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A NOTE ON DECOMPOSITION OF THE DISTRIBUTION ON BMO SPACE

Sadek Gala[1], Amina Lahmar-Benbernou[2]

Abstract. This note is a continuation of the work described in the paper [4]. We prove that there are two bounded complementary projection operators

$$P = \nabla\left(\Delta^{-1}\mathrm{div}\right) \quad \text{and} \quad Q = \mathrm{Div}\left(\Delta^{-1}\mathrm{curl}\right)$$

defined on the class of vectors fields $\overrightarrow{h} \in E$.

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1. Introduction and main result

Recently, S. Gala [3] proved a remarkable theorem to characterize the class of vector fields $\overrightarrow{h}$ which satisfies the commutator inequality

$$\left|\int_{\mathbb{R}^d} \overrightarrow{h}\,.\left(\overline{u}\nabla v - v\nabla\overline{u}\right)dx\right| \leq C\left\|u\right\|_{\dot{H}^1}\left\|v\right\|_{\dot{H}^1} \tag{1.1}$$

for all $u,\ v \in \mathcal{D}\left(\mathbb{R}^d\right)$. Here we use theorem 1 from [3] to decompose $\overrightarrow{h}$ in the form

$$\overrightarrow{h} = \nabla g + \mathrm{Div}\ H$$

in the distributional sense, where $g \in BMO\left(\mathbb{R}^d\right)$, H is a skew-symmetric matrix field such that $H \in BMO\left(\mathbb{R}^d\right)^{d^2}$ and $\mathrm{Div} : \mathcal{D}'\left(\mathbb{R}^d\right)^{d\times d} \to \mathcal{D}'\left(\mathbb{R}^d\right)$ is the row divergence operator defined by

$$\mathrm{Div}\ \left(h_{i,j}\right) = \left(\sum_{j=1}^{d}\partial_j h_{i,j}\right)_{i=1}^{d}.$$

See Stein ([8], Chapter IV) for the theory of BMO.

We start with some prerequisites for our main result. Let B be a ball in $\mathbb{R}^d$, and let $L^p(B)$, $p \geq 1$ be the space of real-valued functions f, defined on B and such that $|f|^p$ is summable with respect to the Lebesgue measure. By $L^\infty(B)$

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[1] University of Mostaganem, Department of Mathematics, P.O. Box 227 Mostaganem (27000), Algeria, e-mail: sadek.gala@gmail.com

[2] University of Mostaganem, Department of Mathematics, P.O. Box 227 Mostaganem (27000), Algeria

we denote the space of real-valued measurable functions f that are essentially bounded on B. The symbol $\mathcal{D}(B) = C_0^\infty(B)$ stands for the set of all real-valued infinitely differentiable functions with a compact support in B.

A vector $\overrightarrow{v} = \{v_1, ..., v_d\}$, $v_i \in L^1(B)$, is said to be the gradient of a function $w \in L^1(B)$, if

$$\int\limits_B w \frac{\partial}{\partial x_i} \varphi dx = -\int\limits_B v_i \varphi dx, \qquad \forall \varphi \in \mathcal{D}(B), \quad i = 1, ..., d.$$

The gradient is denoted either by ∇w or by $\frac{\partial w}{\partial x}$.

Denote by $H^1 = H^1(B)$ the Sobolev space formed by all functions in $L^2(B)$, whose gradients belong to $\mathbb{L}^2(B) = \left(L^2(B)\right)^d$. Equipped with the scalar product

$$\langle w_1, w_2 \rangle = \int\limits_B \frac{\partial w_1}{\partial x_i} \frac{\partial w_2}{\partial x_i} dx + \int\limits_B w_1 w_2 dx,$$

$H^1(B)$ becomes a Hilbert space. The norm correponding to the above scalar product is

$$\|w\|_{H^1(B)} = \|w\|_{L^2(B)} + \|\nabla w\|_{L^2(B)}.$$

Among the subspaces of the space $H^1(B)$ that will be used in the sequel is the space $\dot{H}^1(B)$ which is the closure of the set $\mathcal{D}(B)$ in $H^1(B)$. The space $\dot{H}^1(B)$ is naturally associated with the Dirichlet problem, since the inclusion $w \in \dot{H}^1(B)$ represents an equivalent formulation for the boundary condition $w\,|_{\partial B} = 0$. The imbedding $\dot{H}^1(B) \subset L^2(B)$ is compact. Hereafter, immaterial constants are denoted by $C, c, ...$; they are not necessarily the same on the way of two consecutive occurences.

For any bounded domain B the Friedrichs inequality

$$\|w\|_{L^2(B)} \leq c \|\nabla w\|_{L^2(B)}, \quad \forall w \in \dot{H}^1(B), \tag{1.2}$$

holds with a constant c independent of w. The inequality (1.2) implies that the functional $\|w\|_{H^1(B)}$ can be taken as an equivalent norm in $\dot{H}^1(B)$, and indeed, we shall always consider $\|w\|_{H^1(B)}$ as a norm in this space. The dual space of $\dot{H}^1(B)$, i.e. the set of all continuous linear functionals on $\dot{H}^1(B)$, is denoted by $H^{-1}(B)$. If f is an element of $H^{-1}(B)$, then $\langle f, \varphi \rangle$ stands for the value of the functional f applied to the element $\varphi \in \dot{H}^1(B)$.

