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AN INTEGRAL UNIVALENT OPERATOR OF THE CLASS $S(p)$ AND T_2

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Abstract. We present a few conditions of univalency for the operator $F_{\alpha,n}$ on the classes of the univalent functions $S(p)$ and T_2. These are actually generalizations(extensions) of certain results published in the paper [1].

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1. Introduction

Let A be a class of the analytic functions $\left\{f : f(z) = z + a_2 z^2 + ...\right\}, z \in U$, where U is the unit disk, $U = \{z : |z| < 1\}$. By S we denote the subclass of A consisting of function univalent in the unit disk.

Let p be a real number with the property $0 < p \leq 2$. We define the class $S(p)$ as the class of the functions $f \in A$ that satisfy the conditions $f(z) \neq 0$ and $\left|(z/f(z))''\right| \leq p, z \in U$. Also, if $f \in S(p)$ then the following property is true $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq p\,|z|^2, z \in U$. This relation was proved in [4].

We denote by T_2 a class of the univalent functions that satisfy the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, z \in U$, and also have the property $f''(0) = 0$. These functions have the form $f(z) = z + a_3 z^3 + a_4 z^4 + ...$ For $0 < \mu < 1$ we define a subclass of T_2 containing the functions that satisfy the property $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \mu < 1, z \in U$. We denote that class by $T_{\mu,2}$.

The Schwarz Lemma. *Let the analytic function g be regular in the unit disk U and $g(0) = 0$. If $|g(z)| \leq 1, \forall z \in U$, then*

$$|g(z)| \leq |z|, \forall z \in U \tag{1}$$

and the equality holds only if $g(z) = \varepsilon z$, where $|\varepsilon| = 1$.

Theorem 1. **([2])** *Let $\alpha \in \mathbf{C}$,* Re $\alpha > 0$ *and $f \in A$. If*

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$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \forall z \in U \tag{2}$$

then $\forall \beta \in \mathbf{C}, \mathrm{Re}\ \beta \geq \mathrm{Re}\ \alpha$, *the function*

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)\, dt\right]^{1/\beta} \tag{3}$$

is univalent.

In the paper [3] Pescar proved the following result:

Theorem 2. ([3]) *Assume that* $g \in A$ *satisfies the condition* $\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| < 1, z \in U$, *and let* α *be a complex number with*

$$|\alpha - 1| \leq \frac{\mathrm{Re}\,\alpha}{3}. \tag{4}$$

If

$$|g(z)| \leq 1, \forall z \in U \tag{5}$$

then the function

$$G_\alpha(z) = \left(\alpha \int_0^z g^{(\alpha-1)}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \tag{6}$$

is univalent.

In the sequel by $\mathbb{N}^*$ we denote the set of strictly positive integers.

Theorem 3. ([1]) *Let* $g_i \in T_2,\ g_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., i \in \{1, ..., n\}, n \in N^*$, *so that it satisfies the properties*

$$\left|\frac{z^2 g_i'(z)}{g_i^2(z)} - 1\right| < 1, \forall z \in U, \forall i \in \{1, ..., n\}. \tag{7}$$

If $|g_i(z)| \leq 1,\ \forall z \in U,\ \forall i \in \{1, ..., n\}$, *then for every complex number* α, *such that*

$$\mathrm{Re}\ \alpha \geq 1,\ and\ |\alpha - 1| \leq \frac{\mathrm{Re}\ \alpha}{3n} \tag{8}$$

the function

$$F_{\alpha,n}(z) = \left((n(\alpha-1)+1)\int_0^z g_1^{\alpha-1}(t)\ldots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}} \tag{9}$$

is univalent.

Theorem 4. ([1]) *Let $g_i \in T_{2,\mu}$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., i \in \{1,...,n\}, n \in N^*$. Also let $\alpha \in \mathbf{C}$, be such that*

$$|\alpha - 1| \le \frac{\mathrm{Re}\alpha}{n(\mu+2)}, \ \mathrm{Re}\ \alpha \ge 1. \tag{10}$$

If $|g_i(z)| \le 1, \forall z \in U, i \in \{1,...,n\}$ then the function

$$F_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z g_1^{\alpha-1}(t) \dots g_n^{\alpha-1}(t)\,\mathrm{d}t \right)^{\frac{1}{n(\alpha-1)+1}} \in S. \tag{11}$$

Theorem 5. ([1]) *Let $g_i \in S(p), 0 < p < 2$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + ...,\ i \in \{1,...,n\}, n \in N^*$. Let $\alpha \in \mathbf{C}$ such that*

$$|\alpha - 1| \le \frac{\mathrm{Re}\alpha}{n(p+2)}, \mathrm{Re}\ \alpha \ge 1. \tag{12}$$

