Novi Sad J. Math.
Vol. 37, No. 1, 2007, 33-37

SPACES WITH σ-WEAKLY HEREDITARILY CLOSURE-PRESERVING sn-NETWORKS[1]

Xun Ge[2], Jianhua Shen[3], Ge Ying [4]

Abstract. We prove that a space with a σ-weakly hereditarily closure-preserving sn-network is sn-first countable. As an application of this result, we prove that a Lindelöf space with a σ-weakly hereditarily closure-preserving sn-network is sn-second countable. The above results answer some questions posed by L. Yan.

AMS Mathematics Subject Classification (2000): 54D20, 54D65, 54E20, 54E99

Key words and phrases: weakly hereditarily closure-preserving family, sn-network, sn-first countable, sn-second countable

1. Introduction

In [6], S. Lin and L. Yan obtained the following results.

Theorem 1.1. *(1) A k-space with a σ-weakly hereditarily closure-preserving weak-base is g-first countable, where "k-" can not be omitted?*

(2) A Lindelöf space with a σ-weakly hereditarily closure-preserving weak-base is g-second countable.

Note that sn-networks are an important generalization of weak-bases, recently the second author of [6] posed the following question in a private communication with the author of this paper.

Question 1.2. *(1) Is a k-space with a σ-weakly hereditarily closure-preserving sn-network sn-first countable? Moreover, can "k-" be omitted here?*

(2) Is a Lindelöf space with a σ-weakly hereditarily closure-preserving sn-network sn-second countable?

In this paper we investigate the above Question 2. We prove that a space with a σ-weakly hereditarily closure-preserving sn-network is sn-first countable. As an application of this result, we prove that a Lindelöf space with a σ-weakly hereditarily closure-preserving sn-network is sn-second countable. The above results give some affirmative answers to Question 2.

---

[1]This project was supported by NSFC(No.10571151 and 10671173) and NSF(06KJD110162).

[2]Department of Mathematics, College of Zhangjiagang, Jiangsu University of Science and Technology, Jiangsu Zhangjiagang, 215600, P.R.China, e-mail: zhugexun@163.com

[3] Department of Mathematics, Suzhou Science-Technique University, Suzhou, 215009, P.R.China, email: jssjh@szcatv.com.cn

[4]Department of Mathematics, Suzhou University, Suzhou, 215006, P.R.China, e-mail: geying@pub.sz.jsinfo.net

2. Notations and Definitions

Throughout this paper, all spaces are assumed to be regular and T_1. $\mathbb{N}$ and ω_1 denote the set of all natural numbers and the first uncountable ordinal respectively. $\{x_n\}$ denotes a sequence, where the n-th term is x_n. Let X be a space and $P \subset X$. The closure of P is denoted by $\overline{P}$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbb{N}$; is frequently in P if there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is eventually in P. Let $\mathcal{P}$ be a family of subsets of X and $x \in X$. Then $(\mathcal{P})_x$ denotes the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$, $\bigcup \mathcal{P}$ and $\bigcap \mathcal{P}$ denote the union $\bigcup \{P : P \in \mathcal{P}\}$ and the intersection $\bigcap \{P : P \in \mathcal{P}\}$ respectively.

Definition 2.1. *[2]. Let $\mathcal{P}$ be a family of subsets of a space X.*

(1) $\mathcal{P}$ is called closure-preserving if $\overline{\bigcup \mathcal{P}'} = \bigcup \{\overline{P} : P \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$.

(2) $\mathcal{P}$ is called hereditarily closure-preserving if any family $\{H(P) : P \in \mathcal{P}\}$ is closure-preserving, where each $H(P) \subset P \in \mathcal{P}$.

(3) $\mathcal{P}$ is called weakly hereditarily closure-preserving if any family $\{\{x_P\} : P \in \mathcal{P}\}$ is closure-preserving, where each $x_P \in P \in \mathcal{P}$.

It is clear that a hereditarily closure-preserving family is closure-preserving and weakly hereditarily closure-preserving.

Definition 2.2. *A subset P of a space X is called a sequential neighborhood of x [3], if each sequence converging to x is eventually in P.*

Remark 2.3. *(1) P is a sequential neighborhood of x iff each sequence converging to x is frequently in P.*

(2) The intersection of finite sequential neighborhoods of x is a sequential neighborhood of x.