For any vector field $\overrightarrow{h} \in \mathbb{L}^2(B)$, the divergence is an element of the space $H^{-1}(B)$ defined by the formula

$$\left\langle \operatorname{div} \overrightarrow{h}, \varphi \right\rangle = -\int\limits_B \overrightarrow{h}.\nabla \varphi dx, \qquad \forall \varphi \in \dot{H}^1(B).$$

The following evident estimate holds :

$$\left\| \text{div } \overrightarrow{h} \right\|_{H^{-1}(B)} = \sup_{\|\varphi\|_{H^1(B)}=1} \int_B \overrightarrow{h}.\nabla\varphi dx \leq \left\| \overrightarrow{h} \right\|_{\mathbb{L}^2(B)}.$$

A vector field $\overrightarrow{\beta}$ is said to be solenoidal if div $\overrightarrow{\beta} = 0$. For any vector field $\overrightarrow{v} \in \mathbb{L}^2(B)$ the relations

$$\langle \text{curl } \overrightarrow{v}, \varphi \rangle_{i,j} = \int_B \left(v_j \frac{\partial \varphi}{\partial x_i} - v_i \frac{\partial \varphi}{\partial x_j} \right) dx, \quad \forall \varphi \in \dot{H}^1(B), \ i,j \in \{1, ..., d\},$$

define a skew-symmetric matrix curl $\overrightarrow{v}$, whose elements belong to the space $H^{-1}(B)$.

A vector field $\overrightarrow{v}$ is said to be irrotational, or vortex-free if curl $\overrightarrow{v} = 0$. We say that a vector field $\overrightarrow{v} \in \mathbb{L}^2(B)$ is potential if $\overrightarrow{v}$ can be represented in the form $\overrightarrow{v} = \nabla w$, where $w \in H^1(B)$. Obviously, any potential vector field is irrotational. For the sake simplicity, we shall often write curl $\overrightarrow{v} \in H^{-1}(B)$ instead of $(\text{curl } \overrightarrow{v})_{i,j} \in H^{-1}(B)$.

We shall also have to deal with unbounded domains, e.g. $\mathbb{R}^d$. The notations $\mathcal{D}(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$ are clear from the previous consideration.

Of particular interest to us will be the divergence operator acting on matrix fields, where it is the formal adjoint to the differential operator

$$\mathbf{D} : \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right) \to \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right).$$

We have

$$\text{Div } : \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right) \to \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right)$$

given explicitly by the formula

$$\text{Div } M = \left(\text{div } M^1, ..., \text{div } M^d\right)$$

where, of course, M^i are the row vectors of $M \in \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right)$ (see [5]). Hence, for $M \in W^{1,1}_{loc}\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right)$, we have

$$\int_{\mathbb{R}^d} \langle \text{Div } M, \Phi \rangle \, dx = -\int_{\mathbb{R}^d} \langle M, D\Phi \rangle \, dx$$

for every test mapping $\Phi \in \mathcal{D}\left(\mathbb{R}^d, \mathbb{R}^d\right)$. Notice that in fact the scalar product of matrices has been used here on the left-hand side :

$$\langle M, N \rangle = \text{Trace } \left(M^t N\right) = \sum_{i=1}^d \left\langle M^i, N^i \right\rangle.$$

The linear operators

$$\begin{aligned}
\nabla &: \mathcal{D}'\left(\mathbb{R}^d\right) \to \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right) \\
\text{div} &: \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right) \to \mathcal{D}'\left(\mathbb{R}^d\right) \\
\text{Div} &: \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right) \to \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right) \\
\text{curl} &: \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right) \to \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right)
\end{aligned}$$

owe much of their importance to the theory of Maxwell's equations [2]. In this connection, we recall that the Laplacian is an operator

$$\Delta : \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right) \to \mathcal{D}'\left(\mathbb{R}^d, \mathbb{R}^d\right)$$

defined coordinatewise on the vector field $\overrightarrow{h} = (h_1, ..., h_d)$ by

$$\Delta \overrightarrow{h} = \left(\Delta h^1, ..., \Delta h^d\right)$$

where Δh^i is the usual Laplacian on functions,

$$\Delta h^i = \sum_{j=1}^{d} h^i_{jj}.$$

We recall the well-known Hodge decomposition of the Laplacian which asserts

$$\Delta \overrightarrow{h} = \nabla \text{div } \overrightarrow{h} + \text{Div curl } \overrightarrow{h}$$

where the first component is curl-free, while the second component is div-free (see e.g. [6]).

The Maxwell equations for vacuum have the form (see e.g. [2])

$$\begin{aligned} \partial_t E &= \text{curl } H, \\ \partial_t H &= -\text{curl } E, \\ \text{div } E &= \text{div } H = 0, \end{aligned} \tag{1.3}$$

where E (resp. H) is the electric (resp. magnetic) field. Recall that $E(t, x)$, $H(t, x)$ are vector-valued functions from Minkowski space in $\mathbb{R}^3$. To pose correctly the Cauchy problem for the Maxwell equations we take the initial conditions :

$$E(0, x) = e(x), \quad H(0, x) = h(x). \tag{1.4}$$

Then the equations div $E =$div $H = 0$ in (1.3) show that the initial data have to satisfy the constraint conditions

$$\text{div } e = \text{div } h = 0. \tag{1.5}$$

Taking the evolution part

$$\begin{aligned} \partial_t E &= \text{curl } H, \\ \partial_t H &= -\text{curl } E, \end{aligned} \tag{1.6}$$

of the Maxwell equations, we see that we can solve the Cauchy problem for (1.6) with initial data (1.4) satisfying the constraint conditions (1.5). Then, taking the div operator in the equations (1.6), we see that

$$\partial_t \text{div } E = \partial_t \text{div } H = 0,$$

so the constraint conditions (1.5) insure the elliptic part div E =div $H = 0$ in the Maxwell equations (1.3).

