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INTEGRAL WITH RESPECT TO FUZZY MEASURE IN FINITE DIMENSIONAL BANACH SPACES[1]

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Abstract. In [9] definition of integral with respect to fuzzy valued measure is introduced. In this paper we continue the investigation of that notion. Some properties concerning the structure of that kind of integral when the fuzzy valued measure has closed levels are given.

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1. Introduction

Fuzzy valued measure ([7], [10], [3], [5]) is a natural generalization of a set valued measure ([6]). Infinite addition is defined ([12]) as a Zadeh's extension principle of continuity. In [9], using an additive fuzzy valued measure, an integral of single valued function is defined and some basic properties are given. In this paper we proceed to investigate specific properties of integrals in fuzzy analysis. Basic space is finite dimensional Banach space.

We shall prove some results concerning the fundamental property of the integral - the property that the integral is a new fuzzy valued measure on the measurable space $(\Omega, \mathcal{A})$ on which the basic fuzzy valued measure is defined. The range of the fuzzy valued measure is a set of fuzzy sets with closed levels. From the definition of the integral it is easy to notice that the "fuzzy" integral is closely related to the "set" integral defined with respect to the set valued measure derived from fuzzy valued measure taking its level set. That connection will be used often in the sequel.

The organization of the paper is as follows: Section 2 contains some basic definitions, notions and notations. In Section 3 the main result of the paper concerning the integral with respect to fuzzy valued measure is given.

2. Preliminaries

Throughout this paper let $\mathcal{X}$ be a real finite dimensional Banach space, $\mathcal{X}^*$ be its dual space and $(\Omega, \mathcal{A}, \mu)$ be a complete measure space where $\mathcal{A}$ is a σ-algebra and μ is a measure. By $\mathcal{P}_{(f)(k)(c)}(\mathcal{X})$ we will denote a subset of $\mathcal{P}(\mathcal{X})$

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whose elements are (closed), (compact), (convex) subsets of $\mathcal{X}$. Hausdorff metric $h : \mathcal{P}_f(\mathcal{X}) \times \mathcal{P}_f(\mathcal{X}) \to \mathbb{R}$ is defined by

$$h(A,B) = \max\{\sup_{y\in B} \inf_{x\in A} \|x-y\|, \sup_{x\in A} \inf_{y\in B} \|x-y\|\},$$

and for $A \subset \mathcal{X}$, $|A| = h(A,\{0\}) = \sup_{x\in A} \|x\|$. By $\sigma_A(\cdot)$ we will denote the support function of a set $A \subset \mathcal{X}$ defined by $\sigma_A(x^*) = \sup_{x\in A}(x^*, x)$, $x^* \in \mathcal{X}^*$.

We shall denote by $\mathcal{F}(\mathcal{X})$ the set of fuzzy sets $u : \mathcal{X} \to [0,1]$ for which the α-level set u_α of u, defined by $u_\alpha = \{x \in \mathcal{X} : u(x) \geq \alpha\}$, $\alpha \in (0,1]$, is a nonempty subset of $\mathcal{X}$ for all $\alpha \in (0,1]$. By $\mathcal{F}_{(f)(k)(c)}(\mathcal{X})$ we denote a subset of $\mathcal{F}(\mathcal{X})$ whose α-levels are (closed), (compact), (convex).

If $X : \Omega \to \mathcal{F}(\mathcal{X})$ is a fuzzy valued functions, then the function $X_\alpha : \Omega \to \mathcal{P}(\mathcal{X})$ defined by $X_\alpha(\omega) = (X(\omega))_\alpha$ is a set valued function. The fuzzy valued measure (see [3], [7], [10]) is a natural generalization of set valued measure (see [4], [6]). Let $(\Omega, \mathcal{A})$ be a measurable space with $\mathcal{A}$ a σ-algebra of measurable subsets of the set Ω. If $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ is a mapping such that for every sequence $\{A_i\}_{i\in N}$ of pairwise disjoint elements of $\mathcal{A}$ the next equality is satisfied $\mathcal{M}(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mathcal{M}(A_i)$, where $(\sum_{i=1}^\infty \mathcal{M}(A_i))(x) = \sup\{\bigwedge_{i=1}^\infty \mathcal{M}(A_i)(x_i) \ : \ x = \sum_{i=1}^\infty x_i(\text{uncond.conv.})\}$, and $\mathcal{M}(\emptyset) = I_{\{0\}}$, then $\mathcal{M}$ is a fuzzy valued measure. We shall denote by $S_{\mathcal{M}_\alpha}$ the collection of all measure selectors of the set valued measure $\mathcal{M}_\alpha$. A fuzzy valued measure $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ is of bounded variation if $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is a set valued measure of bounded variation for all $\alpha \in (0,1]$. A fuzzy valued measure $\mathcal{M}$ is μ-continuous if $A \in \mathcal{A}$ with $\mu(A) = 0$ implies $\mathcal{M}(A) = I_{\{0\}}$.

