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COMPACT LINEAR OPERATORS, A SURVEY

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Abstract. In this paper we approach the Compact Linear Operators Theory by the methods of Nonstandard Analysis. We present new proofs of several known results and some new results as well.

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1. Introduction

Nonstandard Analysis (NSA) was invented by Abraham Robinson in the late 1960s, and among other things, provided an answer to the old question: what is an infinitesimal number? He showed that we can embed the ordered field of real numbers as an ordered subfield of a structure which, beside being a totally ordered field, contains other numbers such as infinitesimal numbers, infinitely large numbers, etc. Plus, all the valid sentences in the real structure continue to be valid in the hyperreal structure. We give here a brief discussion of the matter (for more details on the subject the reader is referred to [2], [8] or [10]).

Let S be a set sufficiently large to contain the elements we work on: reals numbers, vectors, functions, sets, sets of sets, etc. We denote by ${}^*S$ its nonstandard extension.

Unless said otherwise, E and F are two arbitrary normed spaces. We will begin by presenting some basic notions and theorems needed for our work.

Definition 1. *Let x and y be two elements of ${}^*E$. We say that*

1. *x is **infinitesimal** if for all positive standard $r \in \mathbb{R}$ holds $|x| < r$ and we write $x \approx 0$;*

2. *x is **finite** if $|x| < r$ for some positive standard $r \in \mathbb{R}$; the set of the finite elements of ${}^*E$ will be denoted by $fin({}^*E)$;*

3. *x is **infinite** if it is not finite;*

4. *x and y are **infinitely close** if $|x - y|$ is infinitesimal and we denote $x \approx y$;*

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5. *x is **near-standard** if there exists a standard y with $x \approx y$, and we denote $y = st(x)$; the set of the near-standard elements of ${}^*E$ will be denoted by $ns({}^*E)$;*

6. *x is **pre-near-standard** if for every positive standard $\epsilon \in \mathbb{R}$ there exists a standard y with $|x - y| < \epsilon$; the set of the pre-near-standard elements of ${}^*E$ will be denoted by $pns({}^*E)$;*

7. *The **monad** of an element $x \in {}^*E$ is the set*

$$\mu(x) := \{y \in {}^*E |\ |y - x| \approx 0\}. \tag{1}$$

We have presented the previous concepts for normed spaces because they are more elegant and intuitive, but some of them can be presented for metric spaces (replacing $|x - y|$ by $d(x, y)$) or even topological spaces. For example, if $(X, \mathcal{T})$ is a topological space, for $x \in X$, we define

$$\mu(x) = \bigcap_{U \in \mathcal{T}_x} {}^*U. \tag{2}$$

A point $y \in {}^*X$ is infinitely close $x \in X$ if $y \in \mu(x)$. The set of near-standard points is the set

$$ns({}^*X) = \bigcup_{x \in X} \mu(x). \tag{3}$$

Theorem 2. *[2] Let $(X, \mathcal{T})$ be a topological space and $U \subseteq X$ a subset. Then*

1. *X is a Hausdorff space iff monads of distinct points in X are disjoint;*

2. *U is open iff $\mu(x) \subseteq {}^*U$, for all $x \in U$;*

3. *U is closed iff $\mu(x) \cap {}^*U = \emptyset$ for each x in the complement of U; the closure of U consists of those $x \in X$ for which $\mu(x) \cap {}^*U \neq \emptyset$;*

4. *x is an accumulation point of U iff $\mu(x)$ contains a point $y \in {}^*U$ different from x;*

5. *U is compact iff every $y \in {}^*U$ is infinitely close a standard point $x \in U$.*

Theorem 3. *[2] A metric space (X, d) is complete iff $pns({}^*X) = ns({}^*X)$.*

Obviously, for all metric spaces (X, d) holds $ns({}^*X) \subseteq pns({}^*X)$.

Theorem 4. *[2] If E is a normed space, then $ns({}^*E) \subseteq fin({}^*E)$. Moreover, E is finite dimensional iff $ns({}^*E) = fin({}^*E)$.*

A point here: if Φ is a formula in $\mathcal{L}_S$, the $*$-transformation ${}^*\Phi$ is obtained by replacing each constant symbol c in Φ by ${}^*c$. For example, the star transformation of the sentence

$$\forall x \in \mathbb{R} \ \ \forall \epsilon \in \mathbb{R}^+ \ \ \exists \delta \in \mathbb{R}^+ \ \ \forall y \in \mathbb{R} \ \ [|x-y| < \delta \Rightarrow |2x-2y| < \epsilon] \tag{4}$$

is the sentence

$$\forall x \in {}^*\mathbb{R} \ \ \forall \epsilon \in {}^*\mathbb{R}^+ \ \ \exists \delta \in {}^*\mathbb{R}^+ \ \ \forall y \in {}^*\mathbb{R} \ \ [|x-y| < \delta \Rightarrow |2x-2y| < \epsilon]. \tag{5}$$