Again, a simple reduction to the wave equation can be done. In fact, taking the time derivative in the first equation in (1.3) and using the relation

$$\text{curl curl } E = -\Delta E$$

provided div $E = 0$, we get

$$\left(\partial_t^2 - \Delta\right) E = 0.$$

In a similar way one can see that H also satisfies the wave equations.

Definition 1. *Let E be a linear space, and E_1 and E_2 the subspaces of E. We say E is the direct sum of E_1 and E_2 and write*

$$E = E_1 \oplus E_2$$

if any $\overrightarrow{h} \in E$ can be uniquely decomposed as

$$\overrightarrow{h} = \overrightarrow{\alpha} + \overrightarrow{\beta}, \quad \overrightarrow{\alpha} \in E_1, \quad \overrightarrow{\beta} \in E_2.$$

As we will show, this property characterizes projections, so we make the following definition.

Definition 2. *A projection on a linear space E is a linear map $P : E \to E$ such that*

$$P^2 = P.$$

Any projection is associated with a direct sum decomposition. There is one-to-one correspondence between direct sum and linear operators P satisfying $P^2 = P$. Indeed, we have

Lemma 1. *Let E be a linear space. Then*

$$E = E_1 \oplus E_2$$

if and only if there is a linear operator

$$P : E \to E$$

with $P^2 = P$, so that in the decomposition

$$\overrightarrow{h} = \overrightarrow{\alpha} + \overrightarrow{\beta}, \quad \overrightarrow{\alpha} = P\overrightarrow{h}, \quad \overrightarrow{\beta} = (I - P)\overrightarrow{h}.$$

Moreover,

$$E_1 = P(E) \quad \text{and} \quad E_2 = (I - P)(E).$$

As a consequence of theorem 1 in [4], we deduce that if

$$\left|\left\langle \overrightarrow{h}, \overline{u}\nabla v - v\nabla\overline{u}\right\rangle\right| \leq C \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)} \left\|\nabla v\right\|_{L^2(\mathbb{R}^d)}$$

holds for all u, $v \in \mathcal{D}\left(\mathbb{R}^d\right)$, then $\overrightarrow{h}$ can be decomposed in the form

$$\overrightarrow{h} = P\overrightarrow{h} + Q\overrightarrow{h} \tag{1.7}$$

where $P = \nabla\left(\Delta^{-1}\mathrm{div}\right)$ and $Q =$Div $\left(\Delta^{-1}\mathrm{curl}\right)$ are two bounded complementary projection operators.

We now state our main result for arbitrary (complex-valued) distributions $\overrightarrow{h}$. Set E

$$E = \left\{ \begin{array}{c} \overrightarrow{h} \in \mathcal{D}'\left(\mathbb{R}^d\right)^d : \exists C > 0, \forall u, v \in \mathcal{D}\left(\mathbb{R}^d\right) \\ \left|\left\langle \overrightarrow{h}, \overline{u}\nabla v - v\nabla\overline{u}\right\rangle\right| \leq C \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)} \left\|\nabla v\right\|_{L^2(\mathbb{R}^d)} \end{array} \right\}. \tag{1.8}$$

Theorem 1. *Assume $\overrightarrow{h} \in E$. Define P and Q respectively by*

$$P = \nabla\left(\Delta^{-1}div\right) \quad and \quad Q = Div\ \left(\Delta^{-1}curl\right). \tag{1.9}$$

Then

(i)

$$P\overrightarrow{h} \in E, \qquad Q\overrightarrow{h} \in E;$$

(ii)

$$P\left(P\overrightarrow{h}\right) = P\overrightarrow{h}, \qquad Q\left(Q\overrightarrow{h}\right) = Q\overrightarrow{h};$$

(iii)

$$P\overrightarrow{h} + Q\overrightarrow{h} = \overrightarrow{h}.$$

The operators P and Q are called mutually complementary.

As a consequence of this theorem, we obtain a bounded linear operator $P : \overrightarrow{h} \mapsto P\overrightarrow{h}$ from E onto E defined by $P\overrightarrow{h} = \overrightarrow{\alpha}$ with $\overrightarrow{\alpha} = \nabla\left(\Delta^{-1}\mathrm{div}\right)\overrightarrow{h}$.

Corollary 1. *Let $\overrightarrow{h} = \overrightarrow{\alpha} + \overrightarrow{\beta}$ be the decomposition of $\overrightarrow{h} \in E$. Then*

$$P : E \to E,$$

defined by $P\overrightarrow{h} = \overrightarrow{\alpha}$ for all $\overrightarrow{h} \in E$, is a bounded linear operator with the norm $\|P\| \leq C$. Thus

$$\left\|P\overrightarrow{h}\right\|_E \leq C\left\|\overrightarrow{h}\right\|_E, \quad for\ all\ \ \overrightarrow{h} \in E.$$

P has the following properties :

$$\begin{aligned} P\overrightarrow{\beta} &= 0, \quad (I-P)\overrightarrow{h} = \overrightarrow{\beta}, \quad P^2\overrightarrow{h} = P\overrightarrow{h}, \\ (I-P)^2\overrightarrow{h} &= (I-P)\overrightarrow{h}, \quad \left\langle P\overrightarrow{h}, \overrightarrow{g}\right\rangle = \left\langle \overrightarrow{h}, P\overrightarrow{g}\right\rangle, \end{aligned}$$

for all $\overrightarrow{h}, \overrightarrow{g} \in E$, where I denotes the identity.

