Novi Sad J. Math.
Vol. 37, No. 1, 2007, 27-32

A NOTE ABOUT FULL HILBERT MODULES OVER FRÉCHET LOCALLY C*-ALGEBRAS[1]

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Abstract. Let A and B be two Fréchet locally C^*-algebras, let E be a full Hilbert A-module, and let F be a Hilbert B-module. We show that a bijective linear map $\Phi : E \to F$ is a unitary operator from E to F if and only if there is a map $\varphi : A \to B$ with closed range such that $\Phi(\xi a) = \Phi(\xi)\varphi(a)$ and $\varphi(\langle \xi, \eta\rangle) = \langle \Phi(\xi), \Phi(\eta)\rangle$ for all $a \in A$ and for all $\xi, \eta \in E$.

AMS Mathematics Subject Classification (2000): 46L08

Key words and phrases: full Hilbert modules; Fréchet locally C^*-algebras; unitary operators

1. Introduction and preliminaries

A locally C^*-algebra is a complete Hausdorff complex $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for each continuous C^*-seminorm p on A [3], [5]. The set of all continuous C^*-seminorms on A is denoted by $S(A)$. A Fréchet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. Clearly, any C^*-algebra is a Fréchet locally C^*-algebra.

Given two locally C^*-algebras A and B, a morphism of locally C^*-algebras from A to B is a continuous $*$-morphism φ from A to B. An isomorphism of locally C^*-algebras from A to B is a bijective map $\varphi : A \to B$ such that φ and φ^{-1} are morphisms of locally C^*-algebras.

Hilbert modules over locally C^*-algebras are generalizations of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra rather than in a C^*-algebra.

Definition 1.1. *Let A be a locally C^*-algebra. A pre-Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot\rangle : E \times E \to A$ which is $\mathbb{C}$-and A-linear in its second variable and satisfies the following relations:*

1. $\langle \xi, \eta\rangle^* = \langle \eta, \xi\rangle$ *for every* $\xi, \eta \in E$;

[1]This research was supported by grant CNCSIS (Romanian National Council for Research in High Education)-code A 1065/2006

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2. $\langle \xi, \xi \rangle \geq 0$ *for every* $\xi \in E$;

3. $\langle \xi, \xi \rangle = 0$ *if and only if* $\xi = 0$.

We say that E *is a Hilbert* A*-module if* E *is complete with respect to the topology determined by the family of seminorms* $\{\overline{p}\}_{p \in S(A)}$, *where* $\overline{p}(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$ *[5, Definition 4.1].*

Let E be a Hilbert A-module. The $*$-ideal of A generated by $\{\langle \xi, \eta \rangle ; \xi, \eta \in A\}$ is denoted by $\langle E, E \rangle$. We say that E is full if the close of the linear span $\langle E, E \rangle$ in A is the whole of A.

Let A and B be two Fréchet locally C^*-algebras, let E be a full Hilbert module over A, let F be a Hilbert module over B, and let $\Phi : E \to F$ be a bijective linear map such that there is a map $\varphi : A \to B$ with closed range such that $\Phi(\xi a) = \Phi(\xi) \varphi(a)$ and $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $a \in A$ and for all $\xi, \eta \in E$. We show in Proposition 2.2 that φ is an isomorphism of locally C^*-algebras if and only if F is full. As a consequence of this fact we obtain the following: if E is both a full Hilbert A-module and a full Hilbert B-module and there is a map $\varphi : A \to B$ with closed range such that $\xi a = \xi \varphi(a)$ and $\varphi(\langle \xi, \eta \rangle_A) = \langle \xi, \eta \rangle_B$ for all $a \in A$ and for all $\xi, \eta \in E$, then the topologies on E induced by the inner products $\langle \cdot, \cdot \rangle_A$, respectively $\langle \cdot, \cdot \rangle_B$, are equivalent. In Section 3, we extend the definition of unitary operators between Hilbert C^*-modules over different C^*-algebras [1] in the context of Hilbert modules over locally C^*-algebras and we show that the unitary equivalence is an equivalence relation in the set of all full Hilbert modules over Fréchet locally C^*-algebras. Also we prove a necessary and sufficient condition for a linear map between two full Hilbert modules to be a unitary operator, Theorem 3.4.

2. Full Hilbert modules

Let A and B be two Fréchet locally C^*-algebras, let E be a full Hilbert A-module, and let F be a Hilbert B-module.

