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SOME SIGNIFICANT RESULTS OF JANEZ UŠAN[1]

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Abstract. Several of the results of J. Ušan concerning n-ary quasigroups are presented. One open problem is posed.

The article is intended for nonspecialists interested in various aspects of J. Ušan's work in algebra. No completeness is implied.

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1. Introduction

Janez Ušan (1931-2006) was a prolific writer. A casual glance through his papers revels a multitude of topics: Quasigroups – binary, n-ary and infinitary, functional equations, topological n-groups, partial quasigroups, latin rectangles and other combinatorial structures, error correcting codes, geometric nets and their various generalizations, (m, n)-algebras of various kinds, lattices and their generalizations – both binary and n-ary, n-ary relations including generalized equivalences and order relations, generalized implication algebras, etc. Many papers were coauthored by other mathematicians, witnessing to his cooperative and friendly spirit. Most of all, it shows his boundless curiosity and unsatiable urge to do mathematics.

I have chosen to present here several of his results in the field of n-ary quasigroups. For the rest, I am out of my depth.

2. Quasigroups

One way to define a *quasigroup* is that it is a groupoid $(S; \cdot)$ in which for any $a, b \in S$ there are unique solutions x, y to the equations $a \cdot x = b$, $y \cdot a = b$. For more, consult standard references: V. D. Belousov [3], H. O. Pflugfelder [13], O. Chein, H. O. Pflugfelder, J. D. H. Smith [6].

A loop is a quasigroup with *unit* (e) such that

$$ex = xe = x \ . \tag{1}$$

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Groups are *associative* quasigroups, i.e. they satisfy:

(A) $$xy \cdot z = x \cdot yz$$

and they necessarily contain a unit.

As usual, whenever unambiguous, the terms like $x \cdot y$ and $f(x)$ are shortened to xy and fx respectively. For sequences we use the Čupona notation: x_i^j (i, j positive integers) is x_i if $i = j$; is a sequence $x_i, \ldots, x_j$ if $i < j$ and is an empty sequence if $i > j$. The symbol x_i^∞ represents an infinite sequence $x_i, x_{i+1}, \ldots$.

Most of the notions defined for binary quasigroups can be easily generalized to n–ary operations which are called n–quasigroups. An *n-quasigroup* is an *n-groupoid* $(S; A)$ $(A : S^n \longrightarrow S, n > 0)$ in which for every n-sequence a_1^n of elements from S, every $a \in S$ and every i $(1 \leq i \leq n)$, there is a unique solution x of the equation $A(a_1^{i-1}, x, a_{i+1}^n) = a$. For example, 1–quasigroups are just bijections.

For $n > 2$, an n–quasigroup have r–inverse operations $(r = 1, ..., n)$ instead of left and right division. However, not all generalizations are so straightforward. For example, n–groups, defined as associative n–quasigroups, need not have a unit. Also, there are several 'dual' operations (for $n > 2$) and therefore 'dual' of a 'dual' need not be the original operation. Many similar examples suggest that care should be taken when generalizing results to the n–ary case.

3. Generalized associativity on n-ary quasigroups

Most of the early papers of J. Ušan were in the field of quasigroup functional equations. This particular subject arose in early fifties of the previous century, but the cornerstone result which gave rise to vigorous research and an inflow of young mathematicians was published in 1960 in [1]. It is known as The Four Quasigroups Theorem.

Theorem 1. *(J. Aczél, V. D. Belousov, M. Hosszú [1]) If four quasigroup operations A, B, C, D (defined on the same nonempty set S) satisfy the generalized associativity equation:*

(GA) $$A(x, B(y, z)) = C(D(x, y), z)$$

then they are all isotopic to the same group. The general solution of the equation (GA) is given by:

$$\begin{cases} A(x, y) = A_1 x \cdot A_2 y, \\ B(x, y) = {A_2}^{-1}(A_2 B_1 x \cdot A_2 B_2 y), \\ C(x, y) = C_1 x \cdot C_2 y, \\ D(x, y) = {C_1}^{-1}(C_1 D_1 x \cdot C_1 D_2 y) \end{cases}$$

where $\cdot$ is an arbitrary group operation on S and $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are arbitrary permutations of S, such that $A_1 = C_1 D_1$,$A_2 B_1 = C_1 D_2$,$A_2 B_2 = C_2$. The group $\cdot$ is unique up to automorphism and the permutations up to translation by constants.

