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DERIVATIONAL FORMULAS OF A SUBMANIFOLD OF A GENERALIZED RIEMANNIAN SPACE

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Abstract. In the introduction is given basic information on a generalized Riemannian space, as a differentiable manifold endowed with asymmetric basic tensor, and a subspace is defined (in local coordinates).

In §1., for a tensor whose certain indices are related to the space and the others to the subspace, four kinds of covariant derivative are introduced and, in this manner, also four connections.

Derivational formulas for tangents of the submanifold are expressed by means of the unit normals (Theorem 1.1 and Theorem 1.2). It is proved that by applying the third or the fourth kind of covariant derivative one concludes that induced connection is symmetric (Theorem 1.2).

§2. is related to the induced connection of the normal bundle (eq. (2.9)). In this case also are possible four kinds of covariant derivatives on the obtained normal submanifold X^N_{N-M} (eq. (2.10)). In Theorem 2.1. is given the presentation of covariant derivative of the normals, using the first and the second kind of covariant derivatives. Theorem 2.2. is related to the properties of the coefficients of this connection.

In Theorem 2.3. is proved that, applying the third and the fourth kind of covariant derivative at X^N_{N-M}, we express the covariant derivative of normals by means of tangents, and in this case the induced connection at X^N_{N-M} is unique ($\underset{1}{\bar\Gamma} = \underset{2}{\bar\Gamma}$).

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0. Introduction

A generalized Riemannian space GR_N [2, 3, 9] is a differentiable manifold equipped with the asymmetric basic tensor $G_{ij}(x^1, ..., x^N)$ (the components) where x^i are the local coordinates. The symmetric, respectively antisymmetric part of G_{ij} are H_{ij} and K_{ij}.

For the lowering and rasing of indices in GR_N one uses H_{ij}, respectively H^{ij}, where

$$(H^{ij}) = (H_{ij})^{-1}, \quad (det(H_{ij}) \neq 0). \tag{0.1}$$

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Cristoffel symbols at GR_N are

$$(0.2a,b)\qquad \Gamma_{i.jk}=\frac{1}{2}(G_{ji,k}-G_{jk,i}+G_{ik,j}),\quad \Gamma^i_{jk}=H^{ip}\Gamma_{p.jk},$$

where, for example, $G_{ji,k}=\partial G_{ji}/\partial x^k$. Based on the asymmetry of G_{ij}, it follows that the Cristoffel symbols are also asymmetric with respect to j,k in $(2a,b)$.

By equations

$$(0.3)\qquad x^i=x^i(u^1,...,u^M)\equiv x^i(u^\alpha), i=1,..,N,$$

a submanifold X_M is defined in local coordinates. If $rank(B^i_\alpha)=M \quad (B^i_\alpha=\partial x^i/\partial u^\alpha)$ and

$$(0.4)\qquad g_{\alpha\beta}=B^i_\alpha B^j_\beta G_{ij},$$

X_M becomes $GR_M\subset GR_N$, with **induced basic tensor** (0.4), which is generally also asymmetric. Note that in the present work Latin indices $i,j,...$ take values $1,\dots,N$ and refer to the GR_N, while the Greek ones take values $1,\dots,M$ and refer to the GR_M.

In the GR_M are valid the relations similar to (0.1) and (0.2). The symmetric part of $g_{\alpha\beta}$ is denoted with $h_{\alpha\beta}$, and antisymmetric one with $k_{\alpha\beta}$, where e.g.

$$(0.4'a,b)\qquad h_{\alpha\beta}=B^i_\alpha B^j_\beta H_{ij},(h^{\alpha\beta})=(h_{\alpha\beta})^{-1}.$$

Cristoffel symbols $\tilde\Gamma_{\alpha.\beta\gamma},\tilde\Gamma^\alpha_{\beta\gamma}=h^{\alpha\pi}\tilde\Gamma_{\pi.\beta\gamma}$ are expressed by $g_{\alpha\beta}$ analogously to (0.2).

For the unit, mutually orthogonal vectors N^i_A, which are orthogonal to the GR_M too, we have ([4]-[8], [10])

$$(0.5a)\qquad H_{ij}N^i_A N^j_B=e_A\delta^A_B=h_{AB}, e_A\in\{-1,1\},$$

$$(0.5b)\qquad H_{ij}N^i_A B^j_\alpha=0,$$

where $A,B,\dots\in\{M+1,\dots,N\}$.