Integrals of fuzzy valued functions have been studied in connection with problems in probability, statistics, measure theory ([3], [5], [7], [8], [11]). For any measurable set valued function $F : \Omega \to \mathcal{P}(\mathcal{X})$ we can define the set $S_F = \{f \in L : f(\omega) \in F(\omega) \ \mu - a.e.\}$ where L denotes the set of all functions $h : \Omega \to \mathcal{X}$ which are integrable with respect to the measure μ. Using S_F an integral of multivalued mapping F is defined (first introduced by Aumann in [1]) by $\int_\Omega F(\omega)d\mu(\omega) = \{\int_\Omega f(\omega)d\mu(\omega) : f \in S_F\}$. The integrals $\int_\Omega f(\omega)d\mu(\omega)$ are defined in the sense of Bochner. $F : \Omega \to \mathcal{P}(\mathcal{X})$ is called integrably bounded if there exists integrable function $h : \Omega \to \mathbb{R}$ such that $|F(\omega)| = \sup_{x\in F(\omega)} \|x\| \leq h(\omega)$, $\mu - a.e.$ A measurable fuzzy valued function $X : \Omega \to \mathcal{F}(\mathcal{X})$ is integrably bounded if there exists integrable function $h : \Omega \to \mathbb{R}$ such that $|X(\omega)| = \sup_{\alpha\in(0,1]} |X_\alpha(\omega)| \leq h(\omega)$. Let $\mathcal{L}$ denotes the set of all integrably bounded measurable set valued functions $F : \Omega \to \mathcal{P}_c(\mathcal{X})$ and let Λ be the set of all integrably bounded measurable fuzzy valued functions $X : \Omega \to \mathcal{F}(\mathcal{X})$.

Now, the integral with respect to a set valued measure $M : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ will be considered. Let $(\Omega, \mathcal{A})$ be a measurable space, $\mathcal{X}$ be a Banach space and $m : \mathcal{A} \to \mathcal{X}$ be a countable additive measure of bounded variation such that $(\Omega, \mathcal{A}, m)$ is a complete measure space. For the function $f : \Omega \to \mathbb{R}$ integrable with respect to $|m|$, the integral of f with respect to m can be defined. We denote it by $\int_\Omega f(\omega)dm(\omega)$. Detailed construction of this integral can be found in [4]. If $M : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is a set valued measure of bounded variation, then

$\int_\Omega f(\omega)dM(\omega) \overset{\text{def}}{=} \{\int_\Omega f(\omega)dm(\omega),\ m \in S_M\}$, where S_M is the set of measure selectors of M.

If $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ is a fuzzy valued measure and $X : \Omega \to \mathcal{F}(\mathcal{X})$ is a measurable fuzzy valued mapping, then X is said to be a Radon-Nikodým derivative of $\mathcal{M}$ with respect to μ if $\mathcal{M}(A) = \int_A X(\omega)d\mu(\omega)$ for all $A \in \mathcal{A}$ and we write $Xd\mu = d\mathcal{M}$.

Theorem 2.1. *[12] Let $u_i \in \mathcal{F}_k(\mathcal{X})$ and $\sum_{i=1}^{\infty} |(u_i)_\alpha| < \infty$, for every $\alpha \in (0,1]$. Then*

$$\left(\sum_{i=1}^{\infty} u_i\right)_\alpha = \sum_{i=1}^{\infty}(u_i)_\alpha, \quad \text{for every } \alpha \in (0,1],$$

and $\sum_{i=1}^{\infty} u_i \in \mathcal{F}_k(\mathcal{X})$.