If $|g_i(z)| \le 1, \forall z \in U, i \in \{1,...,n\}$ then the function

$$F_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z g_1^{\alpha-1}(t) \dots g_n^{\alpha-1}(t)\,\mathrm{d}t \right)^{\frac{1}{n(\alpha-1)+1}} \in S. \tag{13}$$

2. Main results

Theorem 6. *Let $M \ge 1$, $g_i \in T_2$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., i \in \{1,...,n\}, n \in N^*$, so that it satisfies the properties*

$$\left| \frac{z^2 g_i'(z)}{g_i^2(z)} - 1 \right| < 1, \forall z \in U, \forall i \in \{1,...,n\}. \tag{14}$$

If $|g_i(z)| \le M,\ \forall z \in U,\ \forall i \in \{1,...,n\}$, then for every complex number α, such that

$$\mathrm{Re}\ \alpha \ge 1,\ \textit{and}\ \ |\alpha - 1| \le \frac{\mathrm{Re}\,\alpha}{(2M+1)n}. \tag{15}$$

the function

$$F_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z g_1^{\alpha-1}(t) \dots g_n^{\alpha-1}(t)\,\mathrm{d}t \right)^{\frac{1}{n(\alpha-1)+1}} \tag{16}$$

is univalent.

Proof. We can rewrite $F_{\alpha,n}$ in the form

(17)
$$F_{\alpha,n}(z)=\left((n(\alpha-1)+1)\int_0^z t^{n(\alpha-1)}\left(\frac{g_1(t)}{t}\right)^{\alpha-1}\cdots\left(\frac{g_n(t)}{t}\right)^{\alpha-1}\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}}.$$

Let us consider the function

$$f(z)=\int_0^z\left(\frac{g_1(t)}{t}\right)^{\alpha-1}\cdots\left(\frac{g_n(t)}{t}\right)^{\alpha-1}\mathrm{d}t. \tag{18}$$

The function f is regular in U, and from (18) we obtain

$$f'(z)=\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1} \tag{19}$$

and

$$f''(z)=E_1 f'(z)\frac{z}{g_1(z)}+\cdots+E_n f'(z)\frac{z}{g_1(z)} \tag{20}$$

where, $E_k=(\alpha-1)\frac{zg_k'(z)-g_k(z)}{z^2},\forall k\in\{1,...,n\}$.

Next, we calculate the expresion $\frac{zf''}{f'}$.

$$\frac{zf''(z)}{f'(z)}=(\alpha-1)\left(\frac{zg_1'(z)}{g_1(z)}-1\right)+\cdots+(\alpha-1)\left(\frac{zg_n'(z)}{g_n(z)}-1\right). \tag{21}$$

The modulus $|zf''(z)/f'(z)|$ can then be evaluated as

$$\left|\frac{zf''(z)}{f'(z)}\right|\le|\alpha-1|\left|\frac{zg_1'(z)}{g_1(z)}-1\right|+\cdots+|\alpha-1|\left|\frac{zg_n'(z)}{g_n(z)}-1\right|. \tag{22}$$

Because

$$\left|\frac{zg_k'(z)}{g_k(z)}-1\right|\le\left|\frac{z^2g_k'(z)}{g_k^2(z)}\right|\left|\frac{g_k(z)}{z}\right|+1,\forall k\in\{1,...,n\} \tag{23}$$

then multiplying (22) by $\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}>0$, we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\le AC_1+\cdots+AC_n, \tag{24}$$

where $A = \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha-1|$, and $C_k = \left(\left|\frac{z^2 g_k'(z)}{g_k^2(z)}\right|\left|\frac{g_k(z)}{z}\right|+1\right), \forall k \in \{1,...,n\}$.

By the assumption $|g_i(z)| \leq M, i \in \{1,...,n\}$ then applying Schwarz Lemma we obtain that $\left|\frac{g_i(z)}{z}\right| \leq M, i \in \{1,...,n\}$, so that

$C_k \leq \left(\left|\frac{z^2 g_k'(z)}{g_k^2(z)}\right| M+1\right) = \left(\left|\frac{z^2 g_k'(z)}{g_k^2(z)}-1\right| M+M+1\right) = D_k, \forall k \in \{1,...,n\}$.

Then we have:

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq AD_1+\cdots+AD_n. \tag{25}$$

Because $g_i \in T_2$, we have $\left|\frac{z^2 g_i'(z)}{g_i^2(z)}-1\right| < 1$, $\forall i \in \{1,...,n\}$, then $D_i \leq 2M+1, i \in \{1,...,n\}$.