1. Induced connection and derivational formulas on $X_M\subset GR_N$

1.1. As is known, the following relations between Cristoffel symbols of a generalized Riemannian space and its subspace are valid:

$$(1.1)\qquad \tilde\Gamma_{\alpha.\beta\gamma}=\Gamma_{i.jk}B^i_\alpha B^j_\beta B^k_\gamma+H_{ij}B^i_\alpha B^j_{\beta,\gamma},$$

$$(1.2)\qquad \tilde\Gamma^\alpha_{\beta\gamma}=h^{\pi\alpha}\tilde\Gamma_{\pi.\beta\gamma}=h^{\pi\alpha}(\Gamma_{i.jk}B^i_\pi B^j_\beta B^k_\gamma+H_{ij}B^i_\pi B^j_{\beta,\gamma}),$$

i.e.

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = h^{\pi\alpha} H_{pi} B^p_\pi (\Gamma^i_{jk} B^j_\beta B^k_\gamma + B^i_{\beta,\gamma}). \tag{1.2$'$}$$

Supposing that the both connections, defined by the coefficients Γ and $\tilde{\Gamma}$ are asymmetric, one can define four kinds of covariant derivative for a tensor defined at points of the subspace [4]-[6]. For example, for a tensor $t^{i\alpha}_{j\beta}$ we have

$$\underset{\substack{1\\2\\3\\4}}{\Delta}_\mu t^{i\alpha}_{j\beta} \equiv t^{i\alpha}_{j\beta \underset{\substack{1\\2\\3\\4}}{|} \mu} = t^{i\alpha}_{j\beta,\mu} + \Gamma^i_{\substack{pm\\mp\\pm\\mp}} t^{p\alpha}_{j\beta} B^m_\mu - \Gamma^p_{\substack{jm\\mj\\mj\\jm}} t^{i\alpha}_{p\beta} B^m_\mu + \tilde{\Gamma}^{\alpha}_{\substack{\pi\mu\\\mu\pi\\\pi\mu\\\mu\pi}} t^{i\pi}_{j\beta} - \tilde{\Gamma}^{\pi}_{\substack{\beta\mu\\\mu\beta\\\mu\beta\\\beta\mu}} t^{i\alpha}_{j\pi} \tag{1.3}$$

and in this manner are defined four connections $\underset{\theta}{\nabla}$, $\theta \in \{1, \dots, 4\}$ on the submanifold $X_M \subset GR_N$. The obtained structures we shall denote by $(X_M \subset GR_N, g_{\alpha\beta}, \underset{\theta}{\nabla}, \theta \in \{1, \dots, 4\})$.

1.2. We have to examine the presentation of covariant derivatives of tangent vectors $B^i_\alpha = \partial x^i / \partial u^\alpha$ and of the unit normals N^i_A, with the help of the same magnitudes, so called **derivational formulas** of the subspace for the tangents and normals.

Putting

$$B^i_{\alpha \underset{\theta}{|} \mu} = \underset{\theta}{\Phi}{}^{\pi}_{\alpha\mu} B^i_\pi + \sum_P \underset{\theta}{\Omega}{}_{P\alpha\mu} N^i_P, \theta \in \{1, \dots, 4\}, \tag{1.4}$$

we get

$$\underset{\theta}{\Phi}{}^{\pi}_{\alpha\mu} = H_{ij} B^i_{\alpha \underset{\theta}{|} \mu} B^j_\rho h^{\pi\rho}. \tag{1.5}$$

Let us investigate, firstly, $\underset{1}{\Phi}$. Substituting in (1.5) with respect to (1.3), we obtain

$$\begin{aligned}\underset{1}{\Phi}{}^{\pi}_{\alpha\mu} &= H_{ij} B^j_\rho h^{\pi\rho} (B^i_{\alpha,\mu} + \Gamma^i_{pm} B^p_\alpha B^m_\mu - \tilde{\Gamma}^{\sigma}_{\alpha\mu} B^i_\sigma)\\ &= H_{ij} B^j_\rho h^{\pi\rho} (B^i_{\alpha,\mu} + \Gamma^i_{pm} B^p_\alpha B^m_\mu) - h_{\rho\sigma} h^{\pi\rho} \tilde{\Gamma}^{\sigma}_{\alpha\mu}.\end{aligned}$$

Further, we have

$$\underset{1}{\Phi}{}^{\pi}_{\alpha\mu} = h^{\pi\rho} (H_{ij} B^i_{\alpha,\mu} B^j_\rho + \Gamma_{j.pm} B^j_\rho B^p_\alpha B^m_\mu - \tilde{\Gamma}_{\rho.\alpha\mu}).$$

