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# ON SPIN $Z_6$-ACTIONS ON SPIN 4-MANIFOLDS[1]

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**Abstract.** Let $X$ be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$ and non-positive signature $\sigma(X)$. In this paper we use Seiberg-Witten theory to prove that if $X$ admits a spin $Z_6$ action of even type, then $b_2^+(X) \geq |\sigma(X)|/8 + 2$ under some non-degeneracy conditions.

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## 1. Introduction

Let $X$ be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of $X$. In [12], Y. Matsumoto conjectured the following inequality

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|. \tag{1}$$

This conjecture is well known and has been called the $\frac{11}{8}$-conjecture (see also [7]). All complex surfaces and their connected sums satisfy the conjecture (see [14]).

From the classification of unimodular even integral quadratic forms and the Rochlin's theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold $X$ is

$$-2kE_8 \oplus mH, \qquad k \geq 0,$$

where $E_8$ is the $8 \times 8$ intersection form matrix and $H$ is the hyperbolic matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $m = b_2^+(X)$ and $k = -\sigma(X)/16$ and so the inequality (1) is equivalent to $m \geq 3k$. Since $K3$ surface satisfies the equality with $k = 1$ and $m = 3$, the coefficient $\frac{11}{8}$ is optimal, if the $\frac{11}{8}$-conjecture is true.

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Donaldson has proved that if $k > 0$ then $m \geq 3$ [4]. In early 1995, using the Seiberg-Witten theory introduced by Seiberg and Witten [17], Furuta [8] proved that

$$b_2(X) \geq \frac{5}{4}|\sigma(X)| + 2. \tag{2}$$

This estimate has been dubbed the $\frac{10}{8}$-theorem. In fact, if the intersection form of $X$ is definite, i.e., $m = 0$, then Donaldson proved that $b_2(X)$ and $\sigma(X)$ are zero [4, 5]. Thus, Furuta assumed that $m$ is not zero. Inequality (2) follows by a surgery argument from the non-positive signature, $b_1(X) = 0$ case:

**Theorem 1.1.** *(Furuta [8]). Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ with non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Then,*

$$2k + 1 \leq m$$

*if $m \neq 0$.*

His key idea was to use a finite dimensional approximation of the monopole equation. Later Furuta and Kametani [9] used equivariant $e$-invariants and improved the above $\frac{10}{8}$-theorem, which was also proved by N. Minami [13] by using an equivariant join theorem to reduce the inequality to a theorem of Stolz [16].

In [2] Bryan and also in [6] Fang used Furuta's technique of "finite dimensional approximation" and the equivariant $K$-theory to improve the above bound by $p$ under the assumption that $X$ has a spin odd type $Z/2^p$-action satisfying some non-degeneracy conditions analogous to the condition $m \neq 0$.

In the paper [10], Kim gave the same bound for smooth, spin even type $Z/2^p$-action on $X$ satisfying some non-degeneracy conditions analogous to Bryan and Fang's.

In the paper [11], Kiyono and the second author obtained a bound for smooth spin alternating $A_4$ action on $X$ satisfying some non-degeneracy conditions.

In this paper, we will assume $m \neq 0$ and $b_1(X) = 0$, unless stated otherwise. We study the spin even type $Z_6$-actions on spin 4-manifolds. We prove that if $X$ admits a spin $Z_6$-action of even type, then $b_2^+(X) \geq |\sigma(X)|/8 + 2$ under some non-degeneracy conditions. We also obtain some results about $Ind_{Z_6}D$.

We organize the remainder of this paper as follows. In section 2, we give some preliminaries to prove the main theorem. We refer the readers to the Bryan's excellent exposition [2] for more details. In this section, we also introduce the representation ring and the the character table of cyclic group $Z_6$. In section 3, we use equivariant $K$-theory and representation theory to study the $G$-equivariant properties of the moduli space. In the last section we give our main results.