Remark 2.1. *Let* $a \in A$ *such that* $\xi a = 0$ *for all* $\xi \in E$. *Then* $\langle \eta, \xi \rangle a = 0$ *for all* $\xi, \eta \in E$, *and since* E *is full,* $a = 0$.

Proposition 2.2. *Let* A, B, E *and* F *be as above, let* Φ *be a bijective linear map from* E *onto* F *and let* φ *be a map from* A *to* B *with closed range such that* $\Phi(\xi a) = \Phi(\xi) \varphi(a)$ *and* $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ *for all* a *in* A *and for all* ξ *and* η *in* E. *Then* F *is full if and only if* φ *is an isomorphism of locally* C^**-algebras.*

Proof. First we suppose that F is full. Let $a_1, a_2 \in A$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. It is not difficult to check that

$$\Phi(\xi) (\varphi(\alpha_1 a_1 + \alpha_2 a_2) - \alpha_1 \varphi(a_1) - \alpha_2 \varphi(a_2)) = 0$$

and

$$\Phi(\xi)(\varphi(a_1a_2)-\varphi(a_1)\varphi(a_2))=0$$

for all $\xi\in E$. Since Φ is surjective, from these relations and Remark 2.1 we deduce that φ is a morphism of algebras.

For each $\xi,\eta\in E$, we have

$$\varphi\left(\langle\xi,\eta\rangle^*\right)=\varphi(\langle\eta,\xi\rangle)=\langle\Phi(\eta),\Phi(\xi)\rangle=(\langle\Phi(\xi),\Phi(\eta)\rangle)^*=(\varphi(\langle\xi,\eta\rangle))^*.$$

Let $a\in A$. Then

$$\begin{aligned}
& \left\langle\Phi(\xi)\left(\varphi(a^*)-\varphi(a)^*\right),\Phi(\xi)\left(\varphi(a^*)-\varphi(a)^*\right)\right\rangle\\
= & \ \varphi(a^*)^*\varphi(\langle\xi,\xi\rangle)\varphi(a^*)-\varphi(a^*)^*\varphi(\langle\xi,\xi\rangle)\varphi(a)^*\\
& -\varphi(a)\varphi(\langle\xi,\xi\rangle)\varphi(a^*)+\varphi(a)\varphi(\langle\xi,\xi\rangle)\varphi(a)^*\\
= & \ (\varphi(\langle\xi a^*,\xi\rangle)\varphi(a^*))^*-(\varphi(a)\varphi(\langle\xi,\xi\rangle)\varphi(a^*))^*\\
& -\varphi(\langle\xi a^*,\xi a^*\rangle)+(\varphi(a)\varphi(\langle\xi,\xi a^*\rangle))^*\\
= & \ (\varphi(\langle\xi a^*,\xi a^*\rangle))^*-(\varphi(\langle\xi a^*,\xi a^*\rangle))^*\\
& -\varphi(\langle\xi a^*,\xi a^*\rangle)+(\varphi(\langle\xi a^*,\xi a^*\rangle))^*\\
= & \ 0
\end{aligned}$$

for all $\xi\in E$. This implies that $\Phi(\xi)\left(\varphi(a^*)-\varphi(a)^*\right)=0$ for all $\xi\in E$. Since Φ is surjective, from this fact and Remark 2.1, we conclude that $\varphi(a^*)=\varphi(a)^*$. Therefore φ is a $*$-morphism. Moreover, by Theorem 3.3 [3], φ is continuous.

Let $a\in A$ such that $\varphi(a)=0$. Then $\Phi(\xi a)=0$ for all $\xi\in E$, and since Φ is a linear injective map from E to F, $\xi a=0$ for all $\xi\in E$. From this fact and Remark 2.1 we conclude that $a=0$. Therefore φ is injective.

From

$$\langle\Phi(E),\Phi(E)\rangle=\varphi(\langle E,E\rangle)$$

and taking into account that: Φ is surjective; φ is a continuous $*$-morphism with closed range; E and F are full; we conclude that $\varphi(A)=B$, so φ is surjective. Thus we showed that φ is a bijective $*$-morphism from A and B, and since A and B are Fréchet locally C^*-algebras, φ is an isomorphism of locally C^*-algebras (Corollary 3.4, [3]).