Taking into consideration (1.1), it follows that $\underset{1}{\Phi}{}^{\pi}_{\alpha\mu} = 0$. In the same way one obtains $\underset{2}{\Phi}{}^{\pi}_{\alpha\mu} = 0$. So,

$$\underset{\theta}{\Phi}{}^{\pi}_{\alpha\mu} = 0, \theta \in \{1, 2\}. \tag{1.6}$$

In order to determine $\underset{\theta}{\Omega}$ at (1.4), we shall compose this equation with $H_{ij} N^j_Q$, and, by virtue of (0.5), we get

$$H_{ij} B^i_{\alpha \underset{\theta}{|} \mu} N^j_Q = \sum_P \underset{\theta}{\Omega}{}_{P\alpha\mu} e_P \delta^P_Q = \underset{\theta}{\Omega}{}_{Q\alpha\mu} e_Q, \quad e_Q \in \{-1, 1\} \tag{1.7}$$

(no summing wrp Q), i.e.

$$\underset{\theta}{\Omega}_{P\alpha\mu} = e_P H_{ij} B^i_{\alpha\underset{\theta}{|}\mu} N^j_P, \tag{1.7'}$$

from where, substituting $B^i_{\alpha\underset{\theta}{|}\mu}$ based on (1.3) and taking into consideration (0.5b), one obtains

$$\underset{\substack{1\\2}}{\Omega}_{P\alpha\mu} = \underset{\substack{3\\4}}{\Omega}_{P\alpha\mu} = e_P H_{ij} N^i_P (B^j_{\alpha,\mu} + \Gamma^j_{\substack{pm\\mp}} B^p_\alpha B^m_\mu). \tag{1.8a, b}$$

Based on (1.3),(1.4),(1.6), we have the following theorem.

Theorem 1.1. *Derivational formulas for tangents of submanifold $X_M \subset GR_N$ possessing the structure $(X_M, g_{\alpha\beta}, \underset{\theta}{\nabla}, \theta \in \{1,2\})$, are*

$$B^i_{\alpha\underset{\theta}{|}\mu} \equiv \underset{\theta}{\nabla}_\mu B^i_\alpha = \sum_P \underset{\theta}{\Omega}_{P\alpha\mu} N^i_P, \quad \theta \in \{1,2\}, \tag{1.9}$$

where $\underset{\theta}{\Omega}$ are given at (1.8) .

1.3. Consider now the same structure, but for $\theta \in \{3,4\}$ and find $\underset{\theta}{\Phi}$. Based on (1.5), (1.3), (1.1) we get

$$\begin{aligned} \underset{3}{\Phi}^\pi_{\alpha\mu} &= H_{ij} B^j_\rho h^{\pi\rho} (B^i_{\alpha,\mu} + \Gamma^i_{pm} B^p_\alpha B^m_\mu - \tilde{\Gamma}^\sigma_{\mu\alpha} B^i_\sigma) \\ &= H_{ij} B^j_\rho h^{\pi\rho} (B^i_{\alpha,\mu} + \Gamma^i_{pm} B^p_\alpha B^m_\mu) - h^{\pi\rho} \tilde{\Gamma}_{\rho.\mu\alpha} = h^{\pi\rho}(\tilde{\Gamma}_{\rho.\alpha\mu} - \tilde{\Gamma}_{\rho.\mu\alpha}) = \tilde{T}^\pi_{\alpha\mu}. \end{aligned}$$

In the same manner one finds $\underset{4}{\Phi}^\pi_{\alpha\mu} = -\tilde{T}^\pi_{\alpha\mu}$, i.e.

$$\underset{3}{\Phi}^\pi_{\alpha\mu} = -\underset{4}{\Phi}^\pi_{\alpha\mu} = \tilde{T}^\pi_{\alpha\mu}. \tag{1.10}$$

Composing the equation $H_{ij} B^i_\alpha B^j_\rho = h_{\alpha\rho}$ with $h^{\pi\rho}$, one gets

$$H_{ij} h^{\pi\rho} B^i_\alpha B^j_\rho = h_{\alpha\rho} h^{\pi\rho} = \delta^\pi_\alpha,$$

wherefrom, applying $\underset{3}{\nabla}_\mu$:

$$H_{ij} h^{\pi\rho} (B^i_{\alpha\underset{3}{|}\mu} B^j_\rho + B^i_\alpha B^j_{\rho\underset{3}{|}\mu}) = 0,$$

that is

$$\underset{3}{\Phi}^\pi_{\alpha\mu} + \underset{3}{\hat{\Phi}}^\pi_{\alpha\mu} = 0, \tag{1.11}$$

where $\underset{3}{\Phi}$ is given at (1.5), and

$$\underset{3}{\hat{\Phi}}^\pi_{\alpha\mu} = H_{ij} h^{\pi\rho} B^i_\alpha B^j_{\rho\underset{3}{|}\mu}.$$

