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DECOMPOSITION OF THE DISTRIBUTION ON BMO SPACE

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Abstract. In this paper, we characterize $\overrightarrow{f}$ so that if the inequality

$$\left| \int_{\mathbb{R}^d} \overrightarrow{f} . (\overline{u} \nabla v - v \nabla \overline{u})\, dx \right| \leq C \|u\|_{\dot{H}^1} \|v\|_{\dot{H}^1}$$

holds for all u, $v \in \mathcal{D}(\mathbb{R}^d)$, then $\overrightarrow{f}$ can be represented in the form

$$\overrightarrow{f} = \nabla g + \text{Div } H$$

where $g \in BMO(\mathbb{R}^d)$, H is a skew-symmetric matrix field such that $H \in BMO(\mathbb{R}^d)^{d^2}$.

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1. Introduction

Recently, S. Gala [5] proved a remarkable theorem to characterize the class of vector fields $\overrightarrow{f}$ which satisfies the commutator inequality

$$\left| \int_{\mathbb{R}^d} \overrightarrow{f} . (\overline{u} \nabla v - v \nabla \overline{u})\, dx \right| \leq C \|u\|_{\dot{H}^1} \|v\|_{\dot{H}^1} \tag{1.1}$$

for all u, $v \in \mathcal{D}(\mathbb{R}^d)$. Here we use Theorem 1 from [5] to decompose $\overrightarrow{f}$ in the form

$$\overrightarrow{f} = \nabla g + \text{Div } H$$

in the distributional sense, where $g \in BMO(\mathbb{R}^d)$, H is a skew-symmetric matrix field such that $H \in BMO(\mathbb{R}^d)^{d^2}$ and Div : $\mathcal{D}'(\mathbb{R}^d)^{d \times d} \to \mathcal{D}'(\mathbb{R}^d)$ is the row divergence operator defined by

$$\text{Div } (h_{i,j}) = \left(\sum_{j=1}^{d} \partial_j h_{i,j} \right)_{i=1}^{d}.$$

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We start with some prerequisites for our main result. Let $\mathcal{D}\left(\mathbb{R}^d\right) = C_0^\infty\left(\mathbb{R}^d\right)$ be the class of all infinitely differentiable, compactly supported complex-valued functions, and let $\mathcal{D}'\left(\mathbb{R}^d\right)$ denote the corresponding space of (complex-valued) distributions.

For $\varphi \in \mathcal{D}'\left(\mathbb{R}^d\right)$, consider the multiplication operator on $\mathcal{D}\left(\mathbb{R}^d\right)$ defined by

$$\langle \varphi u, v\rangle = \langle \varphi, \overline{u} v\rangle, \qquad u, v \in \mathcal{D}\left(\mathbb{R}^d\right), \tag{1.2}$$

where $\langle .,.\rangle$ represents the usual pairing between $\mathcal{D}\left(\mathbb{R}^d\right)$ and $\mathcal{D}'\left(\mathbb{R}^d\right)$. If the sesquilinear form $\langle \varphi .,.\rangle$ is bounded on $\dot{H}^1\left(\mathbb{R}^d\right) \times \dot{H}^1\left(\mathbb{R}^d\right)$:

$$|\langle \varphi u, v\rangle| \leq c \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)} \left\|\nabla v\right\|_{L^2(\mathbb{R}^d)}, \quad u, v \in \mathcal{D}\left(\mathbb{R}^d\right), \tag{1.3}$$

where the constant c is independent of u, v, then $\varphi u \in \dot{H}^{-1}\left(\mathbb{R}^d\right)$, where $\dot{H}^{-1}\left(\mathbb{R}^d\right) = \left(\dot{H}^1\left(\mathbb{R}^d\right)\right)^*$ is a dual Sobolev space, and the multiplication operator can be extended by continuity to all of the space $\dot{H}^1\left(\mathbb{R}^d\right)$. Here, the space $\dot{H}^1\left(\mathbb{R}^d\right)$ is defined as the completion of (complex-valued) $\mathcal{D}\left(\mathbb{R}^d\right)$ functions with respect to the norm $\left\|u\right\|_{\dot{H}^1(\mathbb{R}^d)} = \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)}$. As usual, this extension is also denoted by φ. By the polarization identity, (1.3) is equivalent to the boundedness of the corresponding quadratic form :

$$|\langle \varphi u, u\rangle| = \left|\left\langle \varphi, |u|^2\right\rangle\right| \leq c \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)}^2, \quad u \in \mathcal{D}\left(\mathbb{R}^d\right) \tag{1.4}$$

where the constant c is independent of u. If φ is a (complex-valued) Borel measure on $\mathbb{R}^d$, then (1.4) can be recast in the form

$$\int_{\mathbb{R}^d} |u(x)|^2 \, d\varphi(x) \leq c \left\|u\right\|_{\dot{H}^1}^2, \quad u \in \mathcal{D}\left(\mathbb{R}^d\right)$$

which has been studied in a comprehensive way. We refer to [2], [3], [6], [8], where different analytic conditions for the so-called trace inequalities of this type can be found.