## 2. Notations and preliminaries

We assume that we have completely every Banach spaces with suitable Sobolev norms. Let $S = S^+ \oplus S^-$ denote the decomposition of the spinor bundle into the positive and negative spinor bundles. Let $D : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator, and $\rho : \Lambda^*_C \to End_C(S)$ be the Clifford multiplication. The Seiberg-Witten equations are for a pair $(a, \phi) \in \Omega^1(X, \sqrt{-1}R) \times \Gamma(S^+)$ and they are

$$D\phi + \rho(a)\phi = 0, \qquad \rho(d^+a) - \phi \otimes \phi^* + \frac{1}{2}|\phi|^2 id = 0, \qquad d^*a = 0.$$

Let $\quad V = \Gamma(\sqrt{-1}\Lambda^1 \oplus S^+), \quad W' = (S^- \oplus \sqrt{-1}su(S^+) \oplus \sqrt{-1}\Lambda^0).$

We can think of the equation as the zero set of a map

$$\mathcal{D} + \mathcal{Q} : \ V \to W,$$

where $\mathcal{D}(a, \phi) = (D\phi, \rho(d^+a), d^*a))$, $\mathcal{Q}(a, \phi) = (\rho(a)\phi, \phi \otimes \phi^* - \frac{1}{2}|\phi|^2 id, 0)$, and $W$ is defined to be the orthogonal complement to the constant functions in $W'$.

Now it is time to describe the group of symmetries of the equations. Define $Pin(2) \subset SU(2)$ to be the normalizer of $S^1 \subset SU(2)$. Regarding $SU(2)$ as the group of unit quaternions and taking $S^1$ to be elements of the form $e^{\sqrt{-1}\theta}$, $Pin(2)$ then consists of the form $e^{\sqrt{-1}\theta}$ or $e^{\sqrt{-1}\theta}J$. We define the action of $Pin(2)$ on $V$ and $W$ as follows: Since $S^+$ and $S^-$ are $SU(2)$ bundles, $Pin(2)$ naturally acts on $\Gamma(S^{\pm})$ by multiplication on the left. $Z/2$ acts on $\Gamma(\Lambda^*_C)$ by multiplication by $\pm 1$ and this pulls back to an action of $Pin(2)$ by the natural map $Pin(2) \to Z/2$. A calculation shows that this pullback also describes the induced action of $Pin(2)$ on $\sqrt{-1}su(S^+)$. Both $\mathcal{D}$ and $\mathcal{Q}$ are seen to be $Pin(2)$ equivariant maps.

If $X$ is a smooth closed spin 4-manifold. Suppose that $X$ admits a spin structure preserving action by a compact Lie group (or finite group) $G$. We may assume a Riemannian metric on $X$ so that $G$ acts by isometries. If the action is of even type, both $\mathcal{D}$ and $\mathcal{Q}$ are $\tilde{G} = Pin(2) \times G$ equavariant maps.

Now we define $V_\lambda$ to be the subspace of $V$ spanned by the eigenspaces $\mathcal{D}^*\mathcal{D}$ with eigenvalues less than or equal to $\lambda \in R$. Similarly, define $W_\lambda$ using $\mathcal{D}\mathcal{D}^*$. The virtual $G$-representation $[V_\lambda \otimes C] - [W_\lambda \otimes C] \in R(\tilde{G})$ is the $\tilde{G}$-index of $\mathcal{D}$ and can be determined by the $\tilde{G}$-index and is independent of $\lambda \in R$, where $R(\tilde{G})$ is the complex representation of $\tilde{G}$. In particular, since $V_0 = KerD$ and $W_0 = CokerD \oplus Cokerd^+$, we have

$$[V_\lambda \otimes C] - [W_\lambda \otimes C] = [V_0 \otimes C] - [W_0 \otimes C] \in R(\tilde{G}).$$

Note that $Cokerd^+ = H^2_+(X, R)$.

Now let $Z_6 =< \xi >$ be a cyclic group of order 6 generated by $\xi$. Since $Z_6$ is an Abel group, there are 6 irreducible representations of degree 1. Thus we have the following character table for $Z_6$ [15]:

where $\omega = e^{2\pi i/6} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and satisfies $\omega^2 - \omega = -1$.

## 3. The index of $\mathcal{D}$ and the character formula for the $K$-theory degree

The virtual representation $[V_{\lambda,C}] - [W_{\lambda,C}] \in R(\tilde{G})$ is the same as $Ind(\mathcal{D}) = [ker\mathcal{D}] - [Coker\mathcal{D}]$. Furuta determines $Ind(\mathcal{D})$ as a $Pin(2)$ representation; de-

| | 1 | $\xi$ | $\xi^2$ | $\xi^3$ | $\xi^4$ | $\xi^5$ |
|---|---|---|---|---|---|---|
| $\chi_0$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_1$ | 1 | $\omega$ | $\omega^2$ | $-1$ | $-\omega$ | $-\omega^2$ |
| $\chi_2$ | 1 | $\omega^2$ | $-\omega$ | 1 | $\omega^2$ | $-\omega$ |
| $\chi_3$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| $\chi_4$ | 1 | $-\omega$ | $\omega^2$ | 1 | $-\omega$ | $\omega^2$ |
| $\chi_5$ | 1 | $-\omega^2$ | $-\omega$ | $-1$ | $\omega^2$ | $\omega$ |