Definition 2.4. *Let $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$.*

(1) $\mathcal{P}$ is called a network of X [1] if, whenever $x \in U \subset X$ with U open in X, there is $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.

(2) $\mathcal{P}$ is called a wcs*-network of X [9] if, whenever the sequence $S = \{x_n\}$ converges to $x \in U$ with U open in X, there is $P \in \mathcal{P}$ such that $P \subset U$ and for each $n \in \mathbb{N}$, $x_{m_n} \in P$ for some $m_n > n$.

(3) $\mathcal{P}$ is called a k-network of X [8] if, whenever $K \subset U$ with K compact in X and U open in X, there is a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Definition 2.5. *X is called to be an $\aleph_0$-space [7], if X has a countable k-network.*

Definition 2.6. *Let $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ is called an sn-network of X [10], if $\mathcal{P}_x$ satisfies the following* *condition (1),(2) and (3) for each $x \in X$, where $\mathcal{P}_x$ is called an sn-network at x in X.*

(1) $\mathcal{P}_x$ is a network at x in X.

(2) If $P_1, P_2 \in \mathcal{P}_x$, then there is $P \in \mathcal{P}_x$ such that $P \subset P_1 \bigcap P_2$.

(3) Each element of $\mathcal{P}_x$ is a sequential neighborhood of x.

Definition 2.7. *[4]. (1) X is called sn-first countable if X has a countable sn-network at x in X for each $x \in X$.*

(2) X is called sn-second countable if X has a countable sn-network.

3. The Main Results

Theorem 3.1. *If a space X has a σ-weakly hereditarily closure-preserving sn-network, then X is sn-first countable.*

Proof. Let X have a σ-weakly hereditarily closure-preserving sn-network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ is weakly hereditarily closure-preserving. We may assume each $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $x \in X$ and each $n \in \mathbb{N}$, put $\mathcal{P}_{x,n} = \{P \in \mathcal{P}_n : P$ is a sequential neighborhood of $x\}$ and put $P_{x,n} = \bigcap \mathcal{P}_{x,n}$, then $P_{x,n+1} \subset P_{x,n}$ as $\mathcal{P}_{x,n} \subset \mathcal{P}_{x,n+1}$. Put $\mathcal{P}_x = \{P_{x,n} : n \in \mathbb{N}\}$, then $\mathcal{P}_x$ is countable. It suffices to prove that $\mathcal{P}_x$ is an sn-network at x in X.

Claim 1. $\mathcal{P}_x$ is a network at x in X.

Let $x \in U$ with U open in X. Since $\mathcal{P}$ is a sn-network, there is $P \in \mathcal{P}_n$ for some $n \in \mathbb{N}$ such that $P \subset U$, where P is a sequential neighborhood of x, so $P \in \mathcal{P}_{x,n}$. Thus $x \in P_{x,n} \subset P \subset U$. This proves that $\mathcal{P}_x$ is a network at x in X.

Claim 2. If $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \bigcap P_2$ for some $P \in \mathcal{P}_x$.

It is clear because $P_{x,n+1} \subset P_{x,n}$ for each $n \in \mathbb{N}$.

Claim 3. $P_{x,n}$ is a sequential neighborhood of x for each $n \in \mathbb{N}$.

Let $\{x_n\}$ be a sequence converging to x. By Remark 2.3(1), we only need to prove that $\{x_n\}$ is frequently in $P_{x,n}$. If $x_n = x \in P_{x,n}$ for infinitely many $n \in \mathbb{N}$, then $\{x_n\}$ is frequently in $P_{x,n}$. If $x_n \neq x$ for all but finitely many $n \in \mathbb{N}$, we may assume $x_n \neq x$ for all $n \in \mathbb{N}$, then $\mathcal{P}_{x,n}$ is finite. Indeed, suppose $\mathcal{P}_{x,n}$ is infinite. Then there is a infinite subfamily $\{P_k : k \in \mathbb{N}\}$ of $\mathcal{P}_{x,n}$, where $P_k \neq P_l$ if $k \neq l$. Since $\{x_n\}$ converges to x and each P_k is a sequential neighborhood of x, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in P_k$ for each $k \in \mathbb{N}$. Note that $\mathcal{P}_{x,n}$ is weakly hereditarily closure-preserving and $\{x_{n_k}\}$ converges to x, so $x \in \overline{\{x_{n_k} : k \in \mathbb{N}\}} = \{x_{n_k} : k \in \mathbb{N}\}$. This is a contradiction. So $\mathcal{P}_{x,n}$ is finite. By Remark 2.3(2), $P_{x,n}$ is a sequential neighborhood of x, so $\{x_n\}$ is frequently in $P_{x,n}$.

