+1)}| &\leq 26^N \cdot 2 \cdot [(2N-q)^{N-\frac{p+q}{2}} + (2N-q)^{N+1-\frac{p+q}{2}} + (2(N+1)-q)^{N-\frac{p+q}{2}} + \\
&+ (2N-q)^2(2N-q)^{N-\frac{p+q}{2}-1} + 3(2N-q)(2N-q)^{N-\frac{p+q}{2}-1} + \\
&+ 2(2N-q)^{N-\frac{p+q}{2}-1} + 2(2N-q)(2N-q)^{N-\frac{p+q}{2}} + 2(2N-q)^{N-\frac{p+q}{2}} \leq \\
&\leq 26^{N+1} \cdot 2 \cdot 13 \cdot (2(N+1)-q)^{N+1-\frac{p+q}{2}}.
\end{aligned}$$

Thus, by induction (4.10) holds for every $n \in \mathbb{N}$.

Let us now prove (4.11). Using the estimation (4.10), $p+q \le 2N$, condition

$$k^{k/2} \le C\, L^k M_k, \tag{4.13}$$

and the fact that the Gevrey sequence satisfy condition (M.2) with $H = 2$, $A = 1$, and (M.1) and (M.2) we have that

$$\begin{aligned} |c_{p,q}^{(N)}| &\le 26^N (2N-q)^{N-\frac{q}{2}} p^{-\frac{p}{2}} \le 26^N (2N-q)^{N-\frac{q}{2}} p^{-\frac{p}{2}} \frac{M_{2N-(p+q)}}{M_{2N-(p+q)}} \le \\ &\le 26^N \frac{(2N-q)^{N-\frac{q}{2}}}{p^{\frac{p}{2}}(2N-(p+q))^{N-\frac{p+q}{2}}} M_{2N-(p+q)} \le \\ &\le 26^N \cdot 2^{N-\frac{q}{2}} \cdot 1 \cdot M_{2N-(p+q)} \frac{M_p M_q}{M_p M_q} \le 52^N \frac{M_{2N}}{M_p M_q}. \end{aligned} \tag{4.14}$$

□

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