noting the restriction map $r: R(\tilde{G}) \to R(Pin(2))$, Furuta shows

$$r(Ind(\mathcal{D})) = 2kh - m\tilde{1}$$

where $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Thus $Ind(\mathcal{D}) = sh - t\tilde{1}$, where $s$ and $t$ are polynomials such that $s(1) = 2k$ and $t(1) = m$. For a spin even $Z_6$ action, $\tilde{G} = Pin(2) \times Z_6$, we can write

$$s(\eta) = a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + e_0\eta^4 + f_0\eta^5,$$

and

$$t(\eta) = a_1 + b_1\eta + c_1\eta^2 + d_1\eta^3 + e_1\eta^4 + f_1\eta^5,$$

so that $a_0+b_0+c_0+d_0+e_0+f_0 = 2k$ and $a_1+b_1+c_1+d_1+e_1+f_1 = m = b_2^+(X)$.

As an element of $R(Z_6)$, we know that $Ind_{Z_6}D = \overline{Ind_{Z_6}D}$, so from $Ind_{Z_6}D = a_0+b_0\eta+c_0\eta^2+d_0\eta^3+e_0\eta^4+f_0\eta^5$ we have $b_0 = f_0$ and $c_0 = e_0$. Similarly, since $H^+(X,C) = \overline{H^+(X,C)}$, so from $H^+(X,C) = a_1+b_1\eta+c_1\eta^2+d_1\eta^3+e_1\eta^4+f_1\eta^5$ we have $b_1 = f_1$ and $c_1 = e_1$. Thus, we have

$$s(\eta) = a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + c_0\eta^4 + b_0\eta^5,$$

$$t(\eta) = a_1 + b_1\eta + c_1\eta^2 + d_1\eta^3 + c_1\eta^4 + b_1\eta^5,$$

so that $a_0 + 2b_0 + 2c_0 + d_0 = 2k$ and $a_1 + 2b_1 + 2c_1 + d_1 = m = b_2^+(X)$.

Besides, we have

$$dim\,(H^+(X)^{Z_6}) = a_1 = b_2^+(X/Z_6) = b_2^+(X/ < \xi >),$$

$$dim\,(H^+(X)^{<\xi^2>}) = a_1 + d_1 = b_2^+(X/ < \xi^2 >),$$

$$dim\,(H^+(X)^{<\xi^3>}) = a_1 + 2c_1 = b_2^+(X/ < \xi^3 >),$$

The Thom isomorphism theory in equivariant $K$-theory for a general compact Lie group is a deep theory proved using elliptic operator [1]. The subsequent character formula of this section uses only elementary properties of the Bott class.

Let $V$ and $W$ be complex $\Gamma$ representations for some compact Lie group $\Gamma$. Let $BV$ and $BW$ denote balls in $V$ and $W$ and let $f: BV \to BW$ be a $\Gamma$-map preserving the boundaries $SV$ and $SW$. $K_\Gamma(V)$ is by definition $K_\Gamma(BV, SV)$, and by the equivariant Thom isomorphism theorem, $K_\Gamma(V)$ is a free $R(\Gamma)$ module with generator of the Bott class $\lambda(V)$. Applying the $K$-theory functor to $f$ we get a map

$$f^* : K_\Gamma(W) \to K_\Gamma(V)$$

which defines a unique element $\alpha_f \in R(\Gamma)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$. The element $\alpha_f$ is called the $K$-theory degree of $f$.

Let $V_g$ and $W_g$ denote the subspaces if $V$ and $W$ fixed by an element $g \in \Gamma$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g : V_g \to W_g$ be the restriction of $f$ and let $d(f^g)$ denote the ordinary topological degree of $f^g$ (by definition, $d(f^g) = 0$ if $dim\, V_g \neq dim\, W_g$). For any $\beta \in R(\Gamma)$, let $\lambda_{-1}\beta$ denote the alternating sum $\Sigma(-1)^i \lambda^i \beta$ of exterior powers.

Tom Dieck proves the following character formula for the degree $\alpha_f$:

**Theorem 3.1. ([3])** *Let $f : BV \to BW$ be a $\Gamma$-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the $K$-theory degree. Then*

$$tr_g(\alpha_f) = d(f^g) tr_g(\lambda_{-1}(W_g^\perp - V_g^\perp))$$

*where $tr_g$ is the trace of the action of an element $g \in \Gamma$.*

This formula is very useful in the case where $dim\, V_g \neq dim\, W_g$ so that $d(f^g) = 0$.