$\mathcal{H}^1\left(\mathbb{R}^d\right)$ is the Hardy space in the sense of Fefferman and Stein [4] and $BMO\left(\mathbb{R}^d\right)$ is the John-Nirenberg space. $BMO\left(\mathbb{R}^d\right)$ is the Banach space modulo constants with the norm $\left\|.\right\|_*$ defined by

$$\left\|b\right\|_* = \sup_{x \in \mathbb{R}^d} \frac{1}{|Q|} \int_Q |b(y) - m_Q(b)| \, dy$$

where

$$m_Q(b) = \frac{1}{|Q|} \int_Q b(y) dy$$

Fefferman and Stein [4] proved that the Banach space dual of $\mathcal{H}^1\left(\mathbb{R}^d\right)$ is isomorphic to $BMO\left(\mathbb{R}^d\right)$, that is,

$$\|b\|_* \approx \sup_{\|f\|\leq 1} \left| \int_{\mathbb{R}^d} b(x)f(x)dx \right|.$$

As a consequence of Theorem 1 in [5], we deduce that if

$$\left| \int_{\mathbb{R}^d} \overrightarrow{f} . \left(\overline{u}\nabla v - v\nabla\overline{u}\right) dx \right| \leq C \|u\|_{\dot{H}^1} \|v\|_{\dot{H}^1} \tag{1.5}$$

holds for all $u,\, v \in \mathcal{D}\left(\mathbb{R}^d\right)$, then $\overrightarrow{f}$ can be decomposed into the form

$$\overrightarrow{f} = \nabla g + \text{Div } H$$

in the distributional sense, where $g \in BMO\left(\mathbb{R}^d\right)$, H is a skew-symmetric matrix field such that $H \in BMO\left(\mathbb{R}^d\right)^{d^2}$ and Div : $\mathcal{D}'\left(\mathbb{R}^d\right)^{d\times d} \to \mathcal{D}'\left(\mathbb{R}^d\right)$ is the row divergence operator defined by

$$\text{Div }\left(h_{i,j}\right) = \left(\sum_{j=1}^{d} \partial_j h_{i,j}\right)_{i=1}^{d}.$$

We now state our main result for arbitrary (complex-valued) distributions $\overrightarrow{f}$.

Theorem 1. *Let $\overrightarrow{f} \in \mathcal{D}'\left(\mathbb{R}^d\right)$ and $d \geq 3$. If (1.5) is satisfied, then*

$$\overrightarrow{f} = \nabla g + Div\ H \tag{1.6}$$

in the distributional sense where

$$g = \Delta^{-1} div\ \overrightarrow{f} \in BMO\left(\mathbb{R}^d\right) \quad \text{and} \quad H = \Delta^{-1} curl\ \overrightarrow{f} \in BMO\left(\mathbb{R}^d\right)^{d^2}. \tag{1.7}$$

Here g and H are defined respectively by

$$g = \lim_{j\to+\infty} g_j, \quad g_j = \Delta^{-1} div\ \left(\varphi_j \overrightarrow{f}\right), \tag{1.8}$$

$$H = \lim_{j\to+\infty} H_j, \quad H_j = \Delta^{-1} curl\ \left(\varphi_j \overrightarrow{f}\right), \tag{1.9}$$

*in terms of the convergence in the weak-*topology of $BMO\left(\mathbb{R}^d\right)$. The above limits do not depend on the choice of φ_j.*

Moreover,

$$\nabla g = \lim_{j\to+\infty} \nabla g_j, \qquad \text{Div } H = \lim_{j\to+\infty} \text{Div } H_j \quad \text{in} \quad \mathcal{D}'\left(\mathbb{R}^d\right), \tag{1.10}$$

$$\text{curl }(\nabla g) = 0,\ \text{div }(\text{Div } H) = 0,\ \Delta g = \text{div } \overrightarrow{f},\ \Delta H = \text{curl } \overrightarrow{f}. \tag{1.11}$$

The proof of Theorem 1 is rather delicate. We shall need several lemmas for proving some a priori estimates.

Lemma 1. *Let $\overrightarrow{f} \in \mathcal{D}'\left(\mathbb{R}^d\right)$. If (1.5) holds, then we have*

$$\left\| div\ \overrightarrow{f} \right\|_{\dot{H}^{-1}(Q)} \leq C\left|Q\right|^{\frac{1}{2}-\frac{1}{d}} \tag{1.12}$$

for every cube Q in $\mathbb{R}^d$ with C independent of Q.

