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HYPERCLONE LATTICE AND EMBEDDINGS

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Abstract. In this paper, hyperclone lattice is studied via three kinds of embeddings. One is from the clone lattice on A to the hyperclone lattice on A, the second is from the hyperclone lattice on A to the clone lattice on $P(A) \setminus \{\emptyset\}$ and the third one is from the hyperclone lattice on A' to the hyperclone lattice on A, for $A' \subset A$ and the finite set A. The second map has usually been used for the description of hyperclone lattice, but one can see from this paper that hyperclone lattice on A is in some way thin in the clone lattice on $P(A) \setminus \{\emptyset\}$. However, we show that the first and the third embeddings are full order embeddings and they are used to lift several properties.

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1. Introduction

The basic definitions and claims used in Introduction are from [5], [8], [13] and [14]. Many questions which arise from the process of operations extending are studied there. We will follow that work and give an extension of it. Clones of hyperoperations are also studied in [7], [12], [15], etc.

Let A be a nonempty finite set and $P(A)$ the power set of A. For a positive integer n, an n-ary hyperoperation on A is a function $f : A^n \to P(A) \setminus \{\emptyset\}$. We will denote the set of all operations on A by O_A, and the set of all hyperoperations by H_A.

An operation $f^{\#}$ on $P(A) \setminus \{\emptyset\}$ is extended from the hyperoperation f on A if for all $(X_1, X_2, \ldots, X_n) \in (P(A) \setminus \{\emptyset\})^n$ holds $f^{\#}(X_1, \ldots, X_n) = \cup\{f(x_1, \ldots, x_n)|\ x_i \in X_i, 1 \leq i \leq n\}$. For an arbitrary set F of hyperoperations, let $F^{\#} = \{f^{\#} | f \in F\}$.

For a positive integer n, an i-th projection on A of arity n, $1 \leq i \leq n$, is an $n-$ary operation $\pi_i^n : A^n \to A$, $(x_1, \ldots, x_n) \mapsto x_i$. For positive integers n and m, we define the composition $S_m^n : O_A^{(n)} \times (O_A^{(m)})^n \to O_A^{(m)}$, $(f, g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_n)$, where $f(g_1, \ldots, g_n) : A^m \to A$, $(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$.

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Let $f \in O_A^{(n)}$ and $g \in O_A^{(m)}$, then $\varsigma f \in O_A^{(n)}$, $\tau f \in O_A^{(n)}$, $\Delta f \in O_A^{(n-1)}$, $f \circ g \in O_A^{(m+n-1)}$ and $\nabla f \in O_A^{(n+1)}$ are defined by

$(\varsigma f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),\ n \geq 2$
$(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n),\ n \geq 2$
$(\Delta f)(x_1, \ldots, x_{n-1}) = f(x_1, x_1, \ldots, x_{n-1}),\ n \geq 2,$
$(\varsigma f) = (\tau f) = (\Delta f) = f,\ \ n = 1,$
$(f \circ g)(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) = f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1})$
$(\nabla f)(x_1, x_2, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1}),$

where $x_1, \ldots, x_{m+n-1} \in A$. The full algebra of operations is $\mathcal{O}_A = (O_A,\ \circ,\ \varsigma,\ \tau,\ \Delta,\ \pi_1^2)$. Each subuniverse of $\mathcal{O}_A$ is a clone.

A set of operations is a clone iff it contains all projections and is closed with respect to composition.

Let $\rho \subseteq A^h$ be a h-ary relation and f an n-ary operation on A. We say that f preserves ρ if for all h-tuples $(a_{11}, \ldots, a_{1h}), \ldots, (a_{n1}, \ldots, a_{nh})$ from ρ we have $(f(a_{11}, \ldots, a_{n1}), \ldots,\ f(a_{1h}, \ldots, a_{nh})) \in \rho$. $Pol_A\rho$ is the set of all operations on A which preserve ρ.