Recall that $\lambda_{-1}(\Sigma_i a_i r_i) = \prod_i (\lambda_{-1} r_i)^{a_i}$ and that for a one-dimensional representation $r$, we have $\lambda_{-1} r = (1 - r)$. A two-dimensional representation such as $h$ has $\lambda_{-1} h = (1 - h + \Lambda^2 h)$. In this case, since $h$ comes from an $SU(2)$ representation, $\Lambda^2 h = det\, h = 1$ so $\lambda_{-1} h = (2 - h)$.

In the following we use the character formula to examine the $K$-theory degree $\alpha_{f_\lambda}$ of the map $f_\lambda : BV_{\lambda,C} \to BW_{\lambda,C}$ coming from the Seiberg-Witten equations. We will abbreviate $\alpha_{f_\lambda}$ as $\alpha$ and $V_{\lambda,C}$ and $W_{\lambda,C}$ as just $V$ and $W$. Let $\phi \in S^1 \subset Pin(2) \subset G$ be the element generating a dense subgroup of $S^1$, and recall that there is the element $J \in Pin(2)$ coming from the quaternion. Note that the action of $J$ on $h$ has two invariant subspaces on which $J$ acts by multiplication with $\sqrt{-1}$ and $-\sqrt{-1}$.

## 4. The main results

Consider $\alpha = \alpha_{f_\lambda} \in R(Pin(2) \times Z_6)$, it has the following form

$$\alpha = \alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i.$$

where $\alpha_i = l_i + m_i\eta + n_i\eta^2 + p_i\eta^3 + q_i\eta^4 + r_i\eta^5,\ i \geq 0$ and $\tilde{\alpha_0} = \tilde{l_0} + \tilde{m_0}\eta + \tilde{n_0}\eta^2 + \tilde{p_0}\eta^3 + \tilde{q_0}\eta^4 + \tilde{r_0}\eta^5$.

Since $\phi$ acts non-trivially on $h$ and trivially on $\tilde{1}$ , $\xi^2$ acts trivially on $a_1$ and $d_1\xi^3$ and non-trivially on the others, then we have

$$dim(V(\eta)h)_{\phi\xi^2} - dim(W(\eta)\tilde{1})_{\phi\xi^2} = -(a_1 + d_1) = -b_2^+(X/ < \xi^2 >).$$

So if $b_2^+(X/<\xi^2>) \neq 0$, $tr_{\phi\xi^2}\alpha = 0$.

Since $\phi\xi^3$ acts non-trivially on $V(\eta)h$ and $\xi^3$ acts trivially on $a_1$, $c_1\eta^2$ and $c_1\eta^4$ but non-trivially on the others, then we have

$$dim(V(\eta)h)_{\phi\xi^3} - dim(W(\eta)\tilde{1})_{\phi\xi^3} = -(a_1+2c_1) = -b_2^+(X/<\xi^3>).$$

So if $a_1+2c_1 = b_2^+(X/<\xi^3>) \neq 0$, $tr_{\phi\xi^3}\alpha = 0$.

Since $\phi\xi$ acts non-trivially on $V(\eta)h$, and trivially only on $a_1\tilde{1}$ in $W(\eta)\tilde{1}$, then we have

$$dim(V(\eta))_{\phi\xi} - dim(W(\eta))_{\phi\xi} = -a_1 = -b_2^+(X/<\xi>).$$

So if $a_1 = b_2^+(X/\xi) \neq 0$, $tr_{\phi\xi}\alpha = 0$.

From the above analysis, we know if $b_2^+(X/\xi) \neq 0$ that is $a_1 \neq 0$, we have $tr_{\phi\xi}\alpha = tr_{\phi\xi^2}\alpha = tr_{\phi\xi^3}\alpha = 0$, which implies that

$$\begin{aligned}
0 &= tr_{\phi\xi^2}\alpha = tr_{\xi^2}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\phi^i+\phi^{-i})) \\
&= tr_{\xi^2}\alpha_0 + tr_{\xi^2}\tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} tr_{\xi^2}\alpha_i(\phi^i+\phi^{-i}) \\
&= (l_0 + m_0\omega^2 - n_0\omega + p_0 + q_0\omega^2 - r_0\omega) + (\tilde{l_0} + \tilde{m_0}\omega^2 - \tilde{n_0}\omega \\
&\quad + \tilde{p_0} + \tilde{q_0}\omega^2 - \tilde{r_0}\omega) + \sum_{i=1}^{\infty} tr_{\xi^2}\alpha_i(\phi^i+\phi^{-i}),
\end{aligned}$$