The relation (25) and the estimation $A < \frac{|\alpha-1|}{\operatorname{Re}\alpha}$ yield

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq (2M+1)nA \leq \frac{(2M+1)n|\alpha-1|}{\operatorname{Re}\alpha}. \tag{26}$$

By the assumption $|\alpha-1| \leq \frac{\operatorname{Re}\alpha}{(2M+1)n}$ then we arrive at:

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{27}$$

for all $z \in U$. Now, Theorem 1 and (27) imply that the function $F_{\alpha,n}$ is in the class S. □

Remark. Theorem 6 is a generalization of Theorem 3, since for $M=1$ in Theorem 6 we obtain Theorem 3.

Theorem 7. *Let $M \geq 1, g_i \in T_{2,\mu}$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., i \in \{1,...,n\}$, $n \in N^*$. Let $\alpha \in \mathbf{C}$, be such that*

$$|\alpha-1| \leq \frac{\operatorname{Re}\alpha}{n(M\mu+M+1)},\ \operatorname{Re}\alpha \geq 1. \tag{28}$$

If $|g_i(z)| \leq M, \forall z \in U, i \in \{1,...,n\}$ then the function

$$F_{\alpha,n}(z) = \left((n(\alpha-1)+1)\int_0^z g_1^{\alpha-1}(t)\ldots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}} \in S. \tag{29}$$

Proof. Reasoning along the same line as in the proof of Theorem 6 we obtain:

(30)
$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\left(1-|z|^{2\mathrm{Re}\ \alpha}\right)|\alpha-1|}{\mathrm{Re}\ \alpha}\sum_{i=1}^{n}\left(\left|\frac{z^2g_i'(z)}{g_i^2(z)}-1\right|M+M+1\right).$$

But $g_i \in T_{2,\mu}, \forall i \in \{1,...,n\}$, which implies that $\left|\frac{z^2g_i'(z)}{g_i^2(z)}-1\right| < \mu, \forall z \in U, i \in \{1,...,n\}$, then we have

$$\text{(31)}\qquad \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le |\alpha-1|\frac{n(M\mu+M+1)}{\mathrm{Re}\alpha},\ \forall z \in U.$$

Applying the relation (28) we obtain that $\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \forall z \in U$, so that according to Theorem 1 the function $F_{\alpha,n}$ is univalent. □

Remark. Theorem 7 is a generalization of Theorem 4, since for $M = 1$ in Theorem 7 we obtain Theorem 4.

Theorem 8. *Let $M \ge 1, g_i \in S(p), 0 < p < 2$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., \forall i \in \{1,...,n\}, n \in N^*$. Let $\alpha \in \mathbf{C}$, be such that* Re $\alpha \ge 1$, *and*

$$\text{(32)}\qquad |\alpha-1| \le \frac{\mathrm{Re}\alpha}{n(pM+M+1)}.$$

If $|g_i(z)| \le M, \forall z \in U, i \in \{1,...,n\}$ then the function

$$\text{(33)}\qquad F_{\alpha,n}(z) = \left((n(\alpha-1)+1)\int_0^z g_1^{\alpha-1}(t)\dots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}} \in S.$$

Proof. Reasoning along the line as in the proof of Theorem 6 we obtain:
(34)
$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\left(1-|z|^{2\mathrm{Re}\ \alpha}\right)|\alpha-1|}{\mathrm{Re}\ \alpha}\sum_{i=1}^{n}\left(\left|\frac{z^2g_i'(z)}{g_i^2(z)}-1\right|M+M+1\right).$$

Since $g_i \in S(p), i \in \{1,...,n\}$ then

$$\text{(35)}\qquad \left|\frac{z^2g_i'(z)}{g_i^2(z)}-1\right| \le p|z|^2,\ \forall z \in U.$$

In view of (34) and (35) we obtain:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{|\alpha-1|\left(1-|z|^{2\mathrm{Re}\alpha}\right)}{\mathrm{Re}\alpha}\sum_{i=1}^{n}\left(p|z|^2M+M+1\right)$$

$$\leq \frac{|\alpha-1|\,n\,(pM+M+1)}{\mathrm{Re}\alpha},\ \forall z\in U, \tag{36}$$

$$\leq 1.$$

In the last relation we apply the inequality (32). On the basis of Theorem 1 we conclude that the function $F_{\alpha,n}$ is univalent. □

Remark. Theorem 8 is a generalization of the Theorem 5, since for $M=1$ in Theorem 8 we obtain the Theorem 5.

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