Since

$$H_{ij}h^{\pi\rho}B^{j}_{\rho\underset{3}{|}\mu} = (H_{ij}h^{\pi\rho}B^{j}_{\rho})\underset{3}{|}\mu = B^{\pi}_{i\underset{3}{|}\mu},$$

by virtue of (1.3) the previous equation gives

$$\begin{aligned}
\underset{3}{\hat{\Phi}}{}^{\pi}_{\alpha\mu} &= B^{i}_{\alpha}B^{j}_{i\underset{3}{|}\mu} = B^{i}_{\alpha}(B^{\pi}_{i,\mu} - \Gamma^{p}_{mi}B^{\pi}_{p}B^{m}_{\mu} + \tilde{\Gamma}^{\pi}_{\sigma\mu}B^{\sigma}_{i})\\
&= B^{i}_{\alpha}(B^{\pi}_{i,\mu} - \Gamma^{p}_{mi}B^{\pi}_{p}B^{m}_{\mu}) + B^{i}_{\alpha}H_{ij}h^{\sigma\rho}B^{j}_{\rho}\tilde{\Gamma}^{\pi}_{\sigma\mu}\\
&= B^{i}_{\alpha}(B^{\pi}_{i,\mu} - \Gamma^{p}_{mi}B^{\pi}_{p}B^{m}_{\mu}) + \tilde{\Gamma}^{\pi}_{\alpha\mu}\\
&\underset{(1.2')}{=} B^{i}_{\alpha}B^{\pi}_{i,\mu} - \Gamma^{p}_{mi}B^{i}_{\alpha}B^{\pi}_{p}B^{m}_{\mu} + h^{\rho\pi}H_{pi}B^{p}_{\rho}B^{j}_{\alpha}B^{k}_{\mu}\Gamma^{i}_{jk} + h^{\rho\pi}H_{pi}B^{p}_{\rho}B^{i}_{\alpha,\mu}\\
&= B^{i}_{\alpha}(H_{pi}h^{\rho\pi}B^{p}_{\rho})_{,\mu} - \Gamma^{p}_{mi}B^{i}_{\alpha}B^{\pi}_{p}B^{m}_{\mu} + B^{\pi}_{i}B^{j}_{\alpha}B^{k}_{\mu}\Gamma^{i}_{jk} + H_{pi}h^{\rho\pi}B^{p}_{\rho}B^{i}_{\alpha,\mu}
\end{aligned}$$

where $\underset{(1.2')}{=}$ indicates "$=$ based on (1.2′)" .

The first and the last addend give

$$(B^{i}_{\alpha}H_{pi}h^{\rho\pi}B^{p}_{\rho})_{,\mu} = (h^{\rho\pi}h_{\rho\alpha})_{,\mu} = \delta^{\pi}_{\alpha,\mu} = 0,$$

and by corresponding changes of dummy indices at the rest ones, we finally obtain

$$\underset{3}{\hat{\Phi}}{}^{\pi}_{\alpha\mu} = T^{i}_{jk}B^{\pi}_{i}B^{j}_{\alpha}B^{k}_{\mu} \underset{(1.2')}{=} \tilde{T}^{\pi}_{\alpha\mu} \underset{(1.10)}{=} \underset{3}{\Phi}{}^{\pi}_{\alpha\mu}. \tag{1.12}$$

Taking into account (1.10) – (1.12), we obtain

$$\underset{3}{\Phi}{}^{\pi}_{\alpha\mu} = -\underset{4}{\Phi}{}^{\pi}_{\alpha\mu} = \tilde{T}^{\pi}_{\alpha\mu} = 0. \tag{1.13}$$

So, we have proved the following theorem.