One of the main results in Nonstandard Analysis is the *Transfer Principle*, which states that a bounded sentence Ψ is true in $\mathcal{L}_S$ iff ${}^*\Psi$ is true in $\mathcal{L}_{{}^*S}$. For example, we know that

$$\forall x, y \in \mathbb{R} \ \ \exists n \in \mathbb{N} \ \ x + y < n. \tag{6}$$

So, by the Transfer Principle, we have that

$$\forall x, y \in {}^*\mathbb{R} \ \ \exists n \in {}^*\mathbb{N} \ \ x + y < n. \tag{7}$$

One important tool in Nonstandard Analysis are the hyperfinite sets. They may contain infinite points but have all the properties possessed by finite sets.

Definition 5. *Let A be a set. We say that A if finite (resp. hyperfinite) with cardinality $n \in \mathbb{N}$ (resp. $n \in {}^*\mathbb{N}$) if there exists a bijection (resp. internal bijection) $f : \{1, \ldots, n\} \to A$.*

Any elementary mathematical result that holds for finite sets extends to a similar result for hyperfinite sets by the Transfer Principle. For example, every hyperfinite set of hyperreal numbers has a minimum and a maximum element.

The following theorem is also true.

Theorem 6. ***Discretization Principle*** *[10] For any set X there exists an hyperfinite set $\mathcal{H}$ such that*

$$X \subseteq \mathcal{H} \subseteq {}^*X. \tag{8}$$

Furthermore, X is infinite if and only if both inclusions are strict.

2. Compact Operators

We will present several known results related to compact linear operators. All the proofs that we will present here are original and are, in our opinion, easier to understand. When a result is already known we give references where the reader can find a classical proof of it.

In the cases where E or F are finite dimensional spaces, the classical proofs use the Riesz's Lemma that states that a normed space is finite dimensional iff the closed ball $\{x \mid |x| \leq 1\}$ is compact. In our proofs, we will use instead Theorem 4.

We recall that a subset U of a topological space $(X, \mathcal{T})$ is called relatively compact if $\overline{U}$ is a compact set.

Theorem 7. *The set U is relatively compact iff ${}^*\overline{U} \subseteq ns({}^*X)$.*

Proof. The proof is an immediate application of Theorem 2, conditions 3 and 5. □

Theorem 8. *If (X,d) is a metric space and $U \subseteq X$, then ${}^*\overline{U} \subseteq ns({}^*X)$ iff ${}^*U \subseteq ns({}^*X)$.*

Proof. Since ${}^*U \subseteq {}^*\overline{U}$, one of the implications is obvious. Let us prove the other one. To begin with, we may write

$$\overline{U} = \{x \in X \mid \forall n \in \mathbb{N}\, \exists y \in U \;\; d(x,y) < 1/n\}\,, \tag{9}$$

and as a result

$${}^*\overline{U} = \{x \in {}^*X \mid \forall n \in {}^*\mathbb{N}\, \exists y \in {}^*U \;\; d(x,y) < 1/n\}\,. \tag{10}$$

Let us fix $x \in {}^*\overline{U}$ and $n \in {}^*\mathbb{N}_\infty$ (the set of positive infinite hyperintegers). Then there exists $y \in {}^*U$ satisfying the condition $x \approx y$. But ${}^*U \subseteq ns({}^*X)$ and so $x \in ns({}^*X)$. □

For topological spaces the previous theorem is false as can be seen in the next example. Let $X = \mathbb{R}$ with the topology $\mathcal{T}$ where the open sets are $\mathbb{R}$, $\emptyset$ and the intervals $]a,\infty[$, for $a \in \mathbb{R}$. If we take $U = \{1\}$, we have that ${}^*U \subseteq ns({}^*\mathbb{R})$ (all standard elements are near-standard) but ${}^*\overline{U} = {}^*]-\infty,1]$, which contains points that are not near-standard.

From the previous theorems we have the following corollary:

Corollary 9. *If (X,d) is a metric space and $U \subseteq X$ a subset, then U is relatively compact iff ${}^*U \subseteq ns({}^*X)$.*

Again, for topological spaces this result is false.