Lemma 2.1. *[7] Let M be a set and $\{M_\alpha : \alpha \in [0,1]\}$ be a family of subsets of M such that (1) $M_0 = M$, (2) $a < b \Rightarrow M_a \subseteq M_b$, (3) $a_1 < a_2 < \cdots < a_n < \cdots \to a \Rightarrow \cap_{i=1}^{\infty} M_{a_i} = M_a$. Then the function $u : M \to [0,1]$ defined by $u(x) = \sup\{a \in [0,1] : x \in M_a\}$ has the property that $\{x \in M : u(x) \geq a\} = M_a$ for every $a \in [0,1]$.*

3. Integration

In [9] a definition of integral with respect to fuzzy valued measure is introduced. Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space, $\mathcal{M} : \mathcal{A} \to \mathcal{F}(\mathcal{X})$ be a fuzzy valued measure, $S_{\mathcal{M}_\alpha}$ the set of all measure selection of $\mathcal{M}_\alpha$ and $f : \Omega \to \mathbb{R}$, then for every $\alpha \in (0,1]$

$$\mathcal{I}_\alpha(A) = \int_A f(\omega)d\mathcal{M}_\alpha(\omega) \overset{\text{def}}{=} \{\int_A f(\omega)dm(\omega),\ \ m \in S_{\mathcal{M}_\alpha}\}.$$

For every $A \in \mathcal{A}$ the mapping $\mathcal{I}(A) : \mathcal{X} \to (0,1]$ is defined by

$$.\mathcal{I}(A)(x) = \sup\{\alpha \in (0,1] : x \in \mathcal{I}_\alpha\}.$$

We write

$$\mathcal{I}(A) = \int_A f(\omega)d\mathcal{M}(\omega).$$

Theorem 3.1. *Let $(\Omega, \mathcal{A})$ be a measurable space, $\mathcal{X}$ be a finite dimensional Banach space, $\mathcal{M} : \mathcal{A} \to \mathcal{F}_f(\mathcal{X})$ be a nonatomic fuzzy measure of bounded variation. If $f \in L^\infty(\Omega, \mathbb{R})$, then for every $\alpha \in (0,1]$ and every $A \in \mathcal{A}$, $\mathcal{I}_\alpha(A) \in \mathcal{P}_{kc}(\mathcal{X})$.*

Proof. For every $A \in \mathcal{A}$ and $\alpha \in (0,1]$, $\mathcal{M}_\alpha(A)$ is a closed bounded set, which means it is compact. First, we will prove that for every $\alpha \in (0,1]$, $\mathcal{I}_\alpha(\Omega)$ is a closed set. Let $\{x_n\}_{n\in N} \subset \int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega)$ and $\lim_{n\to\infty} x_n = x$. By the definition of the integral $\mathcal{I}$, $x_n = \int_\Omega f(\omega)dm_n(\omega)$, $m_n \in S_{\mathcal{M}_\alpha}$. On the other hand, for every $A \in \mathcal{A}$ the set $\{m(A), m \in S_{\mathcal{M}_\alpha}\} \subseteq \mathcal{M}_\alpha(A)$ is relatively compact. By Th. 7 [2], we have that $S_{\mathcal{M}_\alpha}$ is relatively compact subset of the set $\mathcal{B}$ of all countably additive measures of bounded variation. Then there exists a convergent subsequence $\{m_k\}_{k\in N} \subset \{m_n\}_{n\in N}$, $m_k \overset{w}{\to} m$. From $\mathcal{X}^* \subset \mathcal{B}^*$, $(x^*, m_k) \to (x^*, m)$ for all $x^* \in \mathcal{X}^*$. Since $(x^*, m_k(\cdot))$ and $(x^*, m(\cdot))$ are $\mathbb{R}$-valued measures of bounded variation, we get

$$\left|\int_\Omega f(\omega)d(x^*, m_k)(\omega) - \int_\Omega f(\omega)d(x^*, m)(\omega)\right| =$$

$$= \left|\int_\Omega f(\omega)d(x^*, m_k - m)(\omega)\right| \leq \|f\|_\infty.$$