The Four Quasigroups Theorem was brought to attention of Yugoslav mathematicians by S. B. Prešić who published it (along with a more elegant proof) in an exercise book [15]. Two young beginners at the time, J. Ušan and S. Milić, took the opportunity and started working towards their PhD's. Their theses were defended in 1971 ([19] and [12] respectively), and they both became established experts in the field.

Ušan's first important paper was on the ternary analogue of The Four Quasigroups Theorem [17]. Soon (see [18] and [22]), the n–ary case was solved too. Namely, J. Ušan proved the following:

Theorem 2. *If n–ary $(n \geq 2)$ quasigroups $A_i, B_i (i = 1, ..., n)$ satisfy the system of generalized associativity equations:*

$$\text{(nGA)} \qquad A_1(B_1(x_1^n), x_{n+1}^{2n-1}) = A_i(x_1^{i-1}, B_i(x_i^{n+i-1}), x_{n+i}^{2n-1}) \ (i = 2, ..., n)$$

then all $A_i, B_i (i = 1, ..., n)$ are isotopic to an n–group G with unit. Moreover, there is a binary group $\cdot$ such that $G(x_1, ..., x_n) = x_1 \cdot ... \cdot x_n$.

The formulas of a general solution were also given.

Here we give Ušan's proof in the ternary case, actually a minor variant of the original, using a slightly modernized notation.

Proof. Let us write the equation (3GA) in this form:

$$\text{(3GA)} \qquad A(B(x, y, z), u, v) = C(x, D(y, z, u), v) = E(x, y, F(z, u, v)) \ .$$

This means that the formulas of a general solution of (3GA) should be of the form:

$$A(x, y, z) = G(A_1x, A_2y, A_3z) \ , \ B(x, y, z) = A_1^{-1}G(A_1B_1x, A_1B_2y, A_1B_3z)$$

$$C(x, y, z) = G(C_1x, C_2y, C_3z) \ , \ D(x, y, z) = C_2^{-1}G(C_2D_1x, C_2D_2y, C_2D_3z)$$

$$E(x, y, z) = G(E_1x, E_2y, E_3z) \ , \ F(x, y, z) = E_3^{-1}G(E_3F_1x, E_3F_2y, E_3F_3z)$$

for a suitable 3-group G and permutations $A_1, \ldots, F_3$.

Choose arbitrary elements a, b, c, d, f from S and define:

$$e = A(B(a, b, c), d, f) \ , \ A_{12}(x, y) = A(x, y, f) \ ,$$

$$A_1(x) = A(x, d, f) \ , \ A_2(x) = A(B(a, b, c), x, f) \ ,$$

$$A_3(x) = A(B(a, b, c), d, x) \ , \ B_{12}(x, y) = B(x, y, c) \ ,$$

$$B_1(x) = B(x, b, c) \ , B_2(x) = B(a, x, c) \ ,$$

$$B_3(x) = B(a, b, x) \ , \ C_{12}(x, y) = C(x, y, f) \ ,$$

$$C_1(x) = C(x, D(b, c, d), f) \ , \ C_2(x) = C(a, x, f) \ ,$$

$$C_3(x) = C(a, D(b, c, d), x) \ , \ D_{13}(x, y) = D(x, c, y) \ ,$$

$$D_1(x) = D(x, c, d) \ , \ D_2(x) = D(b, x, d) \ ,$$

$$D_3(x) = D(b, c, x)\ ,\ E_1(x) = E(x, b, F(c, d, f))\ ,$$
$$E_2(x) = E(a, x, F(c, d, f))\ ,\ E_3(x) = E(a, b, x)\ ,$$
$$F_1(x) = F(x, d, f)\ ,\ F_2(x) = F(c, x, f)\ ,\ F_3(x) = F(c, d, x)\ ,$$
$$G(x, y, z) = A(A_1^{-1}x, A_2^{-1}y, A_3^{-1}z)\ ,\ x \cdot y = G(x, y, e)\ .$$

We can express A in terms of G: $A(x, y, z) = G(A_1x, A_2y, A_3z)$.