Theorem 1.2. *Derivational formulas for tangents of a submanifold $X_M \subset GR_N$, possessing the structure $(X_M, g_{\alpha\beta}, \underset{\theta}{\nabla}, \theta \in \{3,4\})$, are*

$$B^{i}_{\alpha\underset{\theta}{|}\mu} \equiv \underset{\theta}{\nabla}_{\mu} B^{i}_{\alpha} = \sum_{P} \underset{\theta}{\Omega}_{P\alpha\mu} N^{i}_{P}, \theta \in \{3,4\}, \tag{1.14}$$

where $\underset{\theta}{\Omega}$ are given at (1.8), *and induced connection $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ in this case is symmetric* $(\tilde{T} = 0)$.

1.4. For the covariant derivative of the normals on X_M, based on (1.3), we have

$$N^{i}_{A\underset{\substack{1\\2}}{|}\mu} = N^{i}_{A\underset{\substack{3\\4}}{|}\mu} = N^{i}_{A,\mu} + \Gamma^{i}_{\underset{mp}{pm}}N^{p}_{A}B^{m}_{\mu}, \tag{1.15}$$

provided that one supposes that the indices $A, B, \dots \in \{M+1, \dots, N\}$ have not a tensor character [7, 8, 4]. Starting from the presentation

$$\underset{\theta}{\nabla}_{\mu}N^{i}_{A} \equiv N^{i}_{A\underset{\theta}{|}\mu} = \underset{\theta}{\Lambda}{}^{\pi}_{A\mu}B^{i}_{\pi} + \sum_{P}\underset{\theta}{\Psi}_{PA\mu}N^{i}_{P} \tag{1.16}$$

one obtains, the known result [7, 8, 4] for **derivational formulas of normals**

$$N^i_{A\underset{\theta}{|}\mu} = -e_A \underset{\theta}{\Omega}_{A\rho\mu} h^{\rho\pi} B^i_\pi + \sum_P \underset{\theta}{\Psi}_{PA\mu} N^i_P, \underset{\theta}{\Psi}_{AA\mu} = 0, \theta \in \{1,2\} \tag{1.17}$$

where $\underset{\theta}{\Omega}$ is given at (1.8), and for $\underset{\theta}{\Psi}$ we have

$$\underset{\theta}{\Psi}_{PA\mu} = e_P H_{ij} N^i_P N^j_{A\underset{\theta}{|}\mu}, \tag{1.18}$$

where $N^j_{A\underset{\theta}{|}\mu}$ is given by virtue of (1.15).

So, the next theorem is valid:

Theorem 1.3. [7, 8, 4] *Derivational formulas for normals of submanifold* $X_M \subset GR_N$ *with structure* $(X_M, g_{\alpha\beta}, \underset{\theta}{\nabla}, \theta \in \{1,2\})$ *are given at* (1.17), *where* $\underset{\theta}{\Psi}$ *have the values* (1.18).

2. Induced connection on the normal bundle (normal subspace)

2.1. The set of normals of the submanifold $X_M \subset GR_N$ make a **normal bundle** for X_M, and we note it X^N_{N-M}. One can introduce a metric tensor on X^N_{N-M} [10, 11, 1]

$$g_{AB} = G_{ij} N^i_A N^j_B, \tag{2.1}$$

which is asymmetric in a general case.

The symmetric part is

$$h_{AB} = H_{ij} N^i_A N^j_B \underset{(0.5a)}{=} e_A \delta^A_B = h_{BA} = \begin{cases} e_a, & \text{A=B}, \\ 0, & \text{otherwise}. \end{cases}, e_A \in \{-1,1\}. \tag{2.2}$$

If

$$(h^{AB}) = (h_{AB})^{-1}, \tag{2.3}$$

we have

$$h^{AB} = e_A \delta^A_B = h_{AB} = h^{AB}. \tag{2.4}$$

2.2. For a vector v^i one says that it belongs to X^N_{N-M}, if it is defined at the points of X_M and is a linear combination of the normals, i.e.

$$v^i = v^P N^i_P \quad (i \in \{1,\dots,N\}, P = M+1.\dots,N, \quad \text{a summation on P}) \tag{2.5}$$

One can define absolute differential δv^i along X_M in two manners

$$\underset{\substack{1\\2}}{\delta} v^i = dv^i + \Gamma^i_{\substack{jk\\kj}} v^j dx^k,$$

from where

$$\underset{\substack{1\\2}}{\delta} v^i = N^i_P dv^P + (N^i_{P,\mu} + \Gamma^i_{\substack{jk\\kj}} N^j_P B^k_\mu) v^P du^\mu. \tag{2.6}$$

Composing the equation (2.6) with

$$N^A_i = H_{ij} h^{AB} N^j_B, \tag{2.7}$$

we obtain the projection of $\underset{\theta}{\delta} v^i$ on X^N_{N-M}:

$$\underset{\substack{1\\2}}{\delta} v^A = dv^A + \underset{\substack{1\\2}}{\bar{\Gamma}}{}^A_{P\mu} v^P du^\mu, \tag{2.8),}$$

where

$$\underset{\substack{1\\2}}{\bar{\Gamma}}{}^A_{P\mu} = N^A_i (N^i_{P,\mu} + \Gamma^i_{\substack{jk\\kj}} N^j_P B^k_\mu) \tag{2.9}$$

are coefficients of **induced connection of the normal bundle** (submanifold, subspace) X^N_{N-M}.