For a positive integer n, an i-th hyperprojection on A of arity n, $1 \leq i \leq n$, is an $n-$ary hyperoperation $e_i^n : A^n \to A$, $(x_1, \ldots, x_n) \mapsto \{x_i\}$. For positive integers n and m, we define the composition $S_m^n : H_A^{(n)} \times (H_A^{(m)})^n \to H_A^{(m)}$, $(f, g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_n)$, where $f(g_1, \ldots, g_n) : A^m \to P(A) \setminus \{\emptyset\}$, $(x_1, \ldots, x_m) \mapsto \cup\{f(y_1, \ldots, y_n) :\ \ y_i \in g_i(x_1, \ldots, x_m), 1, \leq i \leq n\}$. Let $f \in Hp_A^{(n)}$ and $g \in Hp_A^{(m)}$, then $\varsigma f \in Hp_A^{(n)}$, $\tau f \in Hp_A^{(n)}$, $\Delta f \in Hp_A^{(n-1)}$, $f \circ g \in Hp_A^{(m+n-1)}$ and $\nabla f \in Hp_A^{(n+1)}$ are defined by

$(\varsigma f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),\ n \geq 2$
$(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n),\ n \geq 2$
$(\Delta f)(x_1, \ldots, x_{n-1}) = f(x_1, x_1, \ldots, x_{n-1}),\ n \geq 2,$
$(\varsigma f) = (\tau f) = (\Delta f) = f,\ \ n = 1,$
$(f \circ g)(x_1,\ \ldots, x_m,\ x_{m+1},\ \ldots, x_{m+n-1}) = \cup\{f(y,\ x_{m+1},\ \ldots,\ x_{m+n-1}) : y \in g(x_1, \ldots, x_m)\}$
$(\nabla f)(x_1, x_2, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1}),$

where $x_1, \ldots, x_{m+n-1} \in A$. The full algebra of hyperoperations is $\mathcal{H}p_A = (Hp_A,\ \circ,\ \varsigma,\ \tau,\ \Delta,\ e_1^2)$. Each subuniverse of $\mathcal{H}p_A$ is a partial hyperclone.

A set of partial hyperoperations is a partial hyperclone iff it contains all hyperprojections and is closed with respect to composition.

If C is a hyperclone, the set of all n-ary hyperoperations from C will be denoted by $C^{(n)}$. Let us denote the set of all clones on A by L_A and the set of all hyperclones by L_{H_A}. Each of these sets forms a complete algebraic lattice. The atoms (dual atoms) are called minimal (maximal) elements. The least element in both lattices, trivial clone, will be denoted by J_A. For a set F of hyperoperations, the least hyperclone containing F will be denoted by $\langle F \rangle$, and the least clone containing the set F of operations on A will be denoted by $\langle F \rangle_A$.

A hyperclone is minimal if it is not trivial and its only subclone is trivial. A hyperoperation of minimal arity in a minimal hyperclone, that is not a projection, is called minimal hyperoperation.

A ternary majority hyperoperation on A, $ma \in H_A^{(3)}$, is a ternary hyperoperation on A defined by $ma(x,x,y) = ma(x,y,x) = ma(y,x,x) = \{x\}$ for all $x, y \in A$.

A ternary minority hyperoperation on A, $mi \in H_A^{(3)}$, is a ternary hyperoperation on A defined by $mi(x,x,y) = mi(x,y,x) = mi(y,x,x) = \{y\}$ for all $x, y \in A$.

For $n > 2$ and $1 \leq i \leq n$, every n-ary hyperoperation s with $s(x_1, \ldots, x_n) = \{x_i\}$, $|\{x_1, \ldots, x_n\}| < n$ is called semi-hyperprojection.

It is easy to show that the theorem analogous to Rosenberg's classification theorem ([15]) holds for minimal hyperoperations.

Theorem 1. *Every minimal hyperoperation is one of the following types:*

(1) a unary hyperoperation,

(2) a binary idempotent hyperoperation,

(3) a ternary majority hyperoperation,

(4) a ternary minority hyperoperation,

(5) an n-ary semi-hyperprojection, $n > 2$.

2. From the clone lattice to the hyperclone lattice

Let us define a map $\lambda : L_A \to L_{H_A}$ by $\lambda(C) = \bigcup_{n\geq 1} \{f \in H_A^{(n)} : \exists f' \in C \forall (x_1, \ldots, x_n) \in A^n f(x_1, \ldots, x_n) = \{f'(x_1, \ldots, x_n)\}\}$. It is easy to show that $\lambda(C)$ is a hyperclone.

Lemma 1. *The map λ is full-order embedding.*

Proof. Obviously, λ is 1-1 map, and holds $C_1 \leq C_2 \Leftrightarrow \lambda(C_1) \leq \lambda(C_2)$.

Let H be an arbitrary hyperclone with the property $\lambda(J_A) \subseteq H \subseteq \lambda(O_A)$. From the definition of λ immediately follows that there is clone C such that $\lambda(C) = H$. □

Without loss of generality, we will sometimes identify the hyperclone $\lambda(C)$ and the clone C, in order to simplify the presentation.