If we replace y by b and z by c in (3GA), we get $A(B_1x, u, v) = C(x, D_3u, v)$. Also $A_1B_1 = C_1$, $A_2 = C_2D_3$ and $A_3 = C_3$. It follows that $C(x, D_3u, v) = G(A_1B_1x, A_2u, A_3v) = G(C_1x, C_2D_3u, C_3v)$ i.e. $C(x, y, z) = G(C_1x, C_2y, C_3z)$. Analogously, $E(x, y, z) = G(E_1x, E_2y, E_3z)$.

To express B in terms of G, we note that $A_1B_1 = E_1$, $A_1B_2 = E_2$, $A_1B_3 = E_3F_1$ and $A_1B(x, y, z) = E(x, y, F_1z) = G(E_1x, E_2y, E_3F_1z) = G(A_1B_1x, A_1B_2y, A_1B_3z)$ i.e. $B(x, y, z) = A_1^{-1}G(A_1B_1x, A_1B_2y, A_1B_3z)$.

Analogously, $D(x, y, z) = C_2^{-1}G(C_2D_1x, C_2D_2y, C_2D_3z)$ and $F(x, y, z) = E_3^{-1}G(E_3F_1x, E_3F_2y, E_3F_3z)$.

If we express all operations in (3GA) via G, we get:
$G(G(A_1B_1x, A_1B_2y, A_1B_3z), A_2u, A_3v) = G(C_1x, G(C_2D_1y, C_2D_2z, C_2D_3u), C_3v) = G(E_1x, E_2y, G(E_3F_1z, E_3F_2u, E_3F_3v))$. But $A_1B_1x = C_1x = E_1x$, so we can replace all these expressions by a new variable X. Analogously for other variables, so we finally get: $G(G(X, Y, Z), U, V) = G(X, G(Y, Z, U), V) = G(X, Y, G(Z, U, V))$ which tells us that G is a ternary group. Moreover, $G(e, e, x) = A(A_1^{-1}e, A_2^{-1}e, A_3^{-1}x) = A(A_1^{-1}A_1B(a, b, c), A_2^{-1}A_2d, A_3^{-1}x) = A(B(a, b, c), d, A_3^{-1}x) = A_3A_3^{-1}x = x$. Similarly, $G(x, e, e) = G(e, x, e) = x$ so e is a unit of G.

We also have $A_{12}(x, y) = A(x, y, f) = G(A_1x, A_2y, A_3f) = G(A_1x, A_2y, e) = A_1x \cdot A_2y$, and similarly $B_{12}(x, y) = A_1^{-1}(A_1B_1x \cdot A_1B_2y)$, $C_{12}(x, y) = C_1x \cdot C_2y$, $D_{13}(x, y) = C_2^{-1}(C_2D_1x \cdot C_2D_3y)$. If we use these relations in the equality $A_{12}(B_{12}(x, y), u) = C_{12}(x, D_{13}(y, u))$ we get $(A_1B_1x \cdot A_1B_2y) \cdot A_2u = C_1x \cdot (C_2D_1y \cdot C_2D_3u)$. Since $A_1B_1 = C_1$, $A_1B_2 = C_2D_1$ and $A_2 = C_2D_3$, the last relation expresses the associativity of $\cdot$, i.e. $\cdot$ is a group.