$$\begin{aligned}
0 &= tr_{\phi\xi}\alpha = tr_{\xi}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i)(\phi^i+\phi^{-i}) \\
&= tr_{\xi}\alpha_0 + tr_{\xi}\tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} tr_{\xi}\alpha_i(\phi^i+\phi^{-i}) \\
&= (l_0 + m_0\omega + n_0\omega^2 - p_0 - q_0\omega - r_0\omega^2) + (\tilde{l_0} + \tilde{m_0}\omega + \tilde{n_0}\omega^2 \\
&\quad - \tilde{p_0} - \tilde{q_0}\omega - \tilde{r_0}\omega^2) + \sum_{i=1}^{\infty} tr_{\xi}\alpha_i(\phi^i+\phi^{-i}),
\end{aligned}$$

and so on. From these equations we have $\alpha_0 = -\tilde{\alpha_0}$ and $\alpha_i = 0, i > 0$, that is $\alpha = \alpha_0(1-\tilde{1})$.

Next we calculate $tr_J\alpha$. Since $J$ acts non-trivially on both $h$ and $\tilde{1}$, $dimV_J = dimW_J = 0$, so $d(f^J) = 1$ and the character formula gives $tr_J(\alpha) = tr_J(\lambda_{-1}(m\tilde{1} - 2kh) = tr_J((1-\tilde{1})^m(2-h)^{-2k}) = 2^{m-2k}$ using $tr_J h = 0$ and $tr_J\tilde{1} = -1$.

Now we calculate $tr_{J\xi^2}\alpha$. Since $J\xi^2$ acts non-trivially on both $V(\eta)h$ and

$W(\eta)\tilde{1}$, so $d(f^{J\xi^2}) = 1$. By Tom Dieck formula, we have

$$\begin{aligned} tr_{J\xi^2}(\alpha) &= tr_{J\xi^2}[\lambda_{-1}(a_1 + b_1\eta + c_1\eta^2 + d_1\eta^3 + c_1\eta^4 + b_1\eta^5)\tilde{1} \\ &\quad -\lambda_{-1}(a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + c_0\eta^4 + b_0\eta^5)h] \\ &= 2^{a_1}(1+\omega^2)^{b_1}(1-\omega)^{c_1}2^{d_1}(1+\omega^2)^{c_1}(1-\omega)^{b_1} \\ &\quad 2^{-a_0}(1+\omega^2)^{-b_0}(1-\omega)^{-c_0}2^{-d_0}(1+\omega^2)^{-c_0}(1-\omega)^{-b_0} \\ &= 2^{(a_1+d_1)-(a_0+d_0)} \end{aligned}$$

Here the 2-dimensional representation $h$ decomposes into two complex lines on which $J$ acts as $\sqrt{-1}$ and $-\sqrt{-1}$. Besides, $J$ acts on $\tilde{1}$ as $-1$. And $\xi^2$ acts on the 1-dimensional representation $\eta$, $\eta^2$, $\eta^3$, $\eta^4$ and $\eta^5$ as $\omega^2$, $-\omega$, 1, $\omega^2$ and $-\omega$.

Since $\xi^3$ acts trivially on $\eta^2$ and $\eta^4$ but acts on $\eta$, $\eta^3$, $\eta^5$ all as $-1$, which combines with the action of $J$, then tells us that

$$dim(V(\eta)h)_{J\xi^3} - dim(W(\eta)\tilde{1})_{J\xi^3} = -(2b_1 + d_1) = -(b_2^+(X) - b_2^+(X/<\xi^3>))$$

So, if $2b_1 + d_1 \neq 0$, that is $b_2^+(X) \neq b_2^+(X/<\xi^3>)$, then $tr_{J\xi^3}\alpha = 0$

By direct calculation, we have

$$tr_J\alpha_0 = l_0 + m_0 + n_0 + p_0 + q_0 + r_0 = 2^{m-2k-1}, \tag{3}$$

$$tr_{\xi^2}\alpha_0 = l_0 + m_0\omega^2 - n_0\omega + p_0 + q_0\omega^2 - r_0\omega = 2^{(a_1+d_1)-(a_0+d_0)-1}, \tag{4}$$

$$tr_{\xi^3}\alpha_0 = l_0 - m_0 + n_0 - p_0 + q_0 - r_0 = 0. \tag{5}$$

Here we use $tr_{Jg}\alpha = tr_g(2\cdot\alpha_0) = 2\cdot tr_g\alpha_0$ where $g$ is any element of $Z_6$.