Knowing that $\lim_{k\to\infty} |d(x^*, m_k - m)(\Omega)| = 0$, we conclude

$$\lim_{k\to\infty}(x^*, \int_\Omega f(\omega)dm_k(\omega)) = \lim_{k\to\infty}\int_\Omega f(\omega)d(x^*, m_k)(\omega) =$$

$$= \int_\Omega f(\omega)d(x^*, m)(\omega) = (x^*, \int_\Omega f(\omega)dm(\omega)),$$

which means that $\int_\Omega f(\omega)dm_k(\omega)$ converges weakly to $\int_\Omega f(\omega)dm(\omega)$. From the uniqueness of the limit, $x = \int_\Omega f(\omega)dm(\omega)$. Also, since $\mathcal{M}_\alpha$ is compact valued, $m \in S_{\mathcal{M}_\alpha}$ and $x \in \int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega)$. So, we have proved that $\int_\Omega f(\omega)d\mathcal{M}_\alpha(\omega)$ is a closed set for every $\alpha \in (0, 1]$. The preceding arguments are now repeated for any $A \in \mathcal{A}$ instead of Ω.

If $|\mathcal{M}|(\cdot)$ is the total variation of $\mathcal{M}(\cdot)$, then $|\mathcal{M}|(\cdot)$ is a positive finite measure. Then

$$|\mathcal{I}_\alpha| \leq \int_\Omega \|f(\omega)\| d|\mathcal{M}_\alpha|(\omega) \leq \int_\Omega \|f(\omega)\| d|\mathcal{M}|(\omega) < \infty,$$

which, together with closeness in finite Banach space, gives compactness of $\mathcal{I}_\alpha$.

To prove convexity of $\mathcal{I}_\alpha$, we proceed as follows. A diadic structure of the set $D \in \mathcal{A}$ is a collection of measurable sets $D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)$, where $\epsilon_j \in \{0, 1\}$, $j \in N$, such that

$$D(0) \cup D(1) = D, \quad D(0) \cap D(1) = \emptyset,$$

$$D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k, 0) \cup D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k, 1) = D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k),$$

$$D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k, 0) \cap D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k, 1) = \emptyset.$$

To prove convexity of $\mathcal{I}_\alpha(A)$, it is sufficient to consider the case $A = \Omega$. Since $\mathcal{M}_\alpha$ is nonatomic set valued measure of bounded variation, $|\mathcal{M}_\alpha|$ is a nonatomic positive finite measure. Then, there exists a diadic structure $\{D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)\}$ of Ω such that $|\mathcal{M}_\alpha|(D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)) = 2^{-k}|\mathcal{M}_\alpha|(\Omega)$. Let $x_0, x_1 \in \mathcal{I}_\alpha(\Omega)$ and let $\{x_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)\}_{k\in N}$, $j \in \{0, 1\}$ be sequences such that

$$x_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k) \in \mathcal{I}_\alpha(D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)), \quad j \in \{0, 1\},$$

$$x_0(0) + x_0(1) = x_0, \quad x_1(0) + x_1(1) = x_1,$$

$$x_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k, 0) + x_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k, 1) = x_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k), \ \ j \in \{0, 1\}.$$

Let the field $\mathcal{D}_0$ be generated by $\{D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)\}$ and let $\nu_j : \mathcal{D}_0 \to \mathcal{X}$, $j \in \{0, 1\}$, be finitely additive set mappings, such that

$$\nu_j(D(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)) = x_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k).$$

Let $\mathcal{D}_1$ be the σ-field generated by $\mathcal{D}_0$. Since $\|\nu_j(D)\| \leq |\mathcal{I}_\alpha(D)|$ for all $D \in \mathcal{D}_0$, ν_0 and ν_1 have extension on $\mathcal{D}_1$ which are of bounded variation and $|\mathcal{M}_\alpha|$ continuous. Since finite dimensional Banach space has RNP, using results from [14], the closure of the range of $\mathcal{X} \times \mathcal{X}$-valued measure (ν_0, ν_1) is convex. Since, by the definition, $(\nu_0(\Omega), \nu_1(\Omega)) = (x_0, x_1)$, for every $\lambda \in (0, 1)$ there exists $D \in \mathcal{D}_0$ such that $\|\lambda x_j - \nu_j(D)\| < \frac{\delta}{2}$, $j \in \{0, 1\}$. Then we have