Proof. The proof of this fact is straightforward. Let $v \in \mathcal{D}(Q)$ be given and let u be a function in $\mathcal{D}(Q)$ such that $u = 1$ on supp v. Then the following estimate is valid :

$$\begin{aligned} \left|\left\langle \overrightarrow{f}, \overline{u}\nabla v - v\nabla\overline{u}\right\rangle\right| &= \left|\left\langle \overrightarrow{f}, \nabla v\right\rangle\right| = \left|\left\langle \text{div } \overrightarrow{f}, v\right\rangle\right| \\ &\leq C(d)\left\|\nabla u\right\|_{L^2(\mathbb{R}^d)}\left\|\nabla v\right\|_{L^2(Q)}. \end{aligned}$$

Taking the infinimum over all such u on the right-hand side, we get

$$\left|\left\langle \text{div } \overrightarrow{f}, v\right\rangle\right| \leq C\ \sqrt{\text{cap}\,(Q)}\left\|\nabla v\right\|_{L^2(Q)}$$

where the capacity of a compact set $e \subset \mathbb{R}^d$ cap(.) is defined by ([7], sect. 11.15), (see also [1]) :

$$\text{cap }(e) = \inf\left\{\left\|u\right\|^2_{\dot{H}^1(\mathbb{R}^d)} : u \in \mathcal{D}\left(\mathbb{R}^d\right),\quad u \geq 1 \text{ on } e\right\}.$$

Since for a cube Q in $\mathbb{R}^d$,

$$\text{cap}\,(Q) \simeq \left|Q\right|^{1-\frac{2}{d}}$$

the proof of lemma is complete. □

In order to prove our main result, the following lemma will be used.

Lemma 2. *Let $\overrightarrow{f} \in \mathcal{D}'\left(\mathbb{R}^d\right)$. If (1.5) holds, we then have*

$$\left\|\overrightarrow{f}\right\|_{\dot{H}^{-1}(Q)} \leq C\left|Q\right|^{\frac{1}{2}} \tag{1.13}$$

for every cube Q in $\mathbb{R}^d$ with C independent of Q.

Proof. Let Q^* be the cube with the same center as Q but with the side lenght twice as long. Suppose that $v \in \mathcal{D}(Q)$ and let φ be a C^∞ function taking values in $[0,1]$ with support in Q^* and so that $\varphi = 1$ on Q. Let us set $u = (x_i - a_i)\,\varphi$ $\left(i = \overline{1,d}\right)$, where $a = (a_i)$ is the center of Q. Then it is easy to see that

$$\left\|\nabla u\right\|_{L^2(Q^*)} \leq \left\|\nabla u\right\|_{L^2(Q)} \leq C\left|Q\right|^{\frac{1}{2}}.$$

Next note that for such u and v

$$\begin{aligned}\left\langle \overrightarrow{f}, \overline{u}\nabla v - v\nabla \overline{u}\right\rangle &= \left\langle \overrightarrow{f}, \nabla\left(\overline{u}v\right) - 2v\nabla\overline{u}\right\rangle \\ &= -\left\langle \text{div } \overrightarrow{f}, \overline{u}v\right\rangle - 2\left\langle \overrightarrow{f}, v\nabla\overline{u}\right\rangle \\ &= -\left\langle \text{div } \overrightarrow{f}, \left(x_i - a_i\right)v\right\rangle - 2\left\langle f_i, v\right\rangle.\end{aligned}$$

Concerning $\left\langle \text{div } \overrightarrow{f}, (x_i - a_i)\, v\right\rangle$, we observe that by using (1.12), the Poincaré inequality with v replaced by $(x_i - a_i)\, v$

$$\begin{aligned}\left|\left\langle \text{div } \overrightarrow{f}, (x_i - a_i)\, v\right\rangle\right| &\leq C\left|Q\right|^{\frac{1}{2}-\frac{1}{d}}\left\|\nabla\left[(x_i - a_i)\, v\right]\right\|_{L^2(Q)} \\ &\leq C\left|Q\right|^{\frac{1}{2}-\frac{1}{d}}\left(\|v\|_{L^2(Q)} + \|(x_i - a_i)\nabla v\|_{L^2(Q)}\right) \\ &\leq C\left|Q\right|^{\frac{1}{2}-\frac{1}{d}}\left(2\left|Q\right|^{\frac{1}{d}}\|\nabla v\|_{L^2(Q)} + \|(x_i - a_i)\nabla v\|_{L^2(Q)}\right) \\ &\leq C\left|Q\right|^{\frac{1}{2}}\|\nabla v\|_{L^2(Q)}, \quad \forall v \in \mathcal{D}(Q).\end{aligned}$$

