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ALGEBRAS WITH RESTRICTED CARDINALITIES OF CONGRUENCE CLASSES[1]

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Abstract. We show that for a finite algebra $\mathcal{A}$ there exists a function with values in natural numbers assigning to every element of $\mathcal{A}$ and every congruence of $\mathcal{A}$ with a given kernel a number of elements in the corresponding congruence class if and only if $\mathcal{A}$ is weakly regular. This is not true for infinite algebras.

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Let $\mathcal{A} = (A; F)$ be an algebra and $Con\mathcal{A}$ its congruence lattice. Recall that $\mathcal{A}$ is **congruence uniform** (see e.g. [1], [5]) if for each $\Theta \in Con\mathcal{A}$ and every $a, b \in A$, $card[a]_\Theta = card[b]_\Theta$. Examples of congruence uniform algebras are e.g. groups, rings or Boolean algebras. A variety $\mathcal{V}$ is **congruence uniform** if each $\mathcal{A} \in \mathcal{V}$ has this property. It was proved by W. Taylor [5] that every congruence uniform variety is congruence regular (see e.g. [1], [3]). The problem if this assertion remains true for a single algebra was solved in [2] and, in a more advanced version, also by M. Goldstern [4]. The following was proved (see [2])

Proposition 1. *Every finite congruence uniform algebra is congruence regular. For every infinite cardinal there exists a congruence uniform algebra of this cardinality which is not congruence regular.*

The aim of this note is to modify this result for the weak regularity of congruences.

From now on, every algebra will be assumed to have a constant which will be denoted by 1. Recall that $\mathcal{A}$ is **weakly regular** if for each $\Theta, \Phi \in Con\mathcal{A}$ we have $\Theta = \Phi$ whenever $[1]_\Theta = [1]_\Phi$.

The class $[1]_\Theta$ (for some $\Theta \in Con\mathcal{A}$) is called the **kernel** (of Θ). Denote by $K(\mathcal{A})$ the set of all kernels of all congruences of $\mathcal{A}$.

We introduce the following concept:
An algebra $\mathcal{A}$ has **functionally restricted cardinalities of congruence classes** if there exists a function $f_A(x, y)$ from $A \times K(\mathcal{A})$ into the set of cardinal numbers of the congruence classes such that $f_A(a, J) = card[a]_\Theta$ for each $\Theta \in Con\mathcal{A}$ with $[1]_\Theta = J$. Of course, if $\mathcal{A}$ is a finite algebra, then f_A has its values in natural numbers.

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Lemma 2. *Let $\mathcal{A}$ be a weakly regular algebra. Then $\mathcal{A}$ has functionally restricted cardinalities of congruence classes.*

Proof. Let $\mathcal{A}$ be weakly regular and $J \in K(\mathcal{A})$. Then there is a unique $\Theta_J \in Con\mathcal{A}$ with $J = [1]_{\Theta_J}$ and hence we can define $f_A(a, J) = card[a]_{\Theta_J}$. □

For a finite $\mathcal{A}$, we can prove the converse

Theorem 3. *A finite algebra with a constant* 1 *has functionally restricted cardinalities of congruence classes if and only if it is weakly regular.*

Proof. Suppose $\Theta, \Phi \in Con\mathcal{A}$ with $[1]_\Theta = [1]_\Phi = J$. Put $\Psi = \Theta \cap \Phi$. Then it is clear that $[1]_\Psi = [1]_\Theta$ and $\Psi \subseteq \Theta$, $\Psi \subseteq \Phi$. Thus $[a]_\Psi \subseteq [a]_\Theta$, $[a]_\Psi \subseteq [a]_\Phi$ for each a of $\mathcal{A}$. However, we have $card[a]_\Psi = f_A(a, J) = card[a]_\Theta$ for each $a \in A$. Since $\mathcal{A}$ is finite, we conclude that $[a]_\Psi = [a]_\Theta$, and analogously, $[a]_\Psi = [a]_\Phi$. Thus $[a]_\Phi = [a]_\Theta$ for each $a \in A$ whence $\Phi = \Theta$. Thus $\mathcal{A}$ is congruence weakly regular.

The converse follows directly by Lemma 2. □

On the contrary, for infinite algebras, the following assertion can be proven.

Theorem 4. *For every infinite cardinal κ there exists an algebra $\mathcal{A} = (A, F)$ with $cardA = \kappa$ having functionally restricted cardinalities of congruence classes which is not weakly regular.*

Proof. Let A_1, A_2, A_3 be pairwise disjoint sets of cardinality κ and $A = A_1 \cup A_2 \cup A_3$. Since κ is infinite, A is also of cardinality κ. Let us pick up an element of A_1 which will be denoted by 1. Define $f_{ab} : A \to A$ by $f_{ab}(a) = b$ and $f_{ab}(x) = x$ otherwise. Moreover, for $i \in \{1, 2, 3\}$, let g_i be a bijection of A onto A_i. We put

$$\mathcal{A} = (A; \{f_{ab} | \langle a, b\rangle \in A_1^2 \cup A_2^2 \cup A_3^2\} \cup \{g_1, g_2, g_3\})$$

and let $\Theta \in Con\mathcal{A}$. Let $\langle c, d\rangle \in A_j^2$ for some $j \in \{1, 2, 3\}$. If $\Theta \neq \omega_A$ then there exists $\langle p, q\rangle \in \Theta$ with $p \neq q$ and hence

$$c = f_{g_j(p),c}(g_j(p)) \Theta f_{g_j(p),c}(g_j(q)) = g_j(q) =$$

$$= f_{g_j(c),d}(g_j(q)) \Theta f_{g_j(c),d}(g_j(p)) = d$$

thus $\langle c, d\rangle \in \Theta$. Hence $A_j^2 \subseteq \Theta$ for $j = 1, 2, 3$. It is clear that $A_1^2 \cup A_2^2 \cup A_3^2 = \Phi_1$ and $A_1^2 \cup (A_2 \cup A_3)^2 = \Phi_2$ are congruences on $\mathcal{A}$ with $J = [1]_{\Phi_1} = [1]_{\Phi_2} = A_1$ $(1 \in J)$ and $\Phi_1 \neq \Phi_2$ thus $\mathcal{A}$ is not weakly regular. Of course, $Con\mathcal{A} = \{\omega, \Phi_1, \Phi_2, \Phi_3, \Phi_4, A^2\}$ where $\Phi_3 = A_2^2 \cup (A_1 \cup A_3)^2$ and $\Phi_4 = A_3^2 \cup (A_1 \cup A_2)^2$. On the other hand, define $f_A(a, J) = \kappa$ for all $a \in A$ and $J \neq \{1\}$. It is plain that $card[a]_\Theta = \kappa$ for $\Theta \neq \omega$. For $J = \{1\}$ we define $f_A(a, \{1\}) = 1$ (since then $\Theta_J = \omega$). Hence $\mathcal{A}$ has functionally restricted cardinalities of congruence classes. □

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