From (3) and (5) we get $l_0 + n_0 + q_0 = 2^{m-2k-2}$. So, we have the following main result.

**Theorem 1.** *Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. If the cyclic group $Z_6$ acts on $X$ as spin even type, then $2k+2 \leq m$ if $b_2^+(X/<\xi>) \neq 0$ and $b_2^+(X) \neq b_2^+(X/<\xi^3>)$.*

On the other hand, if $b_2^+(X) = b_2^+(X/<\xi^3>)$, i.e., $2b_1 + d_1 = 0$ that is $b_1 = d_1 = 0$, then from the action of $J\xi^3$ and (3), we easily get $l_0 + n_0 + q_0 = 2^{m-2k-1}$, which means $m \geq 2k+1$, and this was proved in Theorem 1.1 by Furuta under a more weaker condition.

Since $\xi$ acts on $\eta^3$ as $-1$, we have

$$dim(V(\eta)h)_{J\xi} - dim(W(\eta)\tilde{1})_{J\xi} = -d_1 = -(b_2^+(X/<\xi^2>) - b_2^+(X/<\xi>)).$$

If $d_1 = b_2^+(X/<\xi^2>) - b_2^+(X/<\xi>) \neq 0$, then $tr_{J\xi}\alpha = 0$.

Then, by direct calculation we have

$$tr_\xi\alpha_0 = l_0 + m_0\omega + n_0\omega^2 - p_0 - q_0\omega - r_0\omega^2 \tag{6}$$

From (4) and (6), we obtain $l_0 = p_0$, $m_0 = n_0 = q_0 = r_0$. Thus (3) and (4) become

$$\begin{aligned} l_0 + 2n_0 &= 2^{m-2k-2} && (7) \\ 2l_0 + 2n_0\omega^2 - 2n_0\omega &= 2^{(a_1+d_1)-(a_0+d_0)-1} && (8) \end{aligned}$$

From the above we get

$$n_0 = \frac{2^{m-2k-2} - 2^{(a_1+d_1)-(a_0+d_0)-2}}{3} \quad , \quad l_0 = \frac{2^{m-2k-2} + 2^{(a_1+d_1)-(a_0+d_0)-1}}{3}$$

Since $n_0 \in Z$, then $2^{m-2k-2} - 2^{(a_1+d_1)-(a_0+d_0)-2} \in 3Z \subset Z$. From Theorem 1, we know $2^{m-2k-2} \in Z$. So $2^{(a_1+d_1)-(a_0+d_0)-2} \in Z$, i.e., $a_1+d_1 \geq (a_0+d_0)-2$. Hence, we have the following proposition.

**Proposition 2.** *Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If $X$ admits a spin $Z_6$ action of even type, then*

$$b_2^+(X/<\xi^2>) \geq dim((Ind_{Z_6} D)^{<\xi^2>}) + 2$$

*if $b_2^+(X/<\xi>) \neq 0$, $b_2^+(X) \neq b_2^+(X/<\xi^3>)$ and $b_2^+(X/<\xi^2>) \neq b_2^+(X/<\xi>)$. Moreover, under this condition, the K-theory degree $\alpha = \alpha_0(1-\tilde{1})$ for some $\alpha_0 = l_0(1+\eta^3) + m_0\eta(1+\eta+\eta^3+\eta^4)$.*

In fact F. Fang obtained the following equivalent version of Furuta's $\frac{10}{8}$-theorem

**Proposition 3.** *(Fang [6]). Let $X$ be a smooth closed spin $G$-manifold of dimension 4, where $G$ is compact. Suppose that $b_1(X) = 0$ and $\sigma(X) \leq 0$. If the $G$-action is of even type so that $ind^G(D) \neq 0$, then*

$$b_2^+(X/G) \geq ind^G(D) + 1,$$

*where $ind^G(D) = dim(ker\, D)^G - dim(coker\, D)^G$.*

On the other hand, if $b_2^+(X/<\xi>) = b_2^+(X/\xi^2)$, i.e., $d_1 = 0$, then $J\xi$ acts non-trivially on both $V(\eta)h$ and $W(\eta)\tilde{1}$, we have $dim(V(\eta)h)_{J\xi} = dim(W(\eta)\tilde{1})_{J\xi}$. From Tom Dieck formula, we have