Let (X,d) be a metric space and $U \subseteq X$. U is called totally bounded if

$$\forall r > 0\, \exists x_1,\ldots,x_n \in X \quad U \subseteq \bigcup_{i=1}^{n} B_r(x_i). \tag{11}$$

We will now prove some known results using nonstandard techniques.

Theorem 10. *[4] Let (X,d) be a metric space and $U \subseteq X$. Then:*

1. *If U is relatively compact, then U is totally bounded.*
2. *If U is totally bounded and X is a complete space, then U is relatively compact.*

Proof.

1. Assume that

$$\exists r > 0 \forall n \in \mathbb{N}\, \forall x_1, \ldots, x_n \in X\, \exists y \in U \;\; y \notin \bigcup_{i=1}^{n} B_r(x_i), \tag{12}$$

   i.e.,

$$\exists r > 0\, \forall n \in \mathbb{N}\, \forall x_1, \ldots, x_n \in X\, \exists y \in U\, \forall i \in \{1, \ldots, n\} \;\; d(x_i, y) \geq r. \tag{13}$$

   Begin by fixing such $r \in \mathbb{R}^+$. By transfer,

$$\forall n \in {}^*\mathbb{N}\, \forall x_1, \ldots, x_n \in {}^*X\, \exists y \in {}^*U\, \forall i \in \{1, \ldots, n\} \;\; d(x_i, y) \geq r. \tag{14}$$

   Let $\mathcal{H} := \{y_1, \ldots y_N\}$ be an hyperfinite set satisfying $X \subseteq \mathcal{H} \subseteq {}^*X$. Therefore there exists $y \in {}^*U$ with $d(y_i, y) \geq r$, for all $y_i \in \mathcal{H}$. Since $X \subseteq \mathcal{H}$, we have that $y \notin ns({}^*X)$.

2. If X is a complete set, if follows from Theorem 3 that $ns({}^*X) = pns({}^*X)$. Let us take $x \in {}^*U$ and a standard $r > 0$. The set U is totally bounded, and so $U \subseteq \cup_{i=1}^{n} B_r(x_i)$, for some $x_i \in X$ and $n \in \mathbb{N}$. Thus

$${}^*U \subseteq {}^*\left(\bigcup_{i=1}^{n} B_r(x_i)\right) = \bigcup_{i=1}^{n} {}^*B_r(x_i) \tag{15}$$

   and $x \in {}^*B_r(x_i)$, for some $x_i \in X$ which proves that $x \in pns({}^*X)$, ending the proof.

□

Definition 11. *Let $T : E \to F$ be a linear operator. We say that T is* ***compact*** *if for all bounded subset M of E, $T(M)$ is relatively compact.*

We will now present a new proof of a theorem due to Robinson:

Theorem 12. *[8] A linear operator $T : E \to F$ is compact iff $T(fin({}^*E)) \subseteq ns({}^*F)$.*

Proof. Suppose that $x \in fin({}^*E)$; therefore $x \in {}^*B_{st(|x|)+1}(0)$. Since T is compact, $T(B_{st(|x|)+1}(0))$ is relatively compact and consequently, by Corollary 9, we obtain the desired.

To prove the reverse, let M be a bounded set. We will prove that ${}^*T(M) \subseteq ns({}^*F)$. Fix $x \in {}^*T(M)$ and let $x = T(y)$, for some $y \in {}^*M$. By the hypothesis $T(y) \in ns({}^*F)$ and so $x \in ns({}^*F)$. □

For our next theorem we will need the following result:

Given a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d), then the sequence has a convergent subsequence iff there exist $L \in X$ and $n \in {}^*\mathbb{N}_\infty$ for which $x_n \approx L$ (see [2]).

Theorem 13. *[4] Let $T : E \to F$ be a linear operator. If T is compact then for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in E, there exists a convergent subsequence of $(Tx_n)_{n\in\mathbb{N}}$ in F.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then there is a standard positive real number R satisfying $|x_n| < R$, for all positive integers n. Hence, by the Transfer Principle, for all $n \in {}^*\mathbb{N}$, x_n is finite and consequently $T(x_n) \in ns({}^*F)$. Fixing any $n \in {}^*\mathbb{N}_\infty$ and defining $L := st(T(x_n))$, we get the desired. □

Theorem 14. *[5] Every compact linear operator is bounded.*

Proof. Let S^1 denote the unit sphere in E and $\mathcal{H} := \{x_1, \ldots, x_N\}$ an hyperfinite set with $S^1 \subseteq \mathcal{H} \subseteq {}^*S^1$. For each $x_i \in \mathcal{H}$ we have that $|x_i| = 1$ and so $T(x_i) \in ns({}^*F) \subseteq fin({}^*F)$. Define $k_i := |T(x_i)| \in fin({}^*\mathbb{R})$ and let $K := \max\{k_i \mid i = 1, \ldots, N\} \in fin({}^*\mathbb{R})$. Thus $|T(S^1)| < st(K) + 1$. □