Conversely, suppose that φ is an isomorphism of locally C^*-algebras. Since E is full, φ is an isomorphism of locally C^*-algebras and $\langle\Phi(E),\Phi(E)\rangle=\varphi(\langle E,E\rangle)$, the closed ideal of B generated by $\langle\Phi(E),\Phi(E)\rangle$ is the whole of B. From this fact and taking into account that Φ is surjective, we conclude that F is full. □

Remark 2.3. *If in Proposition 2.2, A and B are C^*-algebras, $F=E$ and $\Phi=id_E$ (id_E denotes the identity map on E), then we obtain [4, Theorem 2.2].*

Corollary 2.4. *Let E be a full Hilbert A-module, let F be a full Hilbert B-module, and let $\Phi : E \to F$ be a bijective linear map. If there is a map $\varphi : A \to B$ with closed range such that $\Phi(\xi a) = \Phi(\xi)\varphi(a)$ and $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $a \in A$ and for all $\xi, \eta \in E$, then Φ is an isomorphism of locally convex spaces.*

Proof. By Proposition 2.2, φ is an isomorphism of locally C^*-algebras.

Let $q \in S(B)$. Since φ is a continuous morphism of locally C^*-algebras, there is $p_q \in S(A)$ such that

$$q(\varphi(a)) \leq p_q(a)$$

for all $a \in A$. Then

$$\overline{q}(\Phi(\xi))^2 = q(\langle \Phi(\xi), \Phi(\xi) \rangle) = q(\varphi(\langle \xi, \xi \rangle)) \leq p_q(\langle \xi, \xi \rangle) = \overline{p_q}(\xi)^2$$

for all $\xi \in E$. Therefore Φ is continuous.

Let $p \in S(A)$. Since φ is an isomorphism of locally C^*-algebras, there is $q_p \in S(B)$ such that

$$p\left(\varphi^{-1}(b)\right) \leq q_p(b)$$

for all $b \in B$. Then

$$\begin{aligned} \overline{p}\left(\Phi^{-1}(\eta)\right)^2 &= p\left(\left\langle \Phi^{-1}(\eta), \Phi^{-1}(\eta) \right\rangle\right) \\ &= p\left(\varphi^{-1}\left(\varphi\left(\left\langle \Phi^{-1}(\eta), \Phi^{-1}(\eta) \right\rangle\right)\right)\right) \\ &= p\left(\varphi^{-1}(\langle \eta, \eta \rangle)\right) \leq q_p(\langle \eta, \eta \rangle) = \overline{q_p}(\eta)^2 \end{aligned}$$

for all $\eta \in F$, and so Φ^{-1} is continuous too. □

Corollary 2.5. *Let E be both a full Hilbert A-module and a full Hilbert B-module. If there is a map $\varphi : A \to B$ with closed range such that $\xi a = \xi \varphi(a)$ and $\varphi(\langle \xi, \eta \rangle_A) = \langle \xi, \eta \rangle_B$ for all $a \in A$ and for all $\xi, \eta \in E$, where $\langle \cdot, \cdot \rangle_A$ denotes the A valued inner product on E and $\langle \cdot, \cdot \rangle_B$ denotes the B valued inner product on E, then the topologies on E induced by the inner products $\langle \cdot, \cdot \rangle_A$, respectively $\langle \cdot, \cdot \rangle_B$, are equivalent.*

Proof. Putting $F = E$ and $\Phi = \mathrm{id}_E$ in Corollary 2.2, we conclude that id_E is an isomorphism of locally convex spaces. □

3. Unitary operators

Let A and B be two Fréchet locally C^*-algebras, let E be a Hilbert A-module, and let F be a Hilbert B-module.

We extend the definition of unitary operators between Hilbert C^*-modules over different C^*-algebras introduced by Bakic and Guljas [1] in the context of Hilbert modules over locally C^*-algebras.

Definition 3.1. *Let $\Phi : E \to F$ be a linear map. We say that Φ is a unitary operator from E to F if Φ is surjective and there is an injective morphism of locally C^*-algebras $\varphi : A \to B$ with closed range such that $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $\xi, \eta \in E$.*

Remark 3.2. *If E and F are Hilbert modules over A and U is a unitary operator in $L_A(E, F)$, the set of all adjointable module maps from E to F (that is, $UU^* = id_F$ and $U^*U = id_E$), then U is a unitary operator in the sense of Definition 3.1.*

Remark 3.3. *If $\Phi : E \to F$ is a unitary operator, then Φ is a continuous bijective linear map from E to F.*

Theorem 3.4. *Let E be a full Hilbert A-module, let F be a full Hilbert B-module and let $\Phi : E \to F$ be a linear map. Then the following assertions are equivalent:*