$$\begin{aligned} tr_{J\xi}\alpha &= tr_{J\xi}[\lambda_{-1}(a_1 + b_1\eta + c_1\eta^2 + c_1\eta^4 + b_1\eta^5)\tilde{1} - \\ &\quad \lambda_{-1}(a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + c_0\eta^4 + b_0\eta^5)h] \\ &= 2^{a_1}(1+\omega)^{b_1}(1+\omega^2)^{c_1}(1-\omega)^{c_1}(1-\omega^2)^{b_1} \\ &\quad 2^{-a_0}(1+\omega^2)^{-b_0}(1-\omega)^{-c_0}2^{-d_0}(1+\omega^2)^{-c_0}(1-\omega)^{-b_0} \\ &= 2^{a_1-(a_0+d_0)}[(1+\omega)(1-\omega^2)]^{b_1}[(1+\omega^2)(1-\omega)]^{c_1-(b_0+c_0)} \\ &= 2^{a_1-(a_0+d_0)}3^{b_1} \end{aligned}$$

Besides, by direct calculation, we have

$$tr_{J\xi}\alpha = 2[(l_0 - p_0) + (m_0 - q_0)\omega + (n_0 - r_0)\omega^2] \tag{9}$$

So we have $l_0 - p_0 = 2^{a_1-(a_0+d_0)-1}3^{b_1}$, $m_0 = q_0$ and $n_0 = r_0$, for the reason that 1, $\omega$ and $\omega^2$ are linear independent of each other. Thus we get the following proposition.

**Proposition 4.** *Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If $X$ admits a spin $Z_6$ action of even type, then the K-theory degree $\alpha = \alpha_0(1-\tilde{1})$ for some $\alpha_0 = (1+\eta^3)(p_0+m_0\eta+n_0\eta^2)+2^{a_1-(a_0+d_0)-1}3^{b_1}$.*

Now we assume $b_2^+(X/<\xi>) = 0$ and $b_2^+(X) \neq 0$, that is $a_1 = 0$ and $2b_1 + 2c_1 + d_1 \neq 0$. Next we will consider six cases of this condition.

**Case 1.** $b_1 \neq 0$, $a_1 = c_1 = d_1 = 0$

Since $b_2^+(X/<\xi^2>) = a_1 + d_1 = 0$, $dim(V(\eta)h)_{\phi\xi^2} = dim(W(\eta)\tilde{1})_{\phi\xi^2}$, then we have

$$\begin{aligned} tr_{\phi\xi^2}\alpha &= tr_{\phi\xi^2}[\lambda_{-1}(b_1\eta + b_1\eta^5)\tilde{1} - \lambda_{-1}(a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + c_0\eta^4 + b_0\eta^5)h] \\ &= (1+\omega)^{b_1}(1-\omega^2)^{b_1}[(1-\phi)(1-\phi^{-1})]^{-(a_0+d_0)} \\ &\quad [(1+\omega\phi)(1+\omega\phi^{-1})]^{-(b_0+c_0)}[(1-\omega^2\phi)(1-\omega^2\phi^{-1})]^{-(b_0+c_0)} \end{aligned}$$

Since $tr_{\xi^2}\alpha : U(1) \to C$ is a $C^0$-function, $\phi$ is a generic element, so $-(a_0 + d_0) \geq 0$ and $-(b_0 + c_0) \geq 0$.

On the other hand, $IndD = -\frac{\sigma}{8} \in Z$, but we have $IndD = a_0 + 2b_0 + 2c_0 + d_0 \leq 0$, so $a_0 + d_0 = b_0 + c_0 = 0$, and$X$ is homotopic to $\sharp_n S^2 \times S^2$ for some even integer n. Besides, $Ind_{Z_6}D = a_0(1-\eta^3) + b_0\eta(1-\eta-\eta^3+\eta^4)$.

**Case 2.** $c_1 \neq 0$ and $a_1 = b_1 = d_1 = 0$ or $b_1 \neq 0$, $c_1 \neq 0$ and $a_1 = d_1 = 0$

Under the two kinds of conditions, we can obtain the same result as in Case 1.