Theorem 15. *[6] The set of compact operators $\{T : E \to F\}$ is closed if F is a Banach space.*

Proof. Fix any $x \in fin({}^*E)$. Since F is complete, it follows that

$$T(x) \in ns({}^*F) \Leftrightarrow T(x) \in pns({}^*F). \tag{16}$$

Fix now $0 < \epsilon \in \mathbb{R}$. Moreover, $|T_n - T| \to 0$, and so there exists $n \in \mathbb{N}$ such that

$$\forall y \in E \ \ |T_n(y) - T(y)| < \frac{\epsilon}{2}. \tag{17}$$

By the Transfer Principle, $|T_n(x) - T(x)| < \epsilon/2$. Since $T_n(x) \in ns({}^*F)$, there exists $L := st(T_n(x))$ and thus $|L - T(x)| < \epsilon$. □

Theorem 16. *[3] The compact operators $T : E \to F$ form a linear subspace of $L(E, F)$.*

Proof. Straightforward. □

Theorem 17. *[3] If T is a bounded linear operator with finite dimensional range $T(E)$, then T is compact.*

Proof. Fix $x \in fin({}^*E)$; then $T(x) \in fin({}^*T(E)) = ns({}^*T(E)) \subseteq ns({}^*F)$. □

Theorem 18. *[1] If E is finite dimensional, then every linear operator $T : E \to F$ is compact.*

Proof. For $x \in fin(^*E) = ns(^*E)$, since T is a continuous map, it follows that $T(x) \approx T(st(x)) \in F$ and so $T(x) \in ns(^*F)$. □

Theorem 19. *[6] If $T : E \to F$ is an onto compact operator, invertible with continuous inverse, then $dim(E) < \infty$.*

Proof. Fix $x \in fin(^*E)$. So there is $a \in F$ with $T(x) \approx a$. Since T is onto and T^{-1} is continuous, it follows that $x \approx T^{-1}(a)$. Therefore $fin(^*E) = ns(^*E)$. □

Theorem 20. *[3] The identity operator $I : E \to E$ is compact iff E is finite dimensional.*

Proof. Simply note that

$$dim(E) < \infty \Leftrightarrow ns(^*E) = fin(^*E). \tag{18}$$

□

Theorem 21. *[1] Let $T : E \to F$ be a compact operator and $A : E_1 \to E$, $B : F \to F_1$ two bounded linear operators. Then BTA is also compact.*

Proof. Let $x \in fin(^*E_1)$; then $A(x) \in fin(^*E)$. Since T is compact, $TA(x) \in ns(^*F)$ and therefore $BTA(x) \in ns(^*F_1)$ because B is continuous. □

3. Weak Convergence and Weak Cauchy Sequences

Definition 22. *Let $(x_n)_n$ be a sequence in E. We say that*

1. *$(x_n)_n$ is **(strongly) convergent** to x if $|x_n - x| \to 0$ as $n \to \infty$ and we denote $x_n \to x$.*
2. *$(x_n)_n$ is **weakly convergent** to x if for all bounded linear functionals $f : E \to \mathbb{R}$, the sequence $(f(x_n))_n$ is convergent to $f(x)$ and we write $x_n \rightharpoonup x$.*
3. *$(x_n)_n$ is a **weak Cauchy sequence** if for all bounded linear functionals $f : E \to \mathbb{R}$, the sequence $(f(x_n))_n$ is a Cauchy sequence in $\mathbb{R}$.*

Theorem 23. *[2] Given a sequence $(x_n)_n$ in E, we have that $(x_n)_n$ converges to x iff $x_n \approx x$ for all $n \in {}^*\mathbb{N}_\infty$ and $(x_n)_n$ is a Cauchy sequence iff $x_n \approx x_m$ for all $n, m \in {}^*\mathbb{N}_\infty$.*

Let us begin by proving the following theorem:

Theorem 24. *Suppose that $x_n \rightharpoonup x$. Then*

$$\forall n \in {}^*\mathbb{N}_\infty \;\; x_n \in ns({}^*E) \Rightarrow x_n \approx x. \tag{19}$$

Proof. Let $f : E \to \mathbb{R}$ be a bounded linear functional and $n \in {}^*\mathbb{N}_\infty$ with $x_n \in ns({}^*E)$. Then

$$f(x_n) \approx f(x) \Rightarrow f(st(x_n) - x) = 0. \tag{20}$$

Since it is true for all the functionals, it implies that $st(x_n) - x = 0$, *i.e.*, $x_n \approx x$. □