Also, from the equality $E(x, y, F_2u) = A_{12}(B_{12}(x, y), u)$ we get
$G(E_1x, E_2y, E_3F_2u) = A_1A_1^{-1}(A_1B_1x \cdot A_1B_2y) \cdot A_2u = (E_1x \cdot E_2y) \cdot E_3F_2u$ i.e. $G(x, y, z) = x \cdot y \cdot z$. □

One of my first results related to quasigroup functional equations was a new proof of the above Ušan's theorem. It was published in [9] as an example, but despite of its minor status within the paper, I still remember this result with pride. It was my entrance point to the field.

As for the Ušan's theorem itself, it bears this name in the literature, something every scientist strives for – to get the result which is so important to stay remembered by his name.

4. A theorem of Schauffler

Another interesting result by J. Ušan is the generalization of the theorem of R. Schauffler to the n-ary case.

Theorem 3. *(R. Schauffler [16]) Let S be a nonempty set and Ω the set of all (binary) quasigroups on S. If for all $A, B \in \Omega$ there are $C, D \in \Omega$ such that (GA) is true, then S has at most three elements.*

Ušan generalized it to the ternary case first ([20]), but fairly soon, the general case was solved too.

Theorem 4. *(J. Ušan, M. R. Žižović [23]) Let S be a nonempty set and Ω_n the set of all n-ary quasigroups on S. If for all i $(1 \leq i \leq n)$ the following is true: for all $A_i, B_i \in \Omega_n$ there are $A_j, B_j \in \Omega_n$ $(j = 1, ..., n)$ such that (nGA) is true; then S has at most three elements.*

Inspired by these results, I proved the following analogue of the Schauffler theorem:

Theorem 5. *(A. Krapež [10]) For any two groupoids A, B on S, there are groupoids C, D on S such that (GA) holds, iff S is infinite or has one element only.*

I never attempted to prove the groupoid analogue of the Ušan-Žižović theorem. So let this be an opportunity to state it as an unsolved problem:

Problem 4.1. *Let S be a nonempty set and Π_n the set of all n-ary groupoids on S. What are the necessary and sufficient conditions so that for all i $(1 \leq i \leq n)$ the following is true: for all $A_i, B_i \in \Pi_n$ there are $A_j, B_j \in \Pi_n$ $(j = 1, ..., n)$ such that (nGA) is true? In particular, does S has to be either infinite or a singleton?*

5. Functional equations on infinitary quasigroups

V. D. Belousov and Z. Stojaković [5] (see also [4]) proved that there is no nontrivial (i, j)-associative infinitary quasigroup, strengthening the result of Ž. Madevski, B. Trpenovski and Ǵ. Čupona [11] where it was proved that there are no nontrivial infinitary groups. This paper was presented at the important symposium 'Quasigroups and functional equations' held in September of 1974 in Belgrade and Novi Sad, in which J. Ušan also participated.

Definition 5.1. *An* infinitary operation *is a mapping $A : S^\omega \longrightarrow S$, where S^ω is the set of all ω-sequences a_1^∞ of elements from S. The structure $(S; A)$ is called an* ω-groupoid. *An ω-groupoid is an* ω-quasigroup *if for every ω-sequence a_1^∞ from S, every $b \in S$ and every positive integer i, equation $A(a_1^{i-1}, x, a_{i+1}^\infty) = b$ has a unique solution for x.*

An ω-groupoid $(S; A)$ is (i, j)-associative if

$$A(x_1^{i-1}, A(x_i^\infty), y_1^\infty) = A(x_1^{j-1}, A(x_j^\infty), y_1^\infty);$$

it is associative if it is (i, j)-associative for all positive integers $i \neq j$.

An associative ω-quasigroup is an ω-group. *An ω*-loop *is an ω-quasigroup with an element e such that $A(e,\dots,e,x,e,\dots)=x$ for every x and every place in which x can be put.*

V. D. Belousov and Z. Stojaković proved also the existence of ω-quasigroups and ω-loops of all (finite or infinite) orders. They also solved the equation of generalized (i,j)–associativity. In a later paper they solved the functional equation of generalized entropy (bisymmetry) on infinitary quasigroups.