By the above three claims, $\mathcal{P}_x$ is an sn-network at x in X.

It is known that a space is sn-second countable iff it is an sn-first countable, $\aleph_0$-space [5, Theorem 2.1] and a space is an $\aleph_0$-space iff X has a countable wcs^*-network [9, Proposition C]. We have the following result.

Lemma 3.2. *A space X is sn-second countable iff X is an sn-first countable space with a countable wcs*-network.*

Recall a space X is $\aleph_1$-compact if each closed discrete subspace of X is countable.

Theorem 3.3. *Let a space X have a σ-weakly hereditarily closure-preserving sn-network. If X is $\aleph_1$-compact, then X is sn-second countable.*

Proof. Suppose X is an $\aleph_1$-compact space with a σ-weakly hereditarily closure-preserving sn-network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ is weakly hereditarily closure-preserving. By Theorem 3.1, X is sn-first countable. By Lemma 3.2, it suffices to prove that X has a countable wcs^*-network.

For each $n \in \mathbb{N}$, put $D_n = \{x \in X : \mathcal{P}_n$ is not point-countable at $x\}$ and put $\mathcal{P}'_n = \{P - D_n : P \in \mathcal{P}_n\}$.

Claim 1. $\mathcal{P}'_n$ is countable.

If $\mathcal{P}'_n = \{P - D_n : P \in \mathcal{P}_n\}$ is not countable, then there is an uncountable subfamily $\{P_\alpha : \alpha \in \Lambda\}$ of $\mathcal{P}_n$ such that $P_\alpha - D_n \neq \emptyset$ for each $\alpha \in \Lambda$ and $P_\alpha - D_n \neq P_{\alpha'} - D_n$ if $\alpha \neq \alpha'$, where Λ is an uncountable index set. Take $x_\alpha \in P_\alpha - D_n$ for each $\alpha \in \Lambda$. Since $\mathcal{P}_n$ is weakly hereditarily closure-preserving, $\{x_\alpha : \alpha \in \Lambda\}$ is a closed discrete subspace of X. Note that X is $\aleph_1$-compact, $\{x_\alpha : \alpha \in \Lambda\}$ is countable. So, there is an uncountable subset Λ' of Λ such that $\{x_\alpha : \alpha \in \Lambda'\} = \{x\}$ for some $x \in X$. Thus $\mathcal{P}_n$ is not point-countable at x. This contradicts that $x \notin D_n$. So $\{P - D_n : P \in \mathcal{P}_n\}$ is countable.

Claim 2. D_n is a countable closed discrete subspace of X.

If D_n is not countable, then there is an uncountable subset $D'_n = \{y_\beta : \beta < \omega_1\}$ of D_n. Take $y_1 \in P_1$ for some $P_1 \in \mathcal{P}_n$. $\mathcal{P}_n$ is not point-countable at y_2, so $y_2 \in P_2$ for some $P_2 \in \mathcal{P}_n - \{P_1\}$. By transfinite induction we can obtain a subfamily $\{P_\beta : \beta < \omega_1\}$ of $\mathcal{P}_n$ such that $P_\beta \in \mathcal{P}_n - \{P_\gamma : \gamma < \beta\}$ and $y_\beta \in P_\beta$ for each $\beta < \omega_1$. Thus $D'_n = \{y_\beta : \beta < \omega_1\}$ is an uncountable closed discrete subspace of X because $\mathcal{P}_n$ is weakly hereditarily closure-preserving. This contradicts $\aleph_1$-compactness of X. So D_n is countable. In a similar way as in the proof of that D'_n is a closed discrete subspace of X, it is easy to prove that D_n is a closed discrete subspace of X.