1. *Φ is a unitary operator;*
2. *Φ is bijective and there is a map $\varphi : A \to B$ with closed range such that $\Phi(\xi a) = \Phi(\xi)\varphi(a)$ and $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $a \in A$ and for all $\xi, \eta \in E$.*

Proof. 1 $\Rightarrow$ 2. If Φ is a unitary operator, then Φ is bijective and there is an injective morphism of locally C^*-algebras $\varphi : A \to B$ with closed range such that $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $\xi, \eta \in E$. Let $\xi \in E$ and $a \in A$. Then

$$\begin{aligned} & \langle \Phi(\xi a) - \Phi(\xi)\varphi(a), \Phi(\xi a) - \Phi(\xi)\varphi(a) \rangle \\ = \; & \varphi(\langle \xi a, \xi a \rangle) - \varphi(\langle \xi a, \xi \rangle)\varphi(a) - \varphi(a^*)\varphi(\langle \xi, \xi a \rangle) \\ & + \varphi(a^*)\varphi(\langle \xi, \xi a \rangle)\varphi(a) \\ = \; & 0 \end{aligned}$$

and so $\Phi(\xi a) = \Phi(\xi)\varphi(a)$.

2. $\Rightarrow$ 1. It follows from Proposition 2.2. □

Remark 3.5. *Let E be both a full Hilbert A-module and a full Hilbert B-module. Then id_E is a unitary operator if and only if there is a map $\varphi : A \to B$ with closed range such that $\xi a = \xi \varphi(a)$ and $\varphi(\langle \xi, \eta \rangle_A) = \langle \xi, \eta \rangle_B$ for all $a \in A$ and for all $\xi, \eta \in E$.*

Corollary 3.6. *Let E be a full Hilbert module over a Fréchet locally C^*-algebra A, let F be a full Hilbert module over a Fréchet locally C^*-algebra B and let $\Phi : E \to F$ be a linear map. Then Φ is a unitary operator from E to F if and only if there is an isomorphism of locally C^*-algebras $\varphi : A \to B$ such that $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $\xi, \eta \in E$.*

We say that two Hilbert modules E and F are unitarily equivalent if there is a unitary operator from E to F.

Proposition 3.7. *Unitary equivalence in the set of full Hilbert modules over Fréchet locally C^*-algebras is an equivalence relation.*

Proof. Let E be a full Hilbert module over a Fréchet locally C^*-algebra A. By Corollary 3.6, id_E is a unitary operator from E to E. Therefore the relation is reflexive.

To show that the relation is symmetric, let A and B be two Fréchet locally C^*-algebras, let E be a full Hilbert A-module and let F be a full Hilbert B-module. If Φ is a unitary operator from E to F, then Φ is an isomorphism of locally convex spaces and there is an isomorphism of locally C^*-algebras $\varphi : A \rightarrow B$ such that $\varphi\left(\langle\xi,\eta\rangle\right) = \left\langle\Phi\left(\xi\right),\Phi\left(\eta\right)\right\rangle$ for all $\xi,\eta \in E$. It is not difficult to check that Φ^{-1} is a unitary operator from F to E.

Let A, B and C be three Fréchet locally C^*-algebras and let E be a full Hilbert A-module, let F be a full Hilbert B-module, and let G be a full Hilbert C-module. If Φ is a unitary operator from E to F and Ψ is a unitary operator from F to G, then there is an isomorphism of locally C^*-algebras $\varphi : A \rightarrow B$ such that $\varphi\left(\langle\xi,\eta\rangle\right) = \left\langle\Phi\left(\xi\right),\Phi\left(\eta\right)\right\rangle$ for all $\xi,\eta \in E$, and there is an isomorphism of locally C^*-algebras $\psi : B \rightarrow C$ such that $\psi\left(\langle x,y\rangle\right) = \left\langle\Psi\left(x\right),\Psi\left(y\right)\right\rangle$ for all $x,y \in F$. Clearly, $\Psi\circ\Phi$ is an isomorphism of locally convex spaces from E to G and $\psi\circ\varphi$ is an isomorphism of locally C^*-algebras such that $\left(\psi\circ\varphi\right)\left(\langle\xi,\eta\rangle\right) = \left\langle\left(\Psi\circ\Phi\right)\left(\xi\right),\left(\Psi\circ\Phi\right)\left(\eta\right)\right\rangle$ for all $\xi,\eta \in E$. This shows that $\Psi\circ\Phi$ is a unitary operator from E to F and so the relation is transitive. □

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Received by the editors November 23, 2005