Building on these results, J. Ušan and D. Žarkov in [24] solved the (finite) system of generalized $(1,j)$-associativities for $1<j\le n$. This is a system of equations $A_1(A_2(x_1^\infty),y_1^\infty)=A_{2j-1}(x_1^{j-1},A_{2j}(x_j^\infty,y_1^{j-1}),y_j^\infty)$, where j assumes all integer values such that $1<j\le n$. The solution is given in terms of an arbitrary group and two ω-quasigroups on S of which one is an ω-loop.

6. n-groups and their Hosszú–Gluskin algebras

A group $(S;\cdot)$ is usually described as an associative groupoid which possesses a unique distinguished element e called the unit, which is characterized by the neutrality with respect to multiplication, i.e. satisfying the axioms $e\cdot x=x$ and $x\cdot e=x$, and by the existence, for any given element x, of the unique inverse element y (depending on x), such that the product of the element and its inverse (in any order) is the unit. In other words, the axiom $\forall x\exists_1 y(x\cdot y=e\wedge y\cdot x=e)$ is also assumed.

A vigorous research at the turn of 19th century into 20th has shown that above axioms are not independent, but more importantly, that the groups so defined have some 'deficiencies'. Take the group of integers under addition and the subset of all numbers greater than, say 99. The subset is closed under addition, but is not itself a group. In modern parlance, such groups do not constitute a variety.

We are all familiar with the solution given by L. E. Dickson, even though the majority do not recognize the name. Dickson defined groups as algebras $(S;\cdot,^{-1},e)$ with three operations: binary multiplication $\cdot$, unary inverse operation $^{-1}$ and nullary operation (i.e. constant) e satisfying the axioms:

$$(x\cdot y)\cdot z=x\cdot(y\cdot z) \tag{A}$$

$$e\cdot x=x \tag{Lu}$$

$$x^{-1}\cdot x=e \tag{Li}$$

(or their left/right duals). The axioms are independent and the groups now do constitute a variety.

It was much later that T. Evans gave the characterization of groups as associative quasigroups, i.e. the algebras $(S;\cdot,\backslash,/)$ with three binary operations, satisfying axioms (A) and:

$$x\backslash xy=y \tag{Q1}$$

(Q2) $$x(x\backslash y) = y$$

(Q3) $$xy/y = x$$

(Q4) $$(x/y)y = x\ .$$

It is always useful to have different representations of a mathematical object. Witness the lattices, where we use to good advantage both our intuition with lattices defined as ordered sets and nice properties of lattices as algebras following from their being a variety.

In case of groups, Evans' representation incorporates groups within quasigroups as a variety with a special property. Moreover the variety of groups is a subvariety of the variety of quasigroups, which immediately gives us a lot of information about both varieties.

In case of n-ary quasigroups, M. Hosszú in [8] proved the following representation theorem:

Theorem 6. *$(S; A)$ is an n-group iff $A(x_1^n) = (\prod_{i=1}^n \varphi^{i-1}(x_i)) \cdot b$, where*

1. *$\cdot$ is a group operation*
2. *φ is an automorphism of $\cdot$*
3. *$\varphi(b) = b$*
4. *For all $x \in S$ $\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}$.*

Two years later L. M. Gluskin [7] independently proved the same result. Theorem 6 is usually called the Hosszú-Gluskin theorem.

In [25], J. Ušan defined a *Hosszú-Gluskin algebra* of order n ($n \geq 3$) (abbreviated nHG-algebra) as an algebra $(S; \cdot, \varphi, b)$ satisfying conditions (1)–(4) of Theorem 6.