From these properties we easily conclude that P is a selfadjoint operator, and that $P = P'$, where P' means the dual operator of P.

2. Some preliminary lemmas

We now give some lemmas which will be utilized in the following sections. In the sequel, we shall denote by $B = B(z, \rho)$ the ball with its center $z \in \mathbb{R}^d$ and its radius $\rho > 0$.

Lemma 2. *There is a constant C (which depends only on d) such that for all balls B and all $\overrightarrow{h} \in E$,*

$$\left\| div\ \overrightarrow{h} \right\|_{\dot{H}^{-1}(B)} \leq C\, |B|^{\frac{1}{2}-\frac{1}{d}} . \tag{2.1}$$

Proof. Let $v \in \mathcal{D}(B)$ be given and let u be a function in $\mathcal{D}(B)$ such that $u = 1$ on supp v. Then the following estimate is valid :

$$\begin{aligned} \left| \left\langle \overrightarrow{h}, \overline{u} \nabla v - v \nabla \overline{u} \right\rangle \right| &= \left| \left\langle \overrightarrow{h}, \nabla v \right\rangle \right| = \left| \left\langle \text{div } \overrightarrow{h}, v \right\rangle \right| \\ &\leq C(d)\, \|\nabla u\|_{L^2(\mathbb{R}^d)} \, \|\nabla v\|_{L^2(B)} . \end{aligned}$$

Taking the infimum over all such u on the right-hand side, we get

$$\left| \left\langle \text{div } \overrightarrow{h}, v \right\rangle \right| \leq C\ \sqrt{\text{cap}\,(B)}\, \|\nabla v\|_{L^2(B)}$$

where the capacity of a compact set $K \subset \mathbb{R}^d$ cap(.) is defined by ([7], sect. 11.15)

$$\text{cap}\ (K) = \inf \left\{ \|u\|^2_{\dot{H}^1(\mathbb{R}^d)} : u \in \mathcal{D}\left(\mathbb{R}^d\right),\ \ u \geq 1 \ \text{ on } \ K \right\}.$$

Since for a ball B in $\mathbb{R}^d$,

$$\text{cap}\left(B, \dot{H}^1 \right) \simeq |B|^{1-\frac{2}{d}} ,$$

the proof of the lemma is complete. □

In order to prove our main result, the following lemma will be used.

Lemma 3. *There is a constant $C(d)$ so that for all balls B and all $\overrightarrow{h} \in E$,*

$$\left\| \overrightarrow{h} \right\|_{\dot{H}^{-1}(B)} \leq C\ \ |B|^{\frac{1}{2}} . \tag{2.2}$$

Proof. Let B^* be the ball with the same center as B but with the side length twice as long. Suppose that $v \in \mathcal{D}(B)$ and let φ be a C^∞ function taking values in $[0, 1]$ with support in B^* and so that $\varphi = 1$ on B. Let us set $u = (x_i - z_i)\, \varphi$ $\left(i = \overline{1, d}\right)$, where $z = (z_i)$ is the center of B. Then it is easy to see that

$$\|\nabla u\|_{L^2(B^*)} \leq \|\nabla u\|_{L^2(B)} \leq C\, |B|^{\frac{1}{2}} .$$

Next note that for such u and v

$$\begin{aligned}\left\langle \overrightarrow{h}, \overline{u}\nabla v - v\nabla \overline{u}\right\rangle &= \left\langle \overrightarrow{h}, \nabla\left(\overline{u}v\right) - 2v\nabla \overline{u}\right\rangle \\ &= -\left\langle \operatorname{div} \overrightarrow{h}, \overline{u}v\right\rangle - 2\left\langle \overrightarrow{h}, v\nabla \overline{u}\right\rangle \\ &= -\left\langle \operatorname{div} \overrightarrow{h}, \left(x_i - z_i\right)v\right\rangle - 2\left\langle h_i, v\right\rangle.\end{aligned}$$

Concerning $\left\langle \operatorname{div} \overrightarrow{h}, \left(x_i - z_i\right)v\right\rangle$, we observe that by using (2.1), the Poincaré inequality with v replaced by $\left(x_i - z_i\right)v$

$$\begin{aligned}\left|\left\langle \operatorname{div} \overrightarrow{h}, \left(x_i - z_i\right)v\right\rangle\right| &\leq C\left|B\right|^{\frac{1}{2}-\frac{1}{d}}\left\|\nabla\left[\left(x_i - z_i\right)v\right]\right\|_{L^2(B)} \\ &\leq C\left|B\right|^{\frac{1}{2}-\frac{1}{d}}\left(\left\|v\right\|_{L^2(B)} + \left\|\left(x_i - z_i\right)\nabla v\right\|_{L^2(B)}\right) \\ &\leq C\left|B\right|^{\frac{1}{2}-\frac{1}{d}}\left(2\left|B\right|^{\frac{1}{d}}\left\|\nabla v\right\|_{L^2(B)} + \left\|\left(x_i - z_i\right)\nabla v\right\|_{L^2(B)}\right) \\ &\leq C\left|B\right|^{\frac{1}{2}}\left\|\nabla v\right\|_{L^2(B)}, \quad \forall v \in \mathcal{D}\left(B\right).\end{aligned}$$