For a tensor on X_M, whose some indices are related to GR_N and the others to X^N_{N-M}, four kinds of covariant derivative are possible. For example,

$$\begin{aligned} \underset{\substack{1\\2\\3\\4}}{\bar{\nabla}}_\mu t^{iA}_{jB} &\equiv t^{iA}_{jB \underset{\substack{1\\2\\3\\4}}{\perp} \mu} \\ &= t^{iA}_{jB,\mu} + \Gamma^i_{\substack{pm\\mp\\pm\\mp}} t^{pA}_{jB} B^m_\mu - \Gamma^p_{\substack{jm\\mj\\mj\\jm}} t^{iA}_{pB} B^m_\mu + \underset{\substack{1\\2\\1\\2}}{\bar{\Gamma}}{}^A_{P\mu} t^{iP}_{jB} - \underset{\substack{1\\2\\2\\1}}{\bar{\Gamma}}{}^P_{B\mu} t^{iA}_{jP}. \end{aligned} \tag{2.10}$$

In this way, 4 connections $\underset{\theta}{\bar{\nabla}}, \theta \in \{1, \dots, 4\}$ on the submanifold $X^N_{N-M} \subset GR_N$ are defined. We shall denote the obtained structures $(X^N_{N-M} \subset GR_N, g_{AB}, \underset{\theta}{\bar{\nabla}}, \theta \in \{1, \dots, 4\})$.

Derivatives of the type (1.3) and (2.10) are **van der Waeden-Bortoloti derivatives**. Combining these two cases, we can observe also a derivative of a tensor containing simultaneously indices of all three types, e.g. $t^{i\alpha A}_{j\beta B}$.

2.3. Consider now the explanation of $\underset{\theta}{\bar{\nabla}}_\mu N^i_A$. Analogously to (1.16) we have

$$\underset{\theta}{\bar{\nabla}}_\mu N^i_A \equiv N^i_{A \underset{\theta}{\perp} \mu} = \underset{\theta}{\bar{\Lambda}}{}^\pi_{A\mu} B^i_\pi + \sum_P \underset{\theta}{\bar{\Psi}}{}_{PA\mu} N^i_P, \tag{2.11}$$

from where, composing with $H_{ij}B^j_\nu$, one gets

$$H_{ij}B^j_\nu N^i_{A\underset{\theta}{\perp}\mu} = \underset{\theta}{\bar\Lambda}{}^\pi_{A\mu} h_{\pi\nu}. \tag{2.12}$$

In order to determine $\underset{\theta}{\bar\Lambda}$, consider the relation $H_{ij}N^i_A B^j_\nu = 0$ and apply the derivative $\underset{\theta}{\bar\nabla}_\mu \equiv \underset{\theta}{\perp}\mu$, which in the case of B^j_ν is reduced to $\underset{\theta}{\nabla}_\mu$. So,

$$H_{ij}(N^i_{A\underset{\theta}{\perp}\mu}B^j_\nu + N^i_A B^j_{\nu\underset{\theta}{|}\mu}) = 0,$$

wherefrom, in relation to (2.12) and (1.7′): $\underset{\theta}{\bar\Lambda}{}^\pi_{A\mu}h_{\pi\nu} + e_A\underset{\theta}{\Omega}_{A\nu\mu} = 0$,

$$\underset{\theta}{\bar\Lambda}{}^\pi_{A\mu} = -e_A\underset{\theta}{\Omega}_{A\rho\mu}h^{\pi\rho}, \quad \theta\in\{1,\dots,4\}. \tag{2.13}$$

In order to determine $\underset{\theta}{\bar\Psi}$ in (2.11), we are composing with $H_{ij}N^j_Q$, and obtaining

$$H_{ij}N^i_{A\underset{\theta}{\perp}\mu}N^j_Q = \sum_P \underset{\theta}{\bar\Psi}_{PA\mu}e_P\delta^P_Q = e_Q\underset{\theta}{\bar\Psi}_{QA\mu}. \tag{2.14}$$