The reverse of this theorem is false, even if we assume that the sequence is bounded. For example, let $y = (1, 0, 1, 0, 1, \ldots) \in l_\infty$ and $Ty : l_1 \to \mathbb{R}$ defined by $Ty(x) = \sum_{n=1}^{\infty} y_n x_n$; then Ty is a bounded linear functional since $|Ty(x)| \leq |y|_\infty |x|_1$. Define the sequence $(x)_n$ in l_1

$$x_n(i) := \begin{cases} 0, & i \neq n \\ 1, & i = n \end{cases} \tag{21}$$

Note , if $n \in {}^*\mathbb{N}_\infty$ then $x_n \notin ns({}^*l_1)$. So this sequence satisfies the condition

$$\forall n \in {}^*\mathbb{N}_\infty \;\; x_n \in ns({}^*l_1) \Rightarrow x_n \approx x, \tag{22}$$

for any x, but the sequence $(T(x_n))_n$ does not converge to any real number.

Corollary 25. *[4] Let $(x_n)_n$ be a sequence in E. If $dim(E) < \infty$ then $x_n \rightharpoonup x$ iff $x_n \to x$.*

Proof. It is obvious that if $x_n \to x$ then $x_n \rightharpoonup x$. On the other hand, if $x_n \rightharpoonup x$ then the sequence $(x_n)_n$ is bounded (*cf.* [4]) and therefore $x_n \in fin({}^*E) = ns({}^*E)$, for all $n \in {}^*\mathbb{N}$. Thus $x_n \to x$ by Theorem 24. □

Theorem 26. *[7] Let T be a compact linear operator and $x_n \rightharpoonup x$ in E. Then $T(x_n) \to T(x)$ in F.*

Proof. We will begin proving that $T(x_n) \rightharpoonup T(x)$. Let $g : F \to \mathbb{R}$ be a bounded linear function on F and define $f := gT$; then f is also a bounded linear functional (on E). By the hypothesis, $f(x_n) \to f(x)$, *i.e.*, $g(Tx_n) \to g(Tx)$. Since g was any functional, $T(x_n) \rightharpoonup T(x)$. Lastly, since $x_n \rightharpoonup x$ then $(x_n)_n$ is bounded and so $T(x_n) \in ns({}^*F)$, for all $n \in {}^*\mathbb{N}$. By Theorem 24, $T(x_n) \approx T(x)$ for all $n \in {}^*\mathbb{N}_\infty$. □

Theorem 27. *Let $(x_n)_n$ be a weak Cauchy sequence in E. Then*

$$\forall n, m \in {}^*\mathbb{N}_\infty \;\; x_n, x_m \in ns({}^*E) \Rightarrow x_n \approx x_m. \tag{23}$$

Proof. Let $(x_n)_n$ be a weak Cauchy sequence, $n, m \in {}^*\mathbb{N}_\infty$ with $x_n, x_m \in ns({}^*E)$ and $f : E \to \mathbb{R}$ a bounded linear functional. Then

$$f(x_n) \approx f(x_m) \Rightarrow f(st(x_n) - st(x_m)) = 0 \Rightarrow x_n \approx x_m. \tag{24}$$

□

Corollary 28. *If $dim(E) < \infty$, then $(x_n)_n$ is a weak Cauchy sequence iff $(x_n)_n$ is a Cauchy sequence.*

Proof. Observe that if $(x_n)_n$ is a weak Cauchy sequence, then it is bounded (*cf.* [4]) and so, for all $n \in {}^*\mathbb{N}_\infty$, $x_n, x_m \in fin({}^*E) = ns({}^*E)$. □

Theorem 29. *[5] Let T be a compact operator and $(x_n)_n$ a weak Cauchy sequence in E. Then $(T(x_n))_n$ is a Cauchy sequence in F.*

Proof. Let $g : F \to \mathbb{R}$ be a bounded linear funcional on F and $f := gT$. Therefore, for $n, m \in {}^*\mathbb{N}_\infty$, $f(x_n) \approx f(x_m)$. Thus

$$g(T(x_n) - T(x_m)) \approx 0 \Leftrightarrow g(st(T(x_n)) - st(T(x_m))) = 0 \Leftrightarrow \tag{25}$$

$$st(T(x_n)) - st(T(x_m)) = 0. \tag{26}$$

So $T(x_n) \approx T(x_m)$, proving that $(T(x_n))_n$ is a Cauchy sequence. □

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