Since for every $i = \overline{1, d}$,

$$\begin{aligned}2\left|\left\langle h_i, v\right\rangle\right| &\leq \left|\left\langle \overrightarrow{h}, \overline{u}\nabla v - v\nabla \overline{u}\right\rangle\right| + \left|\left\langle \operatorname{div} \overrightarrow{h}, \left(x_i - z_i\right)v\right\rangle\right| \\ &\leq C\left\|\nabla u\right\|_{L^2(2B)}\left\|\nabla v\right\|_{L^2(B)} + C\left|B\right|^{\frac{1}{2}}\left\|\nabla v\right\|_{L^2(B)} \\ &\leq C\left|B\right|^{\frac{1}{2}}\left\|\nabla v\right\|_{L^2(B)},\end{aligned}$$

and we can conclude. □

The $\nu-$th Riesz transform, $1 \leq \nu \leq d$, is a singular integral operator [9]

$$\mathcal{R}_\nu f(x) = p.v. \int_{\mathbb{R}^d} \frac{(x_\nu - y_\nu)}{|x-y|^{d+1}} f(y)dy = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{(x_\nu - y_\nu)}{|x-y|^{d+1}} f(y)dy.$$

The principal value integral above exists for all x if f is a compactly supported smooth function, and one has for such functions the L^p estimate

$$\left\|\mathcal{R}_\nu f\right\|_{L^p(\mathbb{R}^d)} \leq C\left\|f\right\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p < \infty,$$

for some positive constant C independent of f. All higher-order transforms are automatically bounded because the partial differential operators commute, e.g., $\partial_\nu\partial_\mu\Delta^{-1} = \left(\partial_\nu\Delta^{-\frac{1}{2}}\right)\left(\partial_\mu\Delta^{-\frac{1}{2}}\right)$.

Fix $x \in \mathbb{R}^d$ and set

$$\begin{aligned}K_\nu(x,y) &= \frac{\partial K(x,y)}{\partial x_\nu} = c(d)\frac{(x_\nu - y_\nu)}{|x-y|^{d+1}}, \\ K_{\nu,\mu}(x,y) &= \frac{\partial^2 K(x,y)}{\partial x_\nu \partial x_\mu} = c(d)\left\{\frac{\delta_{\nu,\mu}}{|x-y|^d} - \frac{d(x_\nu - y_\nu)(x_\mu - y_\mu)}{|x-y|^{d+2}}\right\}.\end{aligned}$$

For a fixed cube B in $\mathbb{R}^d$, we denote by $\{\omega_j\}_{j=0}^{\infty}$ a smooth partition of unity associated with B, i.e. fix $\omega_0 \in \mathcal{D}(2B)$ with the properties $\omega_j \in \mathcal{D}\left(2^{j+1}B \backslash 2^{j-1}B\right)$, $j \geq 1$ so that

$$0 \leq \omega_j(x) \leq 1, \quad |\nabla \omega_j(x)| \leq C\left(2^j r\right)^{-1}, \;\; j \in \mathbb{N} \tag{2.3}$$

where C depends only on d. Finally, we have for all $x \in \mathbb{R}^d$,

$$\sum_{j=0}^{\infty} \omega_j(x) = 1.$$

In the following $\mathcal{R}_i = (-\Delta)^{-\frac{1}{2}} \partial_i$ $\left(\text{resp. } \mathcal{R}_{i,m} = -\partial_i \partial_m \Delta^{-1}\right)$ $(i, m = 1, ..., d)$ denotes the Riesz transforms (resp. the double Riesz transforms) on $\mathbb{R}^d$ (see [8]) which are given respectively up to a constant multiple by

$$K_i(x-y) = \frac{(x_i - y_i)}{|x-y|^d}, \quad K_{i,m}(x-y) = \frac{|x-y|^2 - d^{-1}(x_i - y_i)(x_m - y_m)}{|x-y|^{d+2}}.$$

From this we derive the following lemma.

Lemma 4. *There is a constant C (which depends only on d) such that if supp $v \subset B$ and $\int_B v dx = 0$, then*

$$\left\| \nabla \left(\omega_j \Delta^{-1} div\ v\right) \right\|_{L^2(2^{j+1}B)} \leq C 2^{-j\frac{d}{2}} \|v\|_{L^2(B)}, \tag{2.4}$$

for all $j \geq 0$.

Proof. To prove (2.4), let $v \in \mathcal{D}(B)$. By the boundedness of $\mathcal{R}_{\nu,\mu}$ on $L^2\left(\mathbb{R}^d\right)$, it is obvious that, for j equal to 0 or 1, we have

$$\begin{aligned}
\left\| \nabla \left(\omega_j \partial_\nu \Delta^{-1} v\right) \right\|_{L^2(2^{j+1}B)} &\leq \left\| \nabla \omega_j \left(\partial_\nu \Delta^{-1} v\right) \right\|_{L^2(2^{j+1}B)} + \left\| \omega_j \nabla \left(\partial_\nu \Delta^{-1} v\right) \right\|_{L^2(2^{j+1}B)} \\
&\leq C \left(|B|^{-\frac{1}{d}} \left\| \nabla \Delta^{-1} v \right\|_{L^2(\mathbb{R}^d)} + \sum_{\mu=1}^{d} \left\| \mathcal{R}_{\nu,\mu} v \right\|_{L^2(\mathbb{R}^d)} \right) \\
&\leq C \left(|B|^{-\frac{1}{d}} \left\| \nabla \Delta^{-1} v \right\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^2(\mathbb{R}^d)} \right).
\end{aligned}$$