Lemma 2. *Let A be a finite set with $|A| \geq 2$. Every hyperclone generated by constant hyperoperation on A is minimal.*

Proof. The proof is trivial. □

Corollary 1. *Let A be a finite set with $|A| \geq 2$. There are least $2^{|A|} - 1$ minimal clones in the lattice L_{H_A}.*

Theorem 2. *On any finite set A, with $|A| \geq 2$, there are three minimal hyperclones such that their join contains all hyperoperations.*

Proof. It is proved in [4] that there are two minimal clones such that their join contains all operations on any finite set A. Romov proved in [12] that O_A is maximal hyperclone. Hence, it is enough to choose the third minimal hyperclone from the set of minimal hyperclones that are not minimal in the clone lattice on A. From previous lemma follows that such a set is not empty. □

Theorem 3. *The interval $[\lambda(\langle O_A^{(1)}\rangle), \lambda(O_A)]$ is a chain.*

Proof. It is the chain obtained by Burle in 1967 [11]. He has shown that the interval $[\langle O_A^{(1)}\rangle, O_A]$ is $(|A|+1)$-element chain

$$\langle O_A^{(1)}\rangle = U_1 \subset L \subset U_2 \subset \ldots \subset U_k = O_A.$$

U_i is the set of all operations depending on at most one variable and operations taking at most j values and L is the set of operations depending of one variable and operations $f(x_1, \ldots, x_n) = \lambda(\psi_1(x_{i_1}) + \ldots \psi_t(x_{i_t}))$, where $\lambda : \{0,1\} \to A$ and $\psi_j : A \to \{0,1\}$, $j \in \{i_1, \ldots, i_t\}$, $1 \leq i_1 < \ldots < i_t \leq n$, are arbitrary maps and $+$ is addition modulo 2. □

Corollary 2. *There are finite maximal chains in the hyperclone lattice.*

Proof. It is known that there are finite maximal chains in the interval $[J_A, \langle O_A^{(1)}\rangle]$. With the maximal chain from previous theorem and the clone H_A (since O_A is maximal in the hyperclone lattice), we get the finite maximal chain in the hyperclone lattice. □

3. From the hyperclone lattice on A to the clone lattice on $P(A) \setminus \{\emptyset\}$

The mapping λ from L_{H_A} into the $L_{P(A)\setminus\{\emptyset\}}$ defined by $\lambda(F) = \langle F^{\#}\rangle_{P(A)\setminus\{\emptyset\}}$ is an order embedding, though not a full one, i.e. there are $F, G \in L_{H_A}$ such that $[\lambda(F), \lambda(G)] \setminus im\lambda \neq \emptyset$ (See [5],[13] and [14]).

3.1. Number of clones in $[\lambda(F), \lambda(G)] \setminus im\lambda$

Let A be $\{0, 1, 2, \ldots\}$, $|A| \geq 3$, $m \geq 2$ and $g_m \in H_A^{(m)}$ the hyperoperation defined by

$$g_m(x_1, \ldots, x_m) = \begin{cases} \{2\}, & (x_1, x_2, \ldots, x_m) \in J_m \\ \{0\}, & \text{otherwise} \end{cases},$$

where J_m is the set of all m-tuples with one coordinate equal 2 and all others equal 1.

Let us define the hyperoperation $f_{m+1} \in H_A^{(m+1)}$ by

$$f_{m+1}(x_1, x_2, \ldots, x_{m+1}) = \begin{cases} A & , x_1 \neq x_2 \\ g_m(x_2, \ldots, x_{m+1}) & , x_1 = x_2. \end{cases}$$

Thus, extended operation from f_{m+1} is the operation $f_{m+1}^{\#} \in P(A) \setminus \{\emptyset\}^{(m+1)}$ defined by

$$\begin{aligned} f_{m+1}^{\#}(X_1, X_2, \ldots, X_{m+1}) &= \cup \{f_{m+1}(x_1, x_2, \ldots, x_{m+1}) | x_i \in X_i\} \\ &= \begin{cases} g_m^{\#}(X_2, \ldots, X_{m+1}) & , X_1 = X_2,\ |X_1| = 1 \\ A & , \text{otherwise} \end{cases} \end{aligned}$$

Lemma 3. *For every clone F of hyperoperations on A and for every $\emptyset \neq Q \subseteq \{f_i \mid i \geq 3\}$ holds*
$lambda(F) \neq \langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}}$.