Since for every $i = \overline{1, d}$,

$$\begin{aligned}2\left|\left\langle f_i, v\right\rangle\right| &\leq \left|\left\langle \overrightarrow{f}, \overline{u}\nabla v - v\nabla\overline{u}\right\rangle\right| + \left|\left\langle \text{div } \overrightarrow{f}, (x_i - a_i)\, v\right\rangle\right| \\ &\leq C\|\nabla u\|_{L^2(2Q)}\|\nabla v\|_{L^2(Q)} + C\left|Q\right|^{\frac{1}{2}}\|\nabla v\|_{L^2(Q)} \\ &\leq C\left|Q\right|^{\frac{1}{2}}\|\nabla v\|_{L^2(Q)},\end{aligned}$$

and we can conclude. □

For a fixed cube Q in $\mathbb{R}^d$, we denote by $\{\omega_j\}_{j=0}^{\infty}$ a smooth partition of unity associated with Q, i.e., fix $\omega_0 \in \mathcal{D}(2Q)$ with the properties $\omega_j \in \mathcal{D}\left(2^{j+1}Q \backslash 2^{j-1}Q\right)$, $j \geq 1$ so that

$$0 \leq \omega_j(x) \leq 1, \quad |\nabla\omega_j(x)| \leq C\left(2^j l(Q)\right)^{-1}, \;\; j \in \mathbb{N} \tag{1.14}$$

where $l(Q)$ denotes the side lenght of Q and C depends only on d. Finally, we have for all $x \in \mathbb{R}^d$,

$$\sum_{j=0}^{\infty}\omega_j(x) = 1.$$

In the following $\mathcal{R}_i$ $\left(\text{resp. } \mathcal{R}_{i,m} = -\partial_i\partial_m\Delta^{-1}\right)$ $(i, m = 1, ..., d)$ denotes the Riesz transforms (resp. the double Riesz transforms) on $\mathbb{R}^d$ (see [9]) which are given respectively up to a constant multiple by

$$K_i(x-y) = \frac{(x_i - y_i)}{|x-y|^d}, \quad K_{i,m}(x-y) = \frac{|x-y|^2 - d^{-1}(x_i - y_i)(x_m - y_m)}{|x-y|^{d+2}}.$$

From this we derive

Lemma 3. *The following estimates hold.*

(i) *For every $v \in \mathcal{D}(Q)$ and $j \geq 0$*

(1.15)
$$\left\|\nabla\left(\omega_j \partial_i \partial_m \Delta^{-1} v\right)\right\|_{L^2(2^{j+1}Q)} \leq C 2^{-j\left(1+\frac{d}{2}\right)} \|\nabla v\|_{L^2(Q)}, \quad i, m = 1, ..., d$$

where C depends only on d.

(ii) *For every $v \in \mathcal{D}(Q)$ such that $\int_Q v dx = 0$ and $j \geq 2$*

(1.16)
$$\left\|\nabla\left(\omega_j \partial_i \Delta^{-1} v\right)\right\|_{L^2(2^{j+1}Q)} \leq C 2^{-j\left(1+\frac{d}{2}\right)} |Q|^{-\frac{1}{2}} \|\nabla v\|_{L^1(Q)}, \quad i = 1, ..., d$$

where C depends only on d.

Proof. To prove (1.15), let $v \in \mathcal{D}(Q)$ and let $a = a_Q$ be the center of Q and $\rho = l(Q)$ its side lenght. For $j = 0, 1$, it follows from Poincaré' s inequality, the boundedness of $\mathcal{R}_{i,m}$ on $L^2\left(\mathbb{R}^d\right)$, that

$$\begin{aligned}
\left\|\nabla\left(\omega_j \partial_i \partial_m \Delta^{-1} v\right)\right\|_{L^2(2^{j+1}Q)} &\leq \left\|\nabla \omega_j \left(\partial_i \partial_m \Delta^{-1} v\right)\right\|_{L^2(2^{j+1}Q)} \\
&\quad + \left\|\omega_j \partial_i \partial_m \left(\Delta^{-1} \nabla v\right)\right\|_{L^2(2^{j+1}Q)} \\
&\leq C\left(\rho^{-1} \|\mathcal{R}_{i,m} v\|_{L^2(\mathbb{R}^d)} + \|\mathcal{R}_{i,m} \nabla v\|_{L^2(\mathbb{R}^d)}\right) \\
&\leq C\left(\rho^{-1} \|v\|_{L^2(Q)} + \|\nabla v\|_{L^2(Q)}\right) \\
&\leq C \|\nabla v\|_{L^2(Q)}.
\end{aligned}$$

On the other hand, we have for $j \geq 2$,

$$|K_i(x-y) - K_i(x-a)| \leq C(d) \frac{|y-a|}{|x-y|^d}, \tag{1.17}$$

$$|K_{i,m}(x-y) - K_{i,m}(x-a)| \leq C(d) \frac{|y-a|}{|x-y|^{d+1}}, \tag{1.18}$$

if $|y-a| < R$, $|y-a| > 2R$. Using the preceding estimates with $R = c(d) 2^j \rho$, we see that for $x \in 2^{j+1}Q \backslash 2^{j-1}Q$:

$$\begin{aligned}
\left|\partial_i \partial_m \Delta^{-1} v(x)\right| &= \left|\int_Q \left(K_i(x-y) - K_i(x-a)\right) \partial_m v(y) dy\right| \\
&\leq \int_Q |K_i(x-y) - K_i(x-a)| \, |\nabla v(y)| \, dy \\
&\leq C 2^{-jd} \rho^{1-d} \|\nabla v\|_{L^1(Q)},
\end{aligned}$$

$$\begin{aligned}
\left|\nabla\partial_i\partial_m\Delta^{-1}v(x)\right| &= \left|\int\limits_Q \left(K_{i,m}(x-y)-K_{i,m}(x-a)\right)\partial_m\nabla v(y)dy\right| \\
&\leq \int\limits_Q \left|K_{i,m}(x-y)-K_{i,m}(x-a)\right|\left|\nabla v(y)\right|dy \\
&\leq C2^{-j(d+1)}\rho^{-d}\left\|\nabla v\right\|_{L^1(Q)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|\nabla\left(\omega_j\partial_i\partial_m\Delta^{-1}v\right)\right\|_{L^2(2^{j+1}Q)} &\leq \left\|\nabla\omega_j\left(\partial_i\partial_m\Delta^{-1}v\right)\right\|_{L^2(2^{j+1}Q)} \\
&\quad+\left\|\omega_j\partial_i\partial_m\left(\Delta^{-1}\nabla v\right)\right\|_{L^2(2^{j+1}Q)} \\
&\leq C2^{-j\left(\frac{d}{2}+1\right)}\rho^{-\frac{d}{2}}\left\|\nabla v\right\|_{L^1(Q)} \\
&\leq C2^{-j\left(1+\frac{d}{2}\right)}\left\|\nabla v\right\|_{L^2(Q)},
\end{aligned}$$

which gives (1.15).

The proof of (1.16) for $j\geq 2$, provided $\int\limits_Q vdx=0$, is similar to that of (1.15). Using the estimates (1.17) and (1.18) we deduce that for $x\in 2^{j+1}Q\backslash 2^{j-1}Q$,

$$\begin{aligned}
\left|\nabla\left(\omega_j\partial_i\Delta^{-1}v\right)(x)\right| &\leq \left|\nabla\omega_j(x)\right|\left|\partial_i\Delta^{-1}v(x)\right|+\left|\omega_j(x)\right|\left|\nabla\partial_i\Delta^{-1}v(x)\right| \\
&\leq C2^{-j}\rho^{-1}\int\limits_Q\left|G_i(x-y)-G_i(x)\right|\left|v(y)\right|dy \\
&\quad+C\sum_{m=1}^{d}\int\limits_Q\left|G_{i,m}(x-y)-G_{i,m}(x)\right|\left|v(y)\right|dy \\
&\leq C2^{-j(1+d)}\left|Q\right|^{-1}\int\limits_Q\left|v(y)\right|dy.
\end{aligned}$$

This yields

$$\left\|\nabla\left(\omega_j\partial_i\Delta^{-1}v\right)\right\|_{L^2(2^{j+1}Q)}\leq C2^{-j\left(1+\frac{d}{2}\right)}\left|Q\right|^{-\frac{1}{2}}\left\|\nabla v\right\|_{L^2(Q)}.$$

This completes the proof. □

If we want to prepare the scaling argument, we consider a function $\varphi\in\mathcal{D}\left(\mathbb{R}^d\right)$ with the properties

$$0\leq\varphi\leq 1,\quad \varphi(x)=1\ \text{ if }\ |x|\leq 1,\ \ \varphi(x)=0\ \text{ if }\ |x|\geq 2,$$

and define the functions

$$\varphi_j\in\mathcal{D}\left(\mathbb{R}^d\right),\quad \varphi_j(x)=\varphi(j^{-1}x),\quad x\in\mathbb{R}^d,\quad j\in\mathbb{N}.$$

It follows that

$$\lim_{j\to+\infty}\varphi_j(x)=1 \text{ for all } x\in\mathbb{R}^d,$$

and setting

$$B_j=\left\{x\in\mathbb{R}^d:|x|<j\right\},\quad G_j=B_{2j}\backslash\overline{B_j},$$

we get Supp $\nabla\varphi_j\subseteq\overline{G_j}$, Supp $\varphi_j\subseteq\overline{B_j}$, $j\in\mathbb{N}$.

With these notations we obtain

Lemma 4. *Let $\overrightarrow{f}\in\mathcal{D}'\left(\mathbb{R}^d\right)$. Then, we have*

$$\left\|\varphi_j\overrightarrow{f}\right\|_{\dot{H}^{-1}(Q)}\leq C\left|Q\right|^{\frac{1}{2}},$$

for every cube Q in $\mathbb{R}^d$ where C does not depend on Q.