An nHG-algebra $(S; \cdot, \varphi, b)$ for which an n-group $(S; A)$ can be represented as $A(x_1^n) = (\prod_{i=1}^n \varphi^{i-1}(x_i)) \cdot b$ is said to *correspond* to the n-group $(S; A)$. Theorem 6 now can be restated as:

Theorem 7. *For every n-group, there is a corresponding nHG-algebra.*

Ušan also proved:

Theorem 8. *Let nHG-algebras $(S; \cdot, \varphi, b)$ and $(S; \circ, \Phi, B)$ both correspond to the given n-group. Then there is an $a \in S$ such that:*

1. *$x \circ y = x \cdot a \cdot y$*
2. *$\Phi(x) = a^{-1} \cdot \varphi(x) \cdot \varphi(a)$*
3. *$B = (\prod_{i=1}^{n-1} \varphi^{i-1}(a^{-1})) \cdot b$.*

Corresponding nHG-algebras can be conveniently used to reveal the properties of their underlying n-groups. Ušan used this to prove the following remarkable theorems:

Theorem 9. (J. Ušan [28]) *An equivalence of S is a congruence of an n-group on S iff it is a congruence of a corresponding nHG-algebra.*

We say that the order $\leq$ on S is *compatible* with an algebra on S iff every operation of the algebra is monotone in every argument.

Theorem 10. (J. Ušan, M. Žižović [27]) *The order $\leq$ on S is compatible with an n-group on S iff it is compatible with a corresponding nHG-algebra.*

The same goes for topological structures:

Theorem 11. (J. Ušan [29]) *Let $(S;A)$ be an n-group and $\mathcal{T}$ a topology on S. Then $(S;A,\mathcal{T})$ is a topological n-group iff $(S;\cdot,\varphi,b,\mathcal{T})$ is a corresponding topological nHG-algebra i.e. iff*

- *$(S;\cdot,\mathcal{T})$ is a topological group*
- *φ is continuous in $\mathcal{T}$*

for a corresponding nHG-algebra $(S;\cdot,\varphi,b)$.

For details see the original papers or the book [30] where all previous results were collected.

The last theorem that I will mention here is Ušan's proof of a generalization of Dickson's theorem:

Theorem 12. (J. Ušan [26]) *An n-groupoid $(S;A)$ is an n-group iff there are operations $I : S^{n-1} \longrightarrow S$ and $E : S^{n-2} \longrightarrow S$ such that the algebra $(S;A,I,E)$ satisfies the axioms:*

1. $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$
2. $A(E(x_1^{n-2}), x_1^{n-1}) = x_{n-1}$
3. $A(I(x_1^{n-1}), x_1^{n-1}) = E(x_1^{n-2})$.

Moreover, the axioms 1. *to* 3. *are mutually independent.*

For $n = 2$ the above theorem reduces to Dickson's theorem. What is particularly interesting in this result is the inovative generalization of a notion of 'unit'. By choosing to use a function (E) and not an element, J. Ušan created an opportunity to apply the notion in other contexts, resulting in many new insights and connections. The book [30] is full of relevant examples which the interested reader might wish to consult.

References

[1] Aczél, J., Belousov, V. D., Hosszú, M., Generalized associativity and bisymmetry on quasigroups. Acta Math. Acad. Sci. Hungar. 11 (1960), 127-136.

[2] Andjelić, T. P. (ed.), Symposium en Quasigroupes et Equations Fonctionnelles (Belgrade-Novi Sad, 1974), Zbornik Rad. Mat. Inst. Beograd (N.S.) 1(9) (1976), Matematički institut, Beograd, (1976).

[3] Belousov, V. D., Foundations of the Theory of Quasigroups and Loops. Moscow: Nauka 1967, (in Russian) (MR 36 # 1569, Zbl 163:01801)

[4] Belousov, V. D., Stojaković, Z., On infinitary quasigroups. Publ. Inst. Math. (Beograd) (N.S.) 16 (30) (1973), 31–42.

[5] Belousov, V. D., Stojaković, Z., Infinitary quasigroups. In [2], 25–29.

[6] Chein, O., Pflugfelder, H. O., Smith, J. D. H., Quasigroups and Loops: Theory and Applications. Sigma Series in Pure Math. 9, Berlin: Heldermann Verlag 1990. (MR 93g:21033, Zbl 0719.20036.)