But, since supp $v \subset B$ and $\int_B v dx = 0$, it follows from Poincaré's inequality that

$$\left\| \nabla \Delta^{-1} v \right\|_{L^2(\mathbb{R}^d)} = \left\| \nabla \Delta^{-1} v \right\|_{L^2(B)} \leq C |B|^{\frac{1}{d}} \|v\|_{L^2(B)}$$

and so

$$\left\| \nabla \left(\omega_j \partial_\nu \Delta^{-1} v\right) \right\|_{L^2(2^{j+1}B)} \leq C \|v\|_{L^2(B)}.$$

On the other hand, we have for $j \geq 2$ and for any $x \in 2^{j+1}B \setminus 2^{j-1}B$,

$$\begin{aligned}
\left|\nabla\left(\omega_j \partial_\nu \Delta^{-1} v\right)(x)\right| &\leq |\nabla \omega_j(x)| \left|\partial_\nu \Delta^{-1} v(x)\right| + |\omega_j(x)| \left|\nabla \partial_\nu \Delta^{-1} v(x)\right| \\
&\leq C\left(2^j r\right)^{-1} \int_B |K_\nu(x-y)| \, |v(y)| \, dy \\
&\quad + \sum_{\mu=1}^{d} \int_B |K_{\nu,\mu}(x-y)| \, |v(y)| \, dy \\
&\leq C\left(2^j r\right)^{-1} \int_B \frac{1}{|x-y|^d} |v(y)| \, dy \\
&\leq C\left(2^j r\right)^{-1} \frac{1}{[\text{dist } (x,B)]^d} \int_B |v(y)| \, dy \\
&\leq C\left(2^j r\right)^{-d} \int_B |v(y)| \, dy \\
&\leq C\left(2^j r\right)^{-d} \left(\int_B |v(y)|^2 \, dy\right)^{\frac{1}{2}} |B|^{\frac{1}{2}} \\
&\leq C 2^{-jd} r^{-\frac{d}{2}} \|v\|_{L^2(B)},
\end{aligned}$$

since

$$|K_\nu(x-y)| \leq \frac{C(d)}{|x-y|^d} \quad \text{and} \quad |K_{\nu,\mu}(x-y)| \leq \frac{C(d)}{|x-y|^d}.$$

Hence,

$$\left\|\nabla\left(\omega_j \partial_\nu \Delta^{-1} v\right)\right\|_{L^2(2^{j+1}B)} \leq C 2^{-j\frac{d}{2}} \|v\|_{L^2(B)}, \quad \text{for all } \ j \geq 0.$$

Summing on ν yields the bound

$$\left\|\nabla\left(\omega_j \Delta^{-1} \text{div } v\right)\right\|_{L^2(2^{j+1}B)} \leq C 2^{-j\frac{d}{2}} \|v\|_{L^2(B)}.$$

We thus get the result. □

We recall the well-know inequality.

Lemma 5. (Poincaré's inequality) *Let* $\delta = \delta(B) = \sup_{x,y \in B} |x-y|$ *denote the diameter of* B*. Then*

$$\|v\|_{L^2(B)} \leq C \|\nabla v\|_{L^2(B)}$$

for all $v \in \dot{H}^1(B)$*, where* $C = C(\delta) > 0$ *depends only on* δ*.*

Proof. See ([1], VI, 6.26). □

Using Lemmas 4 and 5, one obtains as a corollary the following result.

Corollary 2. *There is a constant C (which depends only on d) such that if supp $v \subset B$ and $\int_B v dx = 0$, then*

$$\left\|\nabla \left(\omega_j \Delta^{-1} div\ v\right)\right\|_{L^2(2^{j+1}B)} \leq C 2^{-j\frac{d}{2}} \left\|\nabla v\right\|_{L^2(B)},$$

for all $j \geq 0$.

If we want to prepare the scaling argument, we consider a function $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$ with the properties

$$0 \leq \varphi \leq 1, \quad \varphi(x) = 1 \text{ if } |x| \leq 1, \ \varphi(x) = 0 \text{ if } |x| \geq 2.$$

It follows that for any multi-index γ,

$$|\nabla^\gamma \varphi(x)| \leq M_\gamma \quad \text{for all } x \in \mathbb{R}^d.$$

Now, for any positive integer k, we define $\varphi_k(x) = \varphi\left(\frac{x}{k}\right)$, $x \in \mathbb{R}^d$. Then, $\varphi_k(x)$ satisfies the following properties :

$$\begin{cases} 0 \leq \varphi_k(x) \leq 1, \\ \varphi_k(x) = 1 \quad \text{if} \quad |x| \leq k, \\ \varphi_k(x) = 0 \quad \text{if} \quad |x| \geq 2k, \end{cases}$$

and for any multi-index γ

$$|\nabla^\gamma \varphi_k(x)| = \left|\frac{1}{k^{|\gamma|}} \nabla^\gamma \varphi(\frac{x}{k})\right| \leq \frac{1}{k^{|\gamma|}} M_\gamma$$