Proof. The proof is straightforward. By using (1.13) with $\overrightarrow{f}$ replaced by $\varphi_j\overrightarrow{f}$ we obtain

$$\left\|\varphi_j\overrightarrow{f}\right\|_{\dot{H}^{-1}(Q)}\leq C\left|Q\right|^{\frac{1}{2}},\tag{1.19}$$

for every cube Q where C does not depend on Q and j. This is a consequence of the inequality

$$\left\|\left(\nabla\varphi_j\right)v\right\|_{L^2(\mathbb{R}^d)}\leq c(d)\left\|\nabla v\right\|_{L^2(\mathbb{R}^d)}$$

for $v\in\mathcal{D}\left(\mathbb{R}^d\right)$, which follows from Poincaré' s inequality. □

Remark 1. *We observe that g_j and H_j given respectively by (1.8) and (1.9) are well-defined in the distributional sense. Moreover, by (1.19), $\varphi_j\overrightarrow{f}\in\dot{H}^{-1}(Q)$ and hence $g_j\in L^2\left(\mathbb{R}^d\right)$, $H_j\in L^2\left(\mathbb{R}^d\right)^{d^2}$.*

Next, we have to show that the following lemma.

Lemma 5. *Let $\overrightarrow{f}\in\mathcal{D}'\left(\mathbb{R}^d\right)$. Then*

$$\left\|\partial_i\partial_m\Delta^{-1}\left(\varphi_j\overrightarrow{f}\right)\right\|_{\dot{H}^{-1}(Q)}\leq C\left|Q\right|^{\frac{1}{2}}\tag{1.20}$$

for all $i,m=1,2,...,d$ with a constant C independent of the cube Q and j.

Proof. We know already that $\partial_i\partial_m\Delta^{-1}\left(\varphi_j\overrightarrow{f}\right)$ is well-defined in $\mathcal{D}'\left(\mathbb{R}^d\right)$. Then

$$\left\langle\partial_i\partial_m\Delta^{-1}\left(\varphi_j\overrightarrow{f}\right),\overrightarrow{v}\right\rangle=\left\langle\varphi_j\overrightarrow{f},\Delta^{-1}\partial_i\partial_m\overrightarrow{v}\right\rangle=\sum_{j=0}^{\infty}\left\langle\varphi_j\overrightarrow{f},\omega_j\Delta^{-1}\partial_i\partial_m\overrightarrow{v}\right\rangle,$$

for every $v \in \mathcal{D}(Q)$, where the sum on the right contains only a finite number of non-zero terms. Therefore, it follows from (1.19), statement (i) of Lemma 3, and Schwarz inequality,

$$\begin{aligned}\left|\left\langle \varphi_j \overrightarrow{f}, \Delta^{-1}\partial_i\partial_m \overrightarrow{v}\right\rangle\right| &\leq \sum_{j=0}^{\infty}\left|\left\langle \varphi_j \overrightarrow{f}, \omega_j\Delta^{-1}\partial_i\partial_m \overrightarrow{v}\right\rangle\right| \\ &\leq c\sum_{j=0}^{\infty} 2^{j\frac{d}{2}}|Q|^{\frac{1}{2}}\left\|\nabla\left(\omega_j\partial_i\partial_m\Delta^{-1}\overrightarrow{v}\right)\right\|_{L^2(2^{j+1}Q)} \\ &\leq C\sum_{j=0}^{\infty} 2^{j\frac{d}{2}}|Q|^{\frac{1}{2}}2^{-j\left(1+\frac{d}{2}\right)}\|\nabla v\|_{L^2(Q)} \\ &\leq C|Q|^{\frac{1}{2}}\|\nabla v\|_{L^2(Q)},\end{aligned}$$

which proves (1.20). In particular,

$$\|\nabla g_j\|_{\dot{H}^{-1}(Q)} \leq C|Q|^{\frac{1}{2}}, \quad \|\mathbf{D}(H_j)\|_{\dot{H}^{-1}(Q)} \leq C|Q|^{\frac{1}{2}}.$$

□

From this, we deduce immediately

Corollary 1. *Let $\overrightarrow{f} \in \mathcal{D}'\left(\mathbb{R}^d\right)$. If (1.5) is satisfied, then*

$$\begin{aligned}\|g_j - m_Q(g_j)\|_{L^2(Q)} &\leq c\|\nabla g_j\|_{\dot{H}^{-1}(Q)} \leq C|Q|^{\frac{1}{2}}, \\ \|H_j - m_Q(H_j)\|_{L^2(Q)} &\leq c\|\mathbf{D}(H_j)\|_{\dot{H}^{-1}(Q)} \leq C|Q|^{\frac{1}{2}},\end{aligned}$$

where $m_Q(g_j)$ (resp. $m_Q(H_j)$) denotes the mean value of g_j (resp. H_j) over Q and C does not depend on Q and j. Hence

$$\sup_j \|g_j\|_{BMO(\mathbb{R}^d)} < \infty \qquad \textit{and} \quad \sup_j \|H_j\|_{BMO(\mathbb{R}^d)^{d^2}} < \infty.$$