Proof. Let Q be an arbitrary nonvoid subset of $\{f_i \mid i \geq 3\}$. Then, there is $m \geq 2$ such that $f_{m+1} \in Q$.

Suppose to the contrary, that there is a clone of hyperoperations F such that its λ image $\lambda(F)$ is the clone generated by $Q^{\#}$, i.e. $\lambda(F) = \langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}}$. Then, there is a hyperoperation $h \in F$ with property $f_{m+1}^{\#} = \delta_\alpha h^{\#}$. From $f_{m+1}^{\#} \in H_A^{\#}$, it follows $f_{m+1} \in F$ (see [5]). Since F is a clone, the hyperoperation $g_m \in H_A^{(m)}$, defined by $g_m(x_1, \ldots, x_m) = f_{m+1}(x_1, x_1, \ldots, x_m)$, also belongs to F ($g_m = f_{m+1}(e_m^1, e_m^1, e_m^2, \ldots, e_m^m)$). However, we shall prove that $g_m^{\#} \notin \langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}}$.

For every $i \geq 2$ $f_i^{\#}$ $(A, A, \ldots, A) = A$, and $g_m^{\#} \notin Pol_{P(A)\setminus\{\emptyset\}}(A)$, because $g_m^{\#}(A, A, \ldots, A) = \{0, 2\} \neq A$. $\langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}} \subset Pol_{P(A)\setminus\{\emptyset\}}(A)$,

So, $g_m^{\#} \in F^{\#} \subseteq \lambda(F)$ and $g_m \notin \langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}}$.

(Notice that Q is not a clone of hyperoperations on A.) □

The proof of the following lemma is a modification of Rónyi's proof of Yanov's and Mučnik's statement that there is a countable infinite set of operations on a finite set A with $|A| \geq 3$.

Lemma 4. *For every $i \geq 3$ holds $f_i^{\#} \notin \langle \bigcup_{j \geq 3, j \neq i} \{f_j^{\#}\} \rangle_{P(A)\setminus\{\emptyset\}}$.*

Proof. Let us define for every $m \geq 2$ relation $\rho_m \in P_A^m$ by $\rho_m = A_m \cup B_m$, where A_m is the set of all m-tuples with exactly one coordinate equal $\{2\}$ and all others equal $\{1\}$ and $B_m = \{\{0\}, \{2\}, \{0, 2\}, A\}^m \setminus (\{2\}, \{2\}, \ldots, \{2\})$. We are going to show that $f_{m+1}^{\#} \notin Pol_{P(A)\setminus\{\emptyset\}}\rho_m$ and $f_{i+1}^{\#} \in Pol_{P(A)\setminus\{\emptyset\}}\rho_m, i \neq m$.

For m-tuples $(\{2\}, \{1\}, \ldots, \{1\})$, $(\{2\}, \{1\}, \ldots, \{1\})$, $(\{1\}, \{2\}, \ldots, \{1\})$, $\ldots$, $(\{1\}, \{1\}, \ldots, \{2\}) \in A_m$ (the first one and the second one are equal), holds $(f_{m+1}^{\#}(\{2\}, \{2\}, \{1\}, \ldots, \{1\}), f_{m+1}^{\#}(\{1\}, \{1\}, \{2\}, \ldots, \{1\}), \ldots, f_{m+1}^{\#}(\{1\}, \{1\}, \{1\}, \ldots, \{2\})) = (\{2\}, \{2\}, \ldots, \{2\}) \notin \rho_m$. So, $f_{m+1}^{\#} \notin Pol_{P(A)\setminus\{\emptyset\}}\rho_m$.

Suppose that there is i, $i \neq m$ such that $f_{i+1}^{\#} \notin Pol_{P(A)\setminus\{\emptyset\}}\rho_m$. Then, there are tuples $\mathbf{X}_1 := (X_{11}, \ldots, X_{m1}), \ldots, \mathbf{X}_{i+1} := (X_{1(i+1)}, \ldots, X_{m(i+1)}) \in \rho_m$, such that $(Y_1, Y_2, \ldots, Y_m) := (f^{\#}_{i+1}(X_{11}, \ldots, X_{1(i+1)}), \ldots, f^{\#}_{i+1}(X_{m1}, \ldots, X_{m(i+1)}) \notin \rho_m$. Since $imf^{\#}_{i+1} = \{\{0\}, \{2\}, \{0,2\}, A\}$, it follows that $(Y_1, \ldots, Y_m) = (\{2\}, \{2\}, \ldots, \{2\})$ and it is possible only for $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+1} \in A_m$, $\mathbf{X}_1 = \mathbf{X}_2$ and $i = m$. This is a contradiction. □

Theorem 4. *There are continuum many pairwise distinct clones of operations on $P(A) \setminus \{\emptyset\}$ in the interval $[\lambda(J_A), \lambda(H_A)]$ that are not in the set of all images $im\lambda$ of the operation λ.*

Proof. Let $R = \bigcup_{i \geq 3} \{f_i\}$.