With these notations we obtain

Lemma 6. *Let $\overrightarrow{h} \in E$ and $\overrightarrow{\beta} = Q\overrightarrow{h}$. Then for all $\overrightarrow{v} \in \mathcal{D}\left(\mathbb{R}^d\right)^d$*

$$\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1} div\ \overrightarrow{v}\right\rangle \to 0, \quad as \ \ k \to +\infty. \tag{2.5}$$

Proof. Choose $\overrightarrow{h} \in E$ and $\overrightarrow{\beta} = Q\overrightarrow{h}$. In order to see that

$$\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1} \text{div } \overrightarrow{v}\right\rangle \to 0, \quad \text{as } \ k \to +\infty,$$

we proceed in the following way. Since $\nabla\varphi_k$ vanishes outside $\{k \leq |x| \leq 2k\}$, it follows that

$$\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1} \text{div } \overrightarrow{v}\right\rangle = \sum_{N_1 \leq j \leq N_2} \left\langle \nabla\varphi_k.\overrightarrow{\beta}, \omega_j \Delta^{-1} \text{div } \overrightarrow{v}\right\rangle,$$

where N_s, $s = 1, 2$ is chosen so that $N_s \uparrow \infty$ as $k \to +\infty$ (there is at most one non-zero term in the series, all terms are ≥ 0 and at least one term equals 1). But ω_j is supported on $2^{j+1}B \setminus 2^{j-1}B$ for $j \geq 1$. Thus

$$\text{supp } (\nabla\varphi_k \omega_j) \subset \left\{2^{j+1}B \setminus 2^{j-1}B\right\} \cap \{k \leq |x| \leq 2k\}.$$

We may assume without loss of generality that

$$|z| < 2^j r \quad \text{for} \quad k \text{ large.}$$

Then

$$2^j r \simeq k, \quad \text{i.e.,} \quad 2^j \simeq k\ r^{-1}.$$

Consequently, using Hölder' s inequality, Lemma 3 and the fact that $\|\nabla\varphi_k\|_{L^\infty} \leq \frac{C}{k}$ for all $k = 1, 2, ..,$ we get

$$\begin{aligned}
\left|\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1}\mathrm{div}\ \overrightarrow{v}\right\rangle\right| &\leq \sum_{N_1\leq j\leq N_2} \left|\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \omega_j\Delta^{-1}\mathrm{div}\ \overrightarrow{v}\right\rangle\right| \\
&\leq \sum_{N_1\leq j\leq N_2} \left\|\nabla\varphi_k.\overrightarrow{\beta}\right\|_{\dot{H}^{-1}(2^{j+1}B)} \left\|\omega_j\Delta^{-1}\mathrm{div}\ \overrightarrow{v}\right\|_{\dot{H}^1(2^{j+1}B)} \\
&\leq \sum_{N_1\leq j\leq N_2} \frac{C}{k}\left|2^{j+1}B\right|^{\frac{1}{2}} \left\|\omega_j\Delta^{-1}\mathrm{div}\ \overrightarrow{v}\right\|_{\dot{H}^1(2^{j+1}B)} \\
&\leq \frac{C}{k}|B|^{\frac{1}{2}} \sum_{N_1\leq j\leq N_2} 2^{\frac{j+1}{2}} \left\|\omega_j\Delta^{-1}\mathrm{div}\ \overrightarrow{v}\right\|_{\dot{H}^1(2^{j+1}B)} \\
&= \frac{C}{k}|B|^{\frac{1}{2}} \sum_{N_1\leq j\leq N_2} 2^{\frac{j+1}{2}} \left\|\nabla\left(\omega_j\Delta^{-1}\mathrm{div}\ v\right)\right\|_{L^2(2^{j+1}B)}.
\end{aligned}$$

Now by Lemma 4, we obtain

$$\left|\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1}\mathrm{div}\ \overrightarrow{v}\right\rangle\right| \leq \frac{C}{k}|B|^{\frac{1}{2}} \|\overrightarrow{v}\|_{L^2(B)}. \tag{2.6}$$

The right-hand side of this inequality tends to zero as $k \to \infty$. Consequently,

$$\lim_{k\to\infty} \left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1}\mathrm{div}\ \overrightarrow{v}\right\rangle = 0.$$

This completes the proof of the present lemma. □

3. Proof of Theorem 1

We are in a position to prove the main result.
Proof. Suppose that $\overrightarrow{h} \in E$. Let

$$\overrightarrow{\alpha} = P\overrightarrow{h} \quad \text{and} \quad \overrightarrow{\beta} = Q\overrightarrow{h}.$$

It follows from(1.7) that $\overrightarrow{h}$ can be decomposed as

$$\overrightarrow{h} = \overrightarrow{\alpha} + \overrightarrow{\beta}.$$

where

$$\mathrm{curl}\ \overrightarrow{h} = \mathrm{curl}\ \overrightarrow{\beta} \quad \text{and} \quad \overrightarrow{\beta} = \mathrm{Div}\ M,$$

and $\overrightarrow{\beta}$ satisfies the estimate

$$\begin{aligned} \left|\left\langle \overrightarrow{\beta}, v\nabla\overline{u} - \overline{u}\nabla v\right\rangle\right| &= \left|\langle \text{Div } M, v\nabla\overline{u} - \overline{u}\nabla v\rangle\right| \\ &= \left|\text{trace } \langle M, \mathbf{D}\left[v\nabla\overline{u} - \overline{u}\nabla v\right]\rangle\right| \\ &\leq C\left\|u\right\|_{\dot{H}^1(\mathbb{R}^d)}\left\|v\right\|_{\dot{H}^1(\mathbb{R}^d)}, \end{aligned} \tag{3.1}$$