$$\nu_0(D) + \nu_1(\Omega \setminus D) \in \mathcal{I}_\alpha(D) + \mathcal{I}_\alpha(\Omega \setminus D) = \mathcal{I}_\alpha(\Omega)$$

which implies

$$\begin{aligned} &\|\lambda x_0 + (1 - \lambda) x_1 - \nu_0(D) - \nu_1(\Omega \setminus D)\| = \\ &= \|\lambda x_0 + x_1 - \lambda x_1 - \nu_0(D) - \nu_1(\Omega) + \nu_1(D)\| = \\ &= \|\lambda x_0 + x_1 - \lambda x_1 - \nu_0(D) - x_1 + \nu_1(D)\| \leq \\ &\leq \|\lambda x_0 - \nu_0(D)\| + \|\lambda x_1 - \nu_1(D)\| < \delta. \end{aligned}$$

If $\{D_k\}_{k \in N} \subset \mathcal{D}_0$ is a sequence such that $|\mathcal{M}_\alpha|(D_k \Delta D) \to 0$, then $\lim_{k\to\infty} \|\nu_j(D_k) - \nu_j(D)\| = 0$, meaning that we have the same conclusion when $D \in \mathcal{D}_1$.

So, we have proved that for all $x_0, x_1 \in \mathcal{I}_\alpha(\Omega)$ and every $\lambda \in (0, 1)$, $\lambda x_0 + (1 - \lambda) x_1 \in \mathcal{I}_\alpha(\Omega)$. □

Theorem 3.2. *Let $(\Omega, \mathcal{A})$ be a measurable space, $\mathcal{X}$ be a finite dimensional Banach space, $\mathcal{M} : \mathcal{A} \to \mathcal{F}_f(\mathcal{X})$ be a nonatomic fuzzy measure of bounded variation. If $f \in L^\infty(\Omega, \mathbb{R})$, then $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a fuzzy measure of bounded variation.*

Proof. Using the same arguments as in [9] we get that $\mathcal{I}(A) \in \mathcal{F}_{kc}(\mathcal{X})$ for all $A \in \mathcal{A}$. In order to prove that $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a fuzzy valued measure, we prove first finite additivity. From Lemma 3 [12] $(\mathcal{I}(A) + \mathcal{I}(B))_\alpha = \mathcal{I}_\alpha(A) + \mathcal{I}_\alpha(B)$ for every $\alpha \in (0, 1]$. Since

$$\mathcal{I}_\alpha(A + B) = \int_{A+B} f(\omega) d\mathcal{M}_\alpha(\omega) = \int_{A+B} f(\omega) X_\alpha(\omega) d\mu(\omega) =$$

$$= \int_A f(\omega) X_\alpha(\omega) d\mu(\omega) + \int_B f(\omega) X_\alpha(\omega) d\mu(\omega) = \mathcal{I}_\alpha(A) + \mathcal{I}_\alpha(B),$$

for all $A, B \in \mathcal{A}$, we get the finite additivity for $\mathcal{I}$.

To prove countable additivity of $\mathcal{I}$ we consider first countable additivity of $\mathcal{I}_\alpha$.

If $\{A_n\}_{n\in N}$ is the sequence of pairwise disjoint elements of $\mathcal{A}$, the equality

$$\mathcal{I}_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{k}\mathcal{I}_\alpha(A_n) + \mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right)$$

implies

$$h\left(\mathcal{I}_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right), \sum_{n=1}^{\infty}\mathcal{I}_\alpha(A_n)\right) =$$

$$= h\left(\sum_{n=1}^{k}\mathcal{I}_\alpha(A_n) + \mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right), \sum_{n=1}^{k}\mathcal{I}_\alpha(A_n) + \sum_{n=k+1}^{\infty}\mathcal{I}_\alpha(A_n)\right) \leq$$

$$\leq h\left(\mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right), \sum_{n=k+1}^{\infty}\mathcal{I}_\alpha(A_n)\right) \leq \left|\mathcal{I}_\alpha\left(\bigcup_{n=k+1}^{\infty} A_n\right)\right| + \left|\sum_{n=k+1}^{\infty}\mathcal{I}_\alpha(A_n)\right| \leq$$