(a) Since λ is an order embedding, λ is injective and for $F, G \in L_{H_A}$ $F \leq G$ is equivalent to $\lambda(F) \leq \lambda(G)$. So, for $\langle Q \rangle \leq H_A$ it follows $\langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}} \leq \lambda(\langle Q \rangle) \leq \lambda(H_A), Q \subseteq R$. On the other hand, $\lambda(J_A) \leq \langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}}$, because $\lambda(J_A) = J_A$, this being the result of the following: $(e^n_i)^{\#}(X_1, \ldots, X_n) = \bigcup_{x_i \in X_i} e^n_i(x_1, \ldots, x_n) = \bigcup_{x_i \in X_i} \{x_i\} = X_i = p_i^{n,P(A)\setminus\{\emptyset\}}(X_1, \ldots, X_n)$. (See [13],[14].)

(b) It follows from Lemma 3 that for every $Q \subseteq R$ holds $\langle Q^{\#} \rangle_{P(A)\setminus\{\emptyset\}} \notin im\lambda$.

(c) From Lemma 4 follows that for all $Q_1, Q_2 \subseteq R$ if $Q^{\#}_1 \neq Q^{\#}_2$ then $\langle Q^{\#}_1 \rangle_{P(A)\setminus\{\emptyset\}} \neq \langle Q^{\#}_2 \rangle_{P(A)\setminus\{\emptyset\}}$. □

3.2. Minimal hyperclones

In this subsection we will present results from [10], in order to make observation about hyperclones and embeddings more complete.

Lemma 5. *Let $f, g \in H^{(1)}_A$. Then,*

(a) $(f \circ g)^{\#} = f^{\#} \circ g^{\#}$.

(b) $(\Delta f)^{\#} = \Delta f^{\#}$.

Proof.

(a) Let X be an arbitrary subset of A. Then $(f \circ g)^{\#}(X) = \cup\{(f \circ g)(x) : x \in X\} = \cup\{\cup\{f(y) : y \in g(x)\} : x \in X\} = \cup\{f(y) : y \in \cup\{g(x) \in X\}\} = \cup\{f(y) : y \in g^{\#}(X)\} = f^{\#}(g^{\#}(X))$.

(b) It follows immediately from the definition of Δ that $(\Delta f)^{\#} = f^{\#} = \Delta f^{\#}$. □

Corollary 3. *The mapping $f \mapsto f^{\#}$ is isomorphism from $(H^{(1)}_A; *, \zeta, \tau, \Delta, e^1_1)$ onto $(\lambda(H^{(1)}_A), *, \zeta, \tau, \Delta, \pi^1_1)$.*

Corollary 4. *Let $f \in H_A^{(1)}$. Then, $\langle\{f\}\rangle^{\#} = \langle\{f^{\#}\}\rangle_{P(A)\backslash\emptyset}$.*

Corollary 5. *The restriction to the interval $[J_A, H_A^{(1)}]$ of the mapping $\lambda : L_{H_A} \to L_{P(A)\backslash\{\emptyset\}}$, $C \mapsto \langle C^{\#}\rangle_{P(A)\backslash\{\emptyset\}}$ is a full order embedding.*

Corollary 6. *Let $f \in H_A^{(1)}$. $\langle\{f\}\rangle$ be minimal hyperclone iff $\langle\{f^{\#}\}\rangle_{P(A)\backslash\emptyset}$ is a minimal clone.*

Lemma 6. *Let $f \in H_A^{(1)}$. Then, it holds*

(a) $f^2 = f \Leftrightarrow (f^{\#})^2 = f^{\#}$.

(b) $f^p = e_1^1 \Leftrightarrow (f^{\#})^p = \pi_1^1$, *for some prime p.*

Proof.