With respect to (2.10), (2.2), (2.9) the previous relation yields

$$\begin{aligned}
e_Q\underset{1}{\bar\Psi}_{QA\mu} &= H_{ij}N^j_Q(N^i_{A,\mu} + \Gamma^i_{pm}N^p_A B^m_\mu - \underset{1}{\bar\Gamma}{}^P_{A\mu}N^i_P)\\
&\underset{(2.2)}{=} H_{ij}N^j_Q(N^i_{A,\mu} + \Gamma^i_{pm}N^p_A B^m_\mu) - h_{PQ}\underset{1}{\bar\Gamma}{}^P_{A\mu}\\
&\underset{(2.9)}{=} H_{ij}N^j_Q(N^i_{A,\mu} + \Gamma^i_{pm}N^p_A B^m_\mu) - h_{PQ}N^P_i(N^i_{A,\mu} + \Gamma^i_{jk}N^j_A B^k_\mu)\\
&= (N^i_{A,\mu} + \Gamma^i_{pm}N^p_A B^m_\mu)(H_{ij}N^j_Q - h_{PQ}N^P_i) = 0,
\end{aligned}$$

because

$$h_{PQ}N^P_i = H_{ij}N^j_Q. \tag{2.15}$$

So, $\underset{1}{\bar\Psi}_{QA\mu} = 0$. In the same manner one proves that $\underset{2}{\bar\Psi}_{QA\mu} = 0$, and, based on (2.11) and (2.13), we have proved the following theorem.

Theorem 2.1. *Derivational formulas for normals of a submanifold $X_M \subset GR_N$, considered in a structure $(X^N_{N-M}, g_{AB}, \underset{\theta}{\bar\nabla}, \theta\in\{1,2\})$ are*

$$N^i_{A\underset{\theta}{\perp}\mu} \equiv \underset{\theta}{\bar\nabla}_\mu N^i_A = -e_A\underset{\theta}{\Omega}_{A\rho\mu}h^{\pi\rho}B^i_\pi, \theta\in\{1,2\}, \tag{2.16}$$

where $\underset{\theta}{\Omega}$ are given at (1.8).

2.4. In order to investigate $N^i_{A\underset{\theta}{\perp}\mu}$ for $\theta\in\{3,4\}$, we shall firstly consider properties of the coefficients $\underset{1}{\bar\Gamma}, \underset{2}{\bar\Gamma}$. For the Kronecker symbols, being constants, we have

$$\delta^A_{B\underset{\theta}{\perp}\mu} = \delta_{AB\underset{\theta}{\perp}\mu} = \delta^{AB}_{\underset{\theta}{\perp}\mu} = 0, \forall\theta\in\{1,\dots,4\}. \tag{2.17}$$

From here and because of $(2.2), (2.4)$, we obtain

$$h_{AB\underset{\theta}{\perp}\mu} = h^{AB}_{\underset{\theta}{\perp}\mu} = 0, \forall\theta \in \{1,\dots,4\}. \tag{2.18}$$

On the other hand, from (2.10) one gets

$$\delta^{AB}_{\underset{1}{\perp}\mu} = 0 + \underset{1}{\bar{\Gamma}}{}^{A}_{P\mu}\delta^{PB} + \underset{1}{\bar{\Gamma}}{}^{B}_{P\mu}\delta^{AP} = \underset{1}{\bar{\Gamma}}{}^{A}_{B\mu} + \underset{1}{\bar{\Gamma}}{}^{B}_{A\mu} \underset{(2.17)}{=} 0.$$

The analogous is valid for $\underset{2}{\bar{\Gamma}}$, and we have

$$\underset{\omega}{\bar{\Gamma}}{}^{A}_{B\mu} = -\underset{\omega}{\bar{\Gamma}}{}^{B}_{A\mu}, \forall\omega \in \{1,2\}, \tag{2.19}$$

i.e. an antisymmetry is in force with respect to A, B. Further, we have

$$\delta^{A}_{B\underset{3}{\perp}\mu} \underset{(2.10)}{=} \underset{1}{\bar{\Gamma}}{}^{A}_{B\mu} - \underset{2}{\bar{\Gamma}}{}^{A}_{B\mu} \underset{(2.17)}{=} 0,$$

and the result is analogous by applying $\underset{4}{\bar{\nabla}}$, so

$$\underset{1}{\bar{\Gamma}}{}^{A}_{B\mu} = \underset{2}{\bar{\Gamma}}{}^{A}_{B\mu} \quad \text{for} \underset{\theta}{\bar{\nabla}}, \quad \theta \in \{3,4\}. \tag{2.20}$$