We claim that both $\{g_j\}$ and $\{H_j\}$ converge in the weak-*topology of BMO repectively to $f \in BMO\left(\mathbb{R}^d\right)$ and $H \in BMO\left(\mathbb{R}^d\right)^{d^2}$ defined up to an additive constant. We will deduce that

$$\Delta g = \operatorname{div} \overrightarrow{f} \quad \text{and} \quad \Delta H = \operatorname{curl} \overrightarrow{f} \quad \text{in the distributional sense}$$

and set

$$g = \Delta^{-1}\operatorname{div} \overrightarrow{f} \quad \text{and} \quad H = \Delta^{-1}\operatorname{curl} \overrightarrow{f}.$$

Proof. Since $\{g_j\}$ is uniformly bounded in the $BMO-$norm, it is enough to verify that it forms a Cauchy sequence in the weak-*topology of BMO on a dense family of $C_0^\infty-$functions in $\mathcal{H}^1\left(\mathbb{R}^d\right)$. Suppose that $v \in \mathcal{D}(Q)$ and $\int\limits_Q v dx = 0$. Then one can easily check that

$$\left|\int\limits_{\mathbb{R}^d} (g_n - g_m)\,\overline{v}dx\right| \leq \sum_{j\geq n_0} \left|\left\langle (\varphi_n - \varphi_m)\,\overrightarrow{f}, \omega_j \nabla \Delta^{-1} v \right\rangle\right|,$$

where $n_0 \to +\infty$ as $m, n \to +\infty$. By (1.19), it follows that

$$\left|\left\langle (\varphi_n - \varphi_m)\,\overrightarrow{f}, \omega_j \nabla \Delta^{-1} v \right\rangle\right| \leq c 2^{\frac{j}{2}} |Q|^{\frac{1}{2}} \left\|\nabla\left(\omega_j \nabla \Delta^{-1} v\right)\right\|_{L^2(2^j Q)}.$$

By statement (ii) of Lemma 3,

$$\left\|\nabla\left(\omega_j \nabla \Delta^{-1} v\right)\right\|_{L^2(2^j Q)} \leq C 2^{-j\left(1+\frac{d}{2}\right)} |Q|^{-\frac{1}{2}} \|v\|_{L^1(Q)}, \quad j \geq n_0, \tag{1.21}$$

where C does not depend on j, Q and v. Thus, we get

$$\left|\left\langle (\varphi_n - \varphi_m)\,\overrightarrow{f}, \omega_j \nabla\left(\Delta^{-1} v\right) \right\rangle\right| \leq C 2^{-j} \|v\|_{L^1(Q)}, \quad j \geq n_0$$

and consequently

$$\sum_{j\geq n_0} \left|\left\langle (\varphi_n - \varphi_m)\,\overrightarrow{f}, \omega_j \nabla\left(\Delta^{-1} v\right) \right\rangle\right| \leq C \|v\|_{L^1(Q)} \sum_{j\geq n_0} 2^{-j}, \quad j \geq n_0.$$

Using the preceding inequalities and letting $m, n \to +\infty$ so that $n_0 \to +\infty$, it follows that $\{g_j\}$ is a Cauchy sequence in the weak-*topology of BMO which implies in particular,

$$\lim_{j\to+\infty} \int\limits_{\mathbb{R}^d} g_j \overline{v} dx = \int\limits_{\mathbb{R}^d} g \overline{v} dx, \quad v \in \mathcal{D}\left(\mathbb{R}^d\right), \quad \int\limits_{\mathbb{R}^d} v dx = 0, \tag{1.22}$$

where $g \in BMO\left(\mathbb{R}^d\right)$. □

Furthermore, we have

Lemma 6. *The limit in (1.22) does not depend on the choice of the cut-off functions* φ_j.

Proof. To prove this lemma, we show that for every $v \in \mathcal{D}(Q)$ and $\int\limits_Q v dx = 0$,

$$\int\limits_{\mathbb{R}^d} g \overline{v} dx = -\sum_{j\geq 0} \left\langle \overrightarrow{f}, \omega_j \nabla\left(\Delta^{-1} v\right) \right\rangle. \tag{1.23}$$

which will imply the assertion. By (1.13) and statement (*ii*) of Lemma 3, it follows immediately that

$$\begin{aligned}\sum_{j\geq m}\left|\left\langle \overrightarrow{f},\omega_j\nabla\left(\Delta^{-1}v\right)\right\rangle\right| &\leq C\sum_{j\geq m}2^{\frac{j}{2}}\left|Q\right|^{\frac{1}{2}}\left\|\nabla\left(\omega_j\nabla\Delta^{-1}v\right)\right\|_{L^2(2^jQ)}\\ &\leq C\left\|v\right\|_{L^1(Q)}\sum_{j\geq m}2^{-j},\end{aligned}$$

for every $m\geq 1$. Moreover, by (1.19) a similar estimate holds with $\varphi_j\overrightarrow{f}$ in place of $\overrightarrow{f}$ and C does not depend on m and j.