$$\left|\int_{\cup_{n=k+1}^{\infty} A_n} f(\omega) d\mathcal{M}_\alpha(\omega)\right| + \sum_{n=k+1}^{\infty}\left|\int_{A_n} f(\omega) d\mathcal{M}_\alpha(\omega)\right| \leq$$

$$\int_{\cup_{n=k+1}^{\infty} A_n} |f(\omega)|\, d\,|\mathcal{M}_\alpha|(\omega) + \sum_{n=k+1}^{\infty}\int_{A_n} |f(\omega)|\, d\,|\mathcal{M}_\alpha|(\omega).$$

Since $\mathcal{M}_\alpha$ is a set valued measure of bounded variation, $|\mathcal{M}_\alpha|$ is a finite positive measure absolutely continuous with respect to μ. Letting $n \to \infty$, we get

$$h\left(\mathcal{I}_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right), \sum_{n=1}^{\infty}\mathcal{I}_\alpha(A_n)\right) \leq$$

$$\leq \int_{\cup_{n=k+1}^{\infty} A_n} |f(\omega)|\, d\,|\mathcal{M}_\alpha|(\omega) + \sum_{n=k+1}^{\infty}\int_{A_n} |f(\omega)|\, d\,|\mathcal{M}_\alpha|(\omega) \stackrel{k\to\infty}{\longrightarrow} 0.$$

To prove countable additivity of $\mathcal{I}$ we need to establish first that for every sequence $\{A_n\}_{n\in N}$ of pairwise disjoint elements of $\mathcal{A}$

$$\sum_{n=1}^{\infty}|\mathcal{I}_\alpha(A_n)| \leq \sum_{n=1}^{\infty}\int_{A_n} |f(\omega)|\, d\,|\mathcal{M}_\alpha|(\omega) =$$

$$= \sum_{n=1}^{k}\int_{A_n} |f(\omega)|\, d\,|\mathcal{M}_\alpha|(\omega) + \sum_{n=k+1}^{\infty}\int_{A_n} |f(\omega)|\, d\,|\mathcal{M}_\alpha|(\omega) < \infty.$$

Since $\mathcal{I}_\alpha(A)$ are compact for all $\alpha \in (0,1]$ and all $A \in \mathcal{A}$, the last relation allow to apply Th. 2.1,

$$\left(\sum_{i=1}^{\infty}\mathcal{I}(A_i)\right)_\alpha = \sum_{i=1}^{\infty}\mathcal{I}_\alpha(A_i), \quad \text{for every } \alpha \in (0,1].$$

Further, for every $x \in \mathcal{X}$,

$$\mathcal{I}(\bigcup_{n=1}^{\infty} A_n)(x) = \sup \left\{ \alpha \in (0,1] : x \in \mathcal{I}_\alpha(\bigcup_{n=1}^{\infty} A_n) \right\} =$$

$$= \sup \left\{ \alpha \in (0,1] : x \in \sum_{n=1}^{\infty} \mathcal{I}_\alpha(A_n) \right\} =$$

$$= \sup \left\{ \alpha \in (0,1] : x \in \left(\sum_{n=1}^{\infty} \mathcal{I}(A_n) \right)_\alpha \right\} = \left(\sum_{n=1}^{\infty} \mathcal{I}(A_n) \right)(x)$$

which gives the countable additivity of $\mathcal{I}$.

It is easily seen that $\mathcal{I}$ is absolutely continuous with respect to $\mathcal{M}$, which implies that $\mathcal{I}$ is of bounded variation too. For the same reason $\mathcal{I}(\emptyset) = I_{\{0\}}$. If $\mathcal{M}$ is nonatomic, then all measure selection m of $\mathcal{M}_\alpha$ and all the integrals $\int f(\omega)dm(\omega)$ are nonatomic too. So, $\mathcal{I}$ is also nonatomic.

So, we conclude that $\mathcal{I} : \mathcal{A} \to \mathcal{F}_{kc}(\mathcal{X})$ is a μ-continuous fuzzy valued measure of bounded variation. □

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