Put $\mathcal{U}_n = \mathcal{P}'_n \bigcup\{\{x\} : x \in D_n\}$ for each $n \in \mathbb{N}$ and put $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$. Then $\mathcal{U}$ is countable. We only need to prove the following claim.

Claim 3. $\mathcal{U}$ is a wcs^*-network of X.

Let $\{x_n\}$ be a sequence converging to $x \in U$ with U open in X. Since $\mathcal{P} = \bigcup\{\mathcal{P}_k : k \in \mathbb{N}\}$ is an sn-network of X, $\{x_n\}$ is eventually in $P \subset U$, where $P \in \mathcal{P}_k$ for some $k \in \mathbb{N}$. If $x_n = x$ for infinitely many $n \in \mathbb{N}$, then for each $n \in \mathbb{N}$ there is $m_n > n$ such that $x_{m_n} = x$. If $x \in D_k$, then $\{x\} \in \mathcal{U}$ and $x_{m_n} = x \in \{x\} \subset U$. If $x \notin D_k$, then $P - D_k \in \mathcal{U}$ and $x_{m_n} = x \in P - D_k \subset U$. If $x_n \neq x$ for all but finitely many $n \in \mathbb{N}$, put $S = \{x_n : n \in \mathbb{N}\}$, then $S \bigcap P$ is infinite. Note that $S \bigcap D_k$ is compact in D_k, $S \bigcap D_k$ is finite, so $S \bigcap(P - D_k)$ is infinite. Thus, for each $n \in \mathbb{N}$ there is $m_n > n$ such that $x_{m_n} \in P - D_k$, where $P - D_k \in \mathcal{U}$ and $P - D_k \subset U$. This completes the proof of Claim 3.

Theorem 3.4. *The following statements are equivalent for a space X.*

(1) X is sn-second countable.

(2) X is a hereditarily Lindelöf space with a σ-weakly hereditarily closure-preserving sn-network.

(3) X is a Lindelöf space with a σ-weakly hereditarily closure-preserving sn-network.

(4) X is a hereditarily separable space with a σ-weakly hereditarily closure-preserving sn-network.

(5) X is an $\aleph_1$-compact space with a σ-weakly hereditarily closure-preserving sn-network.

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (4): They hold by [5, Theorem 2.1].

(2) $\Longrightarrow$ (3): It is clear.

(3) $\Longrightarrow$ (5) and (4) $\Longrightarrow$ (5): It is easy to prove that every Lindelöf space or every hereditarily separable space is $\aleph_1$-compact. So (3) $\Longrightarrow$ (5) and (4) $\Longrightarrow$ (5).

(5) $\Longrightarrow$ (1): It holds from Theorem 3.3.

Remark 3.5. *"Hereditarily separable" in Corollary 3.4(4) can not relax to "separable". In fact, [5, Example 2.3] gives a separable space with a σ-discrete sn-network, which is not sn-second countable.*

References

[1] Arhangel'skii, A. V., An addition theorem for the weight of sets lying in bicompacta. Dokl. Akad. Nauk. SSSR. 126 (1959), 239-241.

[2] Burke, D. K., Engelking, R., Lutzer, D., Hereditarily closure-preserving and metrizability. Proc. Amer. Math. Soc. 51 (1975), 483-488.

[3] Franklin, S. P., Spaces in which sequence suffice. Fund. Math., 57 (1965), 107-115.

[4] Ge, Y., On sn-metrizable spaces. Acta Math. Sinica, 45 (2002), 355-360. (in Chinese)

[5] Ge, Y., Spaces with countable sn-networks. Comment Math. Univ. Carolinae, 45 (2004), 169-176.

[6] Lin, S. and Yan, L., A note on spaces with a σ-compact-finite weak base. Tsukuba J. Math. 28 (2004), 85-91.

[7] Michael, E. A., $\aleph_0$-spaces. J. Math. Mech., 15(1966), 983-1002.

[8] Meara, P. O', On paracompactness in function spaces with the compact-open topology. Proc. Amer. Math. Soc., 29(1971), 183-189.

[9] Tanaka, Y., Theory of k-networks. Questions Answers in General Topology, 12 (1994), 139-164.

[10] Yan, P., Lin, S., Point-countable k-networks, cs^*-networks and α_4-spaces. Topology. Proceedings 24 (1999), 345-354.

Received by the editors December 21, 2005