for all u, $v \in \mathcal{D}\left(\mathbb{R}^d\right)$. Consequently, $\overrightarrow{\beta} \in E$. Also,

$$\begin{aligned} \left|\langle \overrightarrow{\alpha}, v\nabla\overline{u} - \overline{u}\nabla v\rangle\right| &= \left|\left\langle \overrightarrow{h} - \overrightarrow{\beta}, v\nabla\overline{u} - \overline{u}\nabla v\right\rangle\right| \\ &\leq \left|\left\langle \overrightarrow{h}, v\nabla\overline{u} - \overline{u}\nabla v\right\rangle\right| + \left|\left\langle \overrightarrow{\beta}, v\nabla\overline{u} - \overline{u}\nabla v\right\rangle\right| \\ &\leq 2C\left\|u\right\|_{\dot{H}^1(\mathbb{R}^d)}\left\|v\right\|_{\dot{H}^1(\mathbb{R}^d)}. \end{aligned}$$

Assertion (i) is proved. It remains therefore to show that

$$P\left(P\overrightarrow{h}\right) = P\overrightarrow{h},\ \ Q\left(Q\overrightarrow{h}\right) = Q\overrightarrow{h} \quad \text{and} \quad P\overrightarrow{h} + Q\overrightarrow{h} = \overrightarrow{h}.$$

Applying Lemma 3 to $\overrightarrow{\beta}$ and using (3.1), we obtain

$$\left\|\overrightarrow{\beta}\right\|_{\dot{H}^{-1}(B)} \leq C\ \left|B\right|^{\frac{1}{2}}. \tag{3.2}$$

Thus,

$$\begin{aligned} \left\|\nabla\varphi_k.\overrightarrow{\beta}\right\|_{\dot{H}^{-1}(B)} &\leq \left\|\nabla\varphi_k\right\|_{L^\infty(\mathbb{R}^d)}\left\|\overrightarrow{\beta}\right\|_{\dot{H}^{-1}(B)} \\ &\leq \frac{C}{k}\left|B\right|^{\frac{1}{2}} \end{aligned}$$

for all $k = 1, 2, ...$ Now for every $\varphi \in C_0^\infty\left(\mathbb{R}^d\right)$, we have

$$\begin{aligned} \left\langle \varphi_k\overrightarrow{\beta}, \nabla\left(\Delta^{-1}\text{div }\overrightarrow{v}\right)\right\rangle &= -\left\langle \text{div}\left(\varphi_k\overrightarrow{\beta}\right), \Delta^{-1}\text{div }\overrightarrow{v}\right\rangle \\ &= -\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1}\text{div }\overrightarrow{v}\right\rangle - \left\langle \varphi_k\text{div }\overrightarrow{\beta}, \Delta^{-1}\text{div }\overrightarrow{v}\right\rangle \\ &= -\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1}\text{div }\overrightarrow{v}\right\rangle, \end{aligned}$$

since div $\overrightarrow{\beta} = 0$. Hence,

$$\begin{aligned} \left\langle P\overrightarrow{\beta}, \overrightarrow{v}\right\rangle &= \lim_{k\to+\infty}\left\langle \nabla\left(\Delta^{-1}\text{div }\varphi_k\overrightarrow{\beta}\right), \overrightarrow{v}\right\rangle \\ &= \lim_{k\to+\infty}\left\langle \varphi_k\overrightarrow{\beta}, \nabla\left(\Delta^{-1}\text{div }\overrightarrow{v}\right)\right\rangle \\ &= -\lim_{k\to+\infty}\left\langle \nabla\varphi_k.\overrightarrow{\beta}, \Delta^{-1}\text{div }\overrightarrow{v}\right\rangle \\ &= 0. \end{aligned}$$

Consequently, $P\overrightarrow{\beta} = 0$. Moreover,

$$P\left(P\overrightarrow{h}\right) = P\left(\overrightarrow{\alpha}\right) = P\left(\overrightarrow{h} - \overrightarrow{\beta}\right) = P\overrightarrow{h}.$$

It follows that P and Q have the desired properties. This completes the proof of Theorem 1. □

References

[1] Adams, R. A., Sobolev spaces. Academic Press 1975.

[2] Evans, L.C., Partial Differential Equations. Providence, RI: Amer. Math. Soc. 1998.

[3] Gala, S., The form Boundedness criterion for the Laplacian operator. J. Math. Anal. Appl. 323 : 2 (2006), 1253-1263.

[4] Gala, S., Lahmar-Benbernou, A., Decomposition of the distribution on BMO space. Novi Sad J. Math. Vol. 36 No. 2 (2006), 1-12.

[5] Giaquinta, M., Modica, G., Souček, J., Cartesian currents in the Calculus of Variations I : Cartesian currents. Ergebnisse der Math. und ihre Grenzgebiete 37, Berlin: Springer 1998.

[6] Iwaniec, I., Martin, G., Riesz transforms and related singular integrals. J. Reine Angew. Math. 473 (1996), 25-57.

[7] Lieb, E. H., Loss, M., Analysis, Second Edition. Providence, RI: Amer. Math. Soc. 2001.

[8] Stein, E. M., Harmonic Analysis : Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton, New Jersey: Princeton Univ. Press 1993.

[9] Stein, E. M., Singular integrals and differentiability of functions. Princeton, New Jersey: Princeton Univ. Press 1970.

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