Clearly, (1.23) holds with $\overrightarrow{f}$ replaced by $\varphi_j\overrightarrow{f}$ and for j large,

$$\sum_{0\leq j\leq m}\left\langle \overrightarrow{f},\omega_j\nabla\left(\Delta^{-1}v\right)\right\rangle=\sum_{0\leq j\leq m}\left\langle \varphi_j\overrightarrow{f},\omega_j\nabla\left(\Delta^{-1}v\right)\right\rangle.$$

By picking m and j large enough, and taking into account the above estimates together with (1.22), we arrive at (1.23).

We observe that (1.23) with div $\overrightarrow{v}$ in place of v yields

$$\left\langle \nabla g,\overrightarrow{v}\right\rangle=-\int_{\mathbb{R}^d}g\,\overline{\text{div }\overrightarrow{v}}\,dx=\sum_{j\geq 0}\left\langle \overrightarrow{f},\omega_j\nabla\left(\Delta^{-1}\text{div }\overrightarrow{v}\right)\right\rangle, \tag{1.24}$$

for every $v\in\mathcal{D}\left(\mathbb{R}^d\right)$ supported on a cube Q. Furthermore, we have $\nabla g\in\mathcal{D}'\left(\mathbb{R}^d\right)^d$ and

$$\nabla g=\lim_{j\to+\infty}\nabla g_j \text{ in } \mathcal{D}'\left(\mathbb{R}^d\right)^d,\quad \text{curl }\left(\nabla g\right)=0,\ \text{ in } \mathcal{D}'\left(\mathbb{R}^d\right)^{d^2}.$$

Moreover, for every $v\in\mathcal{D}\left(\mathbb{R}^d\right)$,

$$\left\langle \Delta g,v\right\rangle=\lim_{j\to+\infty}\left\langle g_j,\Delta v\right\rangle=-\lim_{j\to+\infty}\left\langle \varphi_j\overrightarrow{f},\nabla v\right\rangle=-\left\langle \overrightarrow{f},\nabla v\right\rangle,$$

which gives $\Delta g=$div $\overrightarrow{f}$ in $\mathcal{D}'\left(\mathbb{R}^d\right)$.

In a completely analogous fashion, one verifies that $H_j\to H$ in the weak-*topology of BMO,

$$\text{Div } H=\lim_{j\to+\infty}\text{Div } H_j \text{ in } \mathcal{D}'\left(\mathbb{R}^d\right)^d,$$

and $\Delta H=$curl $\overrightarrow{f}$ in $\mathcal{D}'\left(\mathbb{R}^d\right)^{d^2}$, div (Div H) $=0$. Moreover, H is a skew-symmetric matrix field since H_j is skew-symmetric for every j. □

We are in a position to establish decomposition (1.6) for vector fields which obey (1.5).

Proof. Let us set $\overrightarrow{\alpha} = \nabla g$ and $\overrightarrow{\beta}$ =Div H. Using a standard decomposition for $v \in \mathcal{D}\left(\mathbb{R}^d\right)^d$

$$\overrightarrow{v} = \nabla\left(\Delta^{-1}\text{div } \overrightarrow{v}\right) + \text{Div } \left(\Delta^{-1}\text{curl } \overrightarrow{v}\right), \tag{1.25}$$

we deduce

$$\begin{aligned} \langle \nabla g_j, \overrightarrow{v} \rangle &= -\langle g_j, \text{div } \overrightarrow{v} \rangle = \left\langle \varphi_j \overrightarrow{f}, \nabla\left(\Delta^{-1}\text{div } \overrightarrow{v}\right) \right\rangle \\ &= \left\langle \varphi_j \overrightarrow{f}, \overrightarrow{v} \right\rangle - \left\langle \varphi_j \overrightarrow{f}, \text{Div } \left(\Delta^{-1}\text{curl } \overrightarrow{v}\right) \right\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \langle \overrightarrow{\alpha}, \overrightarrow{v} \rangle &= \lim_{j \to +\infty} \langle \nabla g_j, \overrightarrow{v} \rangle \\ &= \lim_{j \to +\infty} \left\langle \varphi_j \overrightarrow{f}, \overrightarrow{v} \right\rangle - \lim_{j \to +\infty} \left\langle \varphi_j \overrightarrow{f}, \text{Div } \left(\Delta^{-1}\text{curl } \overrightarrow{v}\right) \right\rangle \\ &= \left\langle \overrightarrow{f}, \overrightarrow{v} \right\rangle - \lim_{j \to +\infty} \langle \text{Div } H_j, \overrightarrow{v} \rangle \\ &= \left\langle \overrightarrow{f}, \overrightarrow{v} \right\rangle - \left\langle \overrightarrow{\beta}, \overrightarrow{v} \right\rangle. \end{aligned}$$

This completes the proof. □

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