(a) $(\to)$ If $f^2 = f$ then $(f^{\#})^2 = (f^2)^{\#} = f^{\#}$. $(\leftarrow)$ If $f^{\#}(f^{\#}(X)) = f^{\#}(X)$ holds for every $X \subseteq A$, then it also holds for $|X| = 1$. It means that for every $x \in A$, $f^{\#}(f^{\#}(\{x\})) = f^{\#}(\{x\})$, i.e. $f(f(x) = f(x)$.

(b) $(\to)$ If $f^p = e_1^1$ then $(f^{\#})^p = (f^p)^{\#} = (e_1^1)^{\#} = \pi_1^1$. $(\leftarrow)$ For every $X \subseteq A$ $(f^{\#})^p(X) = X$ implies that for every $x \in A$ holds $(f^{\#})^p(\{x\}) = \{x\}$ i.e. for every $x \in A$ holds $f^p(x) = x$. □

Theorem 5. *Let $f \in H_A^{(1)}$. Then, $\langle\{f\}\rangle$ is minimal iff $f^2 = f$ or $f^p = id_A$, for some prime p.*

Proof. Let $f^2 = f$ or $f^p = id_A$, for some prime p. From Lemma 6 ($f^2 = f$ iff $(f^{\#})^2 = f^{\#}$) and ($f^p = id_A$, for some prime p iff $(f^{\#})^p = id_{P(A)\backslash\emptyset}$, for some prime p). It is known ([15], [3],[17]) that $((f^{\#})^2 = f^{\#})$ or $(f^{\#})^p = id_{P(A)\backslash\emptyset}$, for some prime p) iff $\langle\{f^{\#}\}\rangle_{P(A)\backslash\emptyset}$ is minimal clone in $L_{P(A)\backslash\emptyset}$. From Lemma 6, $\langle\{f^{\#}\}\rangle_{P(A)\backslash\emptyset}$ is a minimal clone in $L_{P(A)\backslash\emptyset}$. iff $\langle\{f\}\rangle$ is minimal hyperclone in L_{H_A}. □

Example 1. *There are* 6 *unary minimal hyperclones on $A = \{0,1\}$.*

Example 2. *There are* 63 *unary minimal hyperclones on $A = \{0,1,2\}$*

Lemma 7. *Let $\langle g\rangle_{P(A)\backslash\{\emptyset\}}$ be a minimal clone on $P(A) \setminus \{\emptyset\}$. If there is a hyperoperation f, such that $g = f^{\#}$, then $\langle f\rangle$ is a minimal hyperclone on A.*

Proof. Since $\langle\{f^{\#}\}\rangle_{P(A)\backslash\{\emptyset\}}$ is a minimal clone in $L_{P(A)\backslash\{\emptyset\}}$, and the mapping λ is an order embedding we prove the claim. □

Lemma 8. *Let f be a hyperoperation on $A = \{0,1\}$. $\langle\{f\}\rangle$ is a minimal hyperclone on A if and only if $\langle\{f^{\#}\}\rangle_{P(A)\backslash\{\emptyset\}}$ is a minimal clone on $P(A) \setminus \{\emptyset\}$.*

Example 3. *There are* 13 *minimal hyperclones on $A = \{0,1\}$ [8].*

4. From the hyperclone lattice on A' to the hyperclone lattice on A

Let $A' \subset A$ be a nonempty set and let us define a mapping $\lambda : L_{H_{A'}} \to L_{H_A}$ by $\lambda(C) = \bigcup_{n \geq 1} \{f \in H_A^{(n)} : f|_{A'} \in C\}$.

For every hyperclone C on A', $\lambda(C)$ is a hyperclone on A.

Theorem 6. *The mapping λ is a full order embedding.*

Proof. It is easy to check that λ is an order embedding.

If H is a hyperclone on A that satisfies $\lambda(J_{A'}) \leq H \leq \lambda(H_{A'})$, there is a hyperclone $H_1 = H|_{A'}$ on A' such that $\lambda(H_1) = H$. □

5. Conclusions and problems

The structure of the lattice of clones of hyperoperations is unknown. One attempt is to embed it into some more familiar lattice. The most studied lattice of clones is the lattice of clones of total operations. So, it is natural to try with embeddings of the lattice of hyperclones into the lattice of clones. Some facts about the lattice of hyperclones could be spotted from the known facts about the lattice of clones. However, we can also see from this paper that the studied embedding will not give answers to many questions (even if we knew more about the lattice of clones of operations than we do). So, one can try with other embeddings from the lattice of hyperclones into the lattice of clones of operations.

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