Based on (2.19) and (2.20), we conclude that applying $\underset{1}{\bar{\nabla}}$ and $\underset{3}{\bar{\nabla}}$ or $\underset{1}{\bar{\nabla}}$ and $\underset{4}{\bar{\nabla}}$ or $\underset{2}{\bar{\nabla}}$ and $\underset{3}{\bar{\nabla}}$ or $\underset{2}{\bar{\nabla}}$ and $\underset{4}{\bar{\nabla}}$ one obtains

$$\underset{1}{\bar{\Gamma}}{}^{A}_{B\mu} = -\underset{2}{\bar{\Gamma}}{}^{B}_{A\mu}. \tag{2.21}$$

From the above, we state the following theorem

Theorem 2.2. *The coefficients $\underset{1}{\bar{\Gamma}}, \underset{2}{\bar{\Gamma}}$ (2.9) of induced connection in the normal submanifold $X^N_{N-M} \subset GR_N$ have the properties:*

a) *the property* (2.19) *in the structures* $(X^N_{N-M}, g_{AB}, \underset{\theta}{\bar{\nabla}}, \theta \in \{1,2\})$,

b) *the property* (2.20) *in the structures* $(X^N_{N-M}, g_{AB}, \underset{\theta}{\bar{\nabla}}, \theta \in \{3,4\})$,

c) *the property* (2.21) *in the structures* $(X^N_{N-M}, g_{AB}, \underset{\theta}{\bar{\nabla}}, \underset{\omega}{\bar{\nabla}}, (\theta,\omega) \in \{(1,3), (1,4), (2,3), (2,4)\})$.

2.5. Let us investigate now $\underset{\theta}{\bar{\Psi}}$ for $\theta \in \{3,4\}$ at (2.11). In relation to (2.9) is

$$h_{PQ}(\underset{1}{\bar{\Gamma}}{}^{p}_{A\mu} - \underset{2}{\bar{\Gamma}}{}^{p}_{A\mu}) = h_{PQ}N^{P}_{i}T^{i}_{jk}N^{j}_{A}B^{k}_{\mu} = H_{ij}T^{i}_{pm}N^{j}_{Q}N^{p}_{A}B^{m}_{\mu}, \tag{2.22}$$

and based on $(2.14), (2.9)$ and (2.15):

$$\begin{aligned} e_Q\underset{3}{\bar{\Psi}}_{QA\mu} &= H_{ij}N^{j}_{Q}(N^{i}_{A,\mu} + \Gamma^{i}_{pm}N^{p}_{A}B^{m}_{\mu}) - h_{PQ}N^{P}_{i}(N^{i}_{A,\mu} + \Gamma^{i}_{mp}N^{p}_{A}B^{m}_{\mu}) \\ &\underset{(2.15)}{=} H_{ij}T^{i}_{pm}N^{j}_{Q}N^{p}_{A}B^{m}_{\mu} \underset{(2.22)}{=} h_{PQ}(\underset{1}{\bar{\Gamma}}{}^{p}_{A\mu} - \underset{2}{\bar{\Gamma}}{}^{p}_{A\mu}). \end{aligned}$$

An analogous equation is valid for $\underset{4}{\bar{\Psi}}$ too.

Taking into account (2.20), we conclude that, from the previous equation

$$\underset{\theta}{\Psi}_{QA\mu} = 0, \forall\theta \in \{3,4\}, \tag{2.23}$$

and, by virtue of (2.11) and (2.13) we have the following theorem.

Theorem 2.3. *In the structure* $(X^N_{N-M}, g_{AB}, \underset{\theta}{\bar{\nabla}}, \theta \in \{3,4\})$ *derivational formulas for normals of submanifold* $X_M \subset GR_N$ *are*

$$N^i_{A\underset{\theta}{\perp}\mu} \equiv \underset{\theta}{\bar{\nabla}}_\mu N^i_A = -e_A \underset{\theta}{\Omega}_{A\rho\mu} h^{\pi\rho} B^i_\pi, \quad \theta\{3,4\}, \tag{2.24}$$

and then in X^N_{N-M} *there exists a unique connection* (2.9) *with the coefficients* $\underset{1}{\bar{\Gamma}} = \underset{2}{\bar{\Gamma}} = \bar{\Gamma}$.

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