**Case 3.** $d_1 \neq 0$ and $a_1 = b_1 = c_1 = 0$. Since $b_2^+(X/\xi^3) = a_1 + 2c_1 = 0$, $dim(V(\eta)h)_{J\xi^3} = dim(W(\eta)\tilde{1})_{J\xi^3} = 0$, then by Tom Dieck formula we have

$$\begin{aligned} tr_{\phi\xi^3}\alpha &= tr_{\phi\xi^3}[\lambda_{-1}(d_1\eta^3\tilde{1}) - \lambda_{-1}(a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + c_0\eta^4 + b_0\eta^5)h] \\ &= 2^{d_1}[(1-\phi)(1-\phi^(-1))]^{-(a_0+2c_0)}[(1+\phi)(1+\phi^{-1})]^{-(2b_0+d_0)} \end{aligned}$$

By the same reason as in Case 1, we have $-(a_0 + 2c_0) \geq 0$, $-(2b_0 + d_0) \geq 0$. Since $0 \leq -\frac{\sigma}{8} \leq IndD = a_0 + 2b_0 + 2c_0 + d_0 \leq 0$, then we get $a_0 + 2c_0 = 2b_0 + d_0 = 0$ and $X$ is homotopic to $\sharp_n S^2 \times S^2$ for some integer $n$. Besides, $Ind_{Z_6}D = c_0(-2+\eta^2+\eta^4) + b_0(\eta - 2\eta^3 + \eta^5)$.

**Case 4.** $b_1 \neq 0$, $d_1 \neq 0$ and $a_1 = c_1 = 0$. We can obtain the same result as in Case 3.

**Case 5.** $c_1 \neq 0$, $d_1 \neq 0$ and $a_1 = b_1 = 0$. Since $b_2^+(X/<\xi>) = a_1 = 0$ and $dim(V(\eta)h)_{\phi\xi} = dim(W(\eta)\tilde{1})_{\phi\xi}$, we have $d(f^{\phi\xi}) = 1$. Then the Tom Dieck formula gives us,

$$
\begin{aligned}
tr_{\phi\xi}\alpha = \; & tr_{\phi\xi}[\lambda_{-1}(c_1\eta^2 + d_1\eta^3 + c_1\eta^4)\tilde{1} - \\
& \lambda_{-1}(a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + c_0\eta^4 + b_0\eta^5)h] \\
= \; & (1-\omega^2)^{c_1}2^{d_1}(1+\omega)^{c_1}[(1-\phi)(1-\phi^{-1})]^{-a_0}[(1-\omega\phi)(1-\omega\phi^{-1})]^{-b_0} \\
& [(1-\omega^2\phi)(1-\omega^2\phi^{-1})]^{-c_0}[(1+\phi)(1+\phi^{-1})]^{-d_0}[(1+\omega\phi)(1+\omega\phi^{-1})]^{-c_0} \\
& [(1+\omega^2\phi)(1+\omega^2\phi^{-1})]^{-b_0}
\end{aligned}
$$

By the same reason as in Case 1, we have $a_0 \leq 0$, $b_0 \leq 0$, $c_0 \leq 0$ and $d_0 \leq 0$, so $a_0 = b_0 = c_0 = d_0 = 0$, which means that $Ind_{Z_6}D = 0$.

**Case 6.** $b_1 \neq 0$, $c_1 \neq 0$, $d_1 \neq 0$ and $a_1 = 0$. We can get the same result as in Case 5.

In summary, we have the following result:

**Proposition 5.** *Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If $X$ admits a spin $Z_6$-action of even type and $b_2^+(X/<\xi>) = 0$ and $b_2^+(X) \neq 0$, then as an element of $R(Z_6)$, $Ind_{Z_6}D$ has the following three cases*

**(1).** *When $b_2^+(X/<\xi^2>) = 0$, $Ind_{Z_6}D = a_0(1-\eta^3) + b_0\eta(1-\eta-\eta^3+\eta^4)$ and $X$ is homotopic to $\sharp_n S^2 \times S^2$ for some even integer $n$.*

**(2).** *When $b_2^+(X/<\xi^2>) \neq 0$ and $b_2^+(X/<\xi^3>) = 0$, $Ind_{Z_6}D = c_0(-2+\eta^2+\eta^4) + b_0\eta(\eta - 2\eta^3 + \eta^5)$ and $X$ is homotopic to $\sharp_n S^2 \times S^2$ for some integer $n$.*

**(3).** *When $b_2^+(X/<\xi^3>) \neq 0$, $Ind_{Z_6}D = 0$ and $X$ is homotopic to $\sharp_n S^2 \times S^2$ for some integer $n$.*

**Remark.** If $b_2^+(X) = 0$, by the same method we can also obtain the same result as in Case 6.

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