[7] Gluskin, L. M., Position operatives. Mat. Sb. t. 68 (110) No. 3 (1965), 444–472. (in Russian)

[8] Hosszú, M., On the explicit form of n-group operations. Publ. Math. Debrecen 10 (1963), 88–92.

[9] Krapež, A., On solving a system of balanced functional equations on quasigroups III. Publ. Inst. Math. (Belgrade) 26 (40) (1979), 145–156.

[10] Krapež, A., Generalized associativity on groupoids. Publ. Inst. Math. (Beograd) (N. S.) 28 (42) (1980), 105–112.

[11] Madevski, Ž., Trpenovski, B., Čupona, Ǵ., On infinitary associative operations. Bilten Društ. matem. i fiz. SRM tome XV (1964), 19–22. (in Macedonian)

[12] Milić, S., A Contribution to the Theory of Quasigroups. PhD thesis, University of Belgrade 1971. (in Serbian)

[13] Pflugfelder, H. O., Quasigroups and Loops: Introduction. Sigma Series in Pure Math. 8, Berlin: Heldermann Verlag 1990. (MR 93g:20132, Zbl 719.20036.)

[14] Pinus, A. G., Ponomaryov, K. N. (eds.), Algebra and Model Theory. Novosibirsk 1997.

[15] Prešić, S. B., Collection of Problems from Abstract Algebra, Faculty of Natural Sciences and Mathematics, University of Belgrade, Belgrade 1962. (in Serbian)

[16] Schauffler, R., Die Assoziativität im Ganzen besonders bei Quasigruppen. Math. Zeitschr. 67 no. 5 (1957), 428–435.

[17] Ušan, J., A generalization of a theorem of V. D. Belousov on four quasigroups to the ternary case. Bull. Soc. Phys. Macedoine 20 (1969), 13–17. (in Russian).

[18] Ušan, J., An n-ary analogue of the Belousov's theorem on four quasigroups and some corollaries of it. Bull. Soc. Phys. Macedoine 21 (1970), 5–17. (in Russian)

[19] Ušan, J., On a Class of Quasigoups, PhD thesis, University of Belgrade 1971. (in Serbian)

[20] Ušan, J., Systems of ternary quasigroups that are associative in the large. A ternary analogue of a theorem of Schauffler. Math. Balkanica 1 (1971), 273–281. (in Russian)

[21] Ušan, J., Globally associative systems of ternary quasigroups. Math. Balkanica 2 (1972), 270–287.

[22] Ušan, J., A certain system of functional equations of general n-ary associativity on an algebra of n-ary quasigroups. Math. Balkanica 2 (1972), 288–295. (in Russian),

[23] Ušan, J., Žižović, M., An n-ary analogue of a theorem of Schauffler. Publ. Inst. Math. (Beograd) (N.S.) 19 (33) (1975), 167–172. (in Russian)

[24] Ušan, J., Žarkov, D., A system of functional equations of general associativity on the algebra of infinitary quasigroups. Zb. Rad. Prirod. Mat. Fak. Univ. u Novom Sadu 7 (1977), 37–44. (in Russian)

[25] Ušan, J., On Hosszú-Gluskin algebras corresponding to the same n-group. Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. 25 no. 1 (1995), 101–119.

[26] Ušan, J., n-groups as variety of type $< n, n-1, n-2 >$. In [14], 182–208.

[27] Ušan, J., Žižović, M., On ordered n-groups. Quasigroups and related systems 4 (1997), 77–87.

[28] Ušan, J., Congruences of n-group and of associated Hosszú-Gluskin algebras. Novi Sad J. Math. 28 (1998), 91–108.

[29] Ušan, J., On topological n-groups. Math. Moravica 2 (1998), 149–159.

[30] Ušan, J., n-groups in the light of the neutral operations. Math. Moravica, Special vol., University of Kragujevac, Technical Faculty of Čačak 2003.

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