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COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS WITH SOME WEAK CONDITIONS OF COMMUTATIVITY

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Abstract. Some results on the common fixed point of two set-valued and two single valued mappings defined on a complete metric space with some weak commutativity conditions have been proved.

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1. Introduction

Imdad, Khan and Sessa [3], generalizing the notion of commutativity for set-valued mappings, established the idea of weak commutativity, quasi commutativity, slight commutativity. Under these concepts, Imdad and Ahmad proved Theorems 3.1-3.4 [6] , for set-valued mappings. Our work generalizes earlier results due to Pathak, Mishra and Kalinde [5] with the proof techniques of Imdad and Ahmad [6] .

2. Preliminaries

Let (X, d) be a metric space, then following [1] we record

(i) $B(X) = \{A : A \text{ is a nonempty bounded subset of } X\}$

(ii) For $A, B \in B(X)$ we define $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$

If $A = \{a\}$, then we write $\delta(A, B) = \delta(a, B)$ and if $B = \{b\}$, then $\delta(a, B) = d(a, b)$.

One can easily prove that for A, B, C in $B(X)$

$$\begin{aligned} \delta(A, B) &= \delta(B, A) \geq 0, \\ \delta(A, B) &\leq \delta(A, C) + \delta(C, B) \\ \delta(A, A) &= \sup\{d(a, b) : a, b \in A\} = \text{diam}A \text{ and} \\ \delta(A, B) &= 0 \text{ implies that } A = B = \{a\}. \end{aligned}$$

If $\{A_n\}$ is a sequence in $B(X)$, we say that $\{A_n\}$ converges to $A \subseteq X$, and write $A_n \to A$, iff

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(i) $a \in A$ implies that $a_n \to a$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in N$, and

(ii) for any $\varepsilon > 0$ $\exists$ $m \in N$ such that $A_n \subseteq A_\varepsilon = \{x \in X : d(x, a) < \varepsilon$ for some $a \in A\}$ for $n > m$.

We need the following lemmas.

Lemma 1. [2] *Suppose* $\{A_n\}$ *and* $\{B_n\}$ *are sequences in* $B(X)$ *and* (X, d) *is a complete metric space. If* $A_n \to A \in B(X)$ *and* $B_n \to B \in B(X)$*, then* $\delta(A_n, B_n) \to \delta(A, B)$.

Lemma 2. [3] *If* $\{A_n\}$ *is a sequence of nonempty bounded subsets in the complete metric space* (X, d) *and if* $\delta(A_n, y) \to 0$ *for some* $y \in X$*, then* $A_n \to \{y\}$.

Definition 1. [7] *The mappings* $F, S : X \to X$ *are weakly commuting if for all* $x \in X$*, we have* $d(FSx, SFx) \leq d(Fx, Sx)$.

Definition 2. *Let* $F : X \to B(X)$ *be a set-valued mapping and* $S : X \to X$ *a single-valued mapping. Then, following* [1, 3]*, we say that the pair* (F, S) *is*

(i) *weakly commuting on* X *if* $\delta(FSx, SFx) \leq max\{\delta(Sx, Fx),\ diamSFx\}$ *for any* x *in* X

(ii) *quasi-commuting on* X *if* $SFx \subseteq FSx$ *for any* x *in* X

(iii) *slightly commuting on* X *if* $\delta(FSx, SFx) \leq \max\{\delta(Sx, Fx),\ diamFx\}$ *for any* x *in* X.

Clearly, two commuting mappings satisfy (i) $-$ (iii) but the converse may not be true. In [3] it is demonstrated by suitable examples that the foregoing three concepts are mutually independent and none of them implies the other two.

3. Fixed Point Theorems

Throughout this section, let $\mathbb{R}^+$ denote the set of non-negative reals, and let Φ be the family of all mappings $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ such that ϕ is upper semi continuous, non-decreasing in each coordinate variable and, for any $t > 0$,

$$\gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t,$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ and $a_1 + a_2 + a_3 = 8$.

We need the following lemma.

Lemma 3. [4] *For any* $t > 0$*,* $\gamma(t) < t$ *if and only if* $\lim_{n\to\infty} \gamma^n(t) = 0$*, where* γ^n *denotes the composition of* γ n*-times with itself.*

Let F, G be two set-valued mappings of a metric space (X, d) into $B(X)$, and A, B, two self-mappings of X such that

$$F(X) \subseteq A(X),\ G(X) \subseteq B(X), \tag{1}$$

$$\begin{aligned} \delta^{2p}(Fx,Gy) \ \leq \ & \phi(d^{2p}(Bx,Ay), \\ & \delta^{q}(Bx,Fx)\ \delta^{q^{\star}}(Ay,Gy), \\ & \delta^{r}(Bx,Gy)\ \delta^{r^{\star}}(Ay,Fx), \\ & \delta^{s}(Bx,Fx)\ \delta^{s^{\star}}(Ay,Fx), \\ & \delta^{l}(Bx,Gy)\ \delta^{l^{\star}}(Ay,Gy)) \end{aligned} \quad (2)$$

for all $x, y \in X$, where $\phi \in \Phi$, $0 < p, q, q^\star, r, r^\star, s, s^\star, l, l^\star \leq 1$ such that $2p = q + q^\star = r + r^\star = s + s^\star = l + l^\star$.

Then by choosing an arbitrary $x_0 \in X$ and using (1), we can define a sequence $\{y_n\}$ in X by

$$\begin{aligned} y_{2n+1} &= Ax_{2n+1} \in Fx_{2n} = X_{2n+1} \text{ and} \\ y_{2n+2} &= Bx_{2n+2} \in Gx_{2n+1} = X_{2n+2},\ n = 0, 1, 2, \ldots \end{aligned} \quad (3)$$

Let $F, G : X \to B(X)$ and $A, B : X \to X$ satisfy conditions (1) and (2), and the sequence $\{y_n\}$ is defined by (3), then following the proof techniques of Imdad et al. [6], we can prove the following

Lemma 4. *If* $d_n = \delta(X_n, X_{n+1})$, *then* $\lim_{n\to\infty} d_n = 0$.

Proof. Let us assume that $d_{2n+1} > d_{2n}$, then

$$\begin{aligned} d_{2n+1} &\leq \{\phi(d_{2n+1}^{2p}, d_{2n+1}^{2p}, 4d_{2n+1}^{2p}, 2d_{2n+1}^{2p}, 2d_{2n+1}^{2p})\}^{\frac{1}{2p}} \\ &\leq \{\gamma(d_{2n+1}^{2p})\}^{\frac{1}{2p}} \\ &< d_{2n+1}, \end{aligned}$$

which is a contradiction. Hence $d_{2n+1} \leq d_{2n}$. Similarly, one can show that $d_{2n+2} \leq d_{2n+1}$. Then $\{d_n\}$ is a decreasing sequence.

Now since

$$\begin{aligned} d_2^{2p} &\leq \phi(d_1^{2p}, d_1^{2p}, 4d_1^{2p}, 2d_1^{2p}, 2d_1^{2p}) \\ &\leq \gamma(d_1^{2p}), \end{aligned}$$

it follows by induction that

$$d_{n+1}^{2p} \leq \gamma^n(d_1^{2p})$$

and if $d_1 > 0$, then Lemma 3 implies that $\lim_{n\to\infty} d_n = 0$. If $d_1 = 0$, then $d_n = 0$, $n = \{1, 2, ...\}$. □

Lemma 5. $\{y_n\}$ *is a Cauchy sequence in* X.

Proof. We show that $\{y_n\}$ is a Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose $\{y_{2n}\}$ is not Cauchy sequence. Then there is an $\varepsilon > 0$ such that for an even integer $2k$ there exists even integers $2m(k) > 2n(k) > 2k$ such that

$$d(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \tag{4}$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (4) and such that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon. \tag{5}$$

Now

$$\begin{aligned} \varepsilon &\leq d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \end{aligned}$$

Then by (4) and (5) it follows that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon. \tag{6}$$

Also, by the triangle inequality, we have

$$\left| d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)}) \right| < d_{2m(k)-1}.$$

By using (6) we get $d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon$ as $k \to \infty$. Now by (2) we get

$$\begin{aligned} d(y_{2n(k)}, y_{2m(k)}) &\leq d_{2n(k)} + \delta(Fx_{2n(k)}, Gx_{2m(k)-1}) \\ &\leq d_{2n(k)} + \{\phi(d^{2p}(y_{2n(k)}, y_{2m(k)-1}), d^{q}_{2n(k)} d^{q^\star}_{2m(k)-2}, \\ &\quad [d^{r}(y_{2n(k)}, y_{2m(k)-1}) + d^{r}_{2m(k)-2}] \times \\ &\quad [d^{r^\star}(y_{2m(k)-1}, y_{2n(k)}) + d^{r^\star}_{2n(k)}], \\ &\quad d^{s}_{2n(k)}[d^{s^\star}(y_{2m(k)-1}, y_{2n(k)}) + d^{s^\star}_{2n(k)}], \\ &\quad [d^{l}(y_{2m(k)-1}, y_{2n(k)}) + d^{l}_{2m(k)-2}] d^{l^\star}_{2m(k)-2}\}^{\frac{1}{2p}} \end{aligned}$$

which on letting $k \to \infty$ reduces to

$$\begin{aligned} \varepsilon &\leq \{\phi(\varepsilon^{2p}, 0, \varepsilon^{2p}, 0, 0)\}^{\frac{1}{2p}} \\ &\leq \{\gamma(\varepsilon^{2p}\}^{\frac{1}{2p}} \\ &< \varepsilon, \end{aligned}$$

giving a contradiction. Thus $\{y_{2n}\}$ is a Cauchy sequence. □

Theorem 1. *Let F, G be two set-valued mappings of a complete metric space (X, d) into $B(X)$, and A, B two self-mappings of X satisfying conditions* (1), (2), *(F, B) and (G, A) are slightly commuting and any one of these four mappings is continuous, then F, G, A and B have a unique common fixed point in X.*

Proof. By Lemma 5, the sequence $\{y_n\}$ defined by (3) is a Cauchy sequence in X. Therefore $y_n \to z$ for some $z \in X$. Hence the subsequences $\{y_{2n}\} = \{Bx_{2n}\}$ and $\{y_{2n+1}\} = \{Ax_{2n+1}\}$ of $\{y_n\}$ also converge to z, whereas the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to the set $\{z\}$.

Since (F, B) commute slightly, we have

$$\delta(BFx_{2n}, FBx_{2n}) \leq \max\{\delta(Bx_{2n}, Fx_{2n}),\ \mathrm{diam} Fx_{2n}\}$$

which on letting $n \to \infty$ gives (by Lemma 1)

$$\lim_{n\to\infty} \delta(BFx_{2n}, FBx_{2n}) = 0.$$

Now suppose that B is continuous, then we have $BBx_{2n} = By_{2n} \to Bz$ .Thus

$$\begin{aligned} d(By_{2n+1}, y_{2n+2}) &\leq \delta(BFx_{2n}, Gx_{2n+1}) \\ &\leq \delta(BFx_{2n}, FBx_{2n}) + \delta(FBx_{2n}, Gx_{2n+1}) \\ &\leq \delta(BFx_{2n}, FBx_{2n}) + \{\phi(d^{2p}(BBx_{2n}, Ax_{2n+1}), \\ &\quad [\delta^q(BBx_{2n}, BFx_{2n}) + \delta^q(BFx_{2n}, FBx_{2n})] \times \\ &\quad \delta^{q^\star}(Ax_{2n+1}, Gx_{2n+1}), \\ &\quad \delta^r(BBx_{2n}, Gx_{2n+1}) \times \\ &\quad [\delta^{r^\star}(Ax_{2n+1}, BFx_{2n}) + \delta^{r^\star}(BFx_{2n}, FBx_{2n})], \\ &\quad [\delta^s(BBx_{2n}, BFx_{2n}) + \delta^s(BFx_{2n}, FBx_{2n})] \\ &\quad [\delta^{s^\star}(Ax_{2n+1}, BFx_{2n}) + \delta^{s^\star}(BFx_{2n}, FBx_{2n})], \\ &\quad \delta^l(BBx_{2n}, Gx_{2n+1})\delta^{l^\star}(Ax_{2n+1}, Gx_{2n+1}))\}^{\frac{1}{2p}}. \end{aligned}$$

Suppose $Bz \neq z$. Then letting $n \to \infty$ and using Lemma 1 and Lemma 2 we obtain

$$\begin{aligned} d(Bz, z) &\leq \{\phi(d^{2p}(Bz, z), 0, d^{2p}(Bz, z), 0, 0)\}^{\frac{1}{2p}} \\ &\leq \{\gamma(d^{2p}(Bz, z))\}^{\frac{1}{2p}} \\ &< d(Bz, z) \end{aligned}$$

a contradiction. We must therefore have $Bz = z$. Similarly, applying the condition (2) to

$$\delta(FSz, y_{2n+2}) \leq \delta(FSz, GTx_{2n+1})$$

and letting $n \to \infty$, we can prove that $Fz = \{z\}$, which means that z is in the range of F. Since $F(X) \subseteq A(X)$, there exist a point $z^{\prime}$ in X such that $Az^{\prime} = z$. Suppose that $Gz^{\prime} \neq z$. Then

$$\begin{aligned} \delta(z, Gz^{\prime}) &= \delta(Fz, Gz^{\prime}) \\ &\leq \{\phi(0, 0, 0, 0, \delta^{2p}(z, Gz^{\prime}))\}^{\frac{1}{2p}} \\ &\leq \{\gamma(\delta^{2p}(z, Gz^{\prime}))\}^{\frac{1}{2p}} \\ &< \delta(z, Gz^{\prime}) \end{aligned}$$

a contradiction. We must therefore have $Gz' = \{z\}$. Since (G, A) is slightly commuting, we have

$$\begin{aligned} \delta(Gz, Az) &= \delta(GAz', AGz') \\ &\leq 0, \end{aligned}$$

proving that $Gz = Az$. If $Gz \neq z$, then

$$\begin{aligned} \delta(z, Gz) &= \delta(Fz, Gz) \\ &\leq \{\phi(\delta^{2p}(z, Gz), 0, \delta^{2p}(z, Gz), 0, 0)\}^{\frac{1}{2p}} \\ &\leq \{\gamma(\delta^{2p}(z, Gz))\}^{\frac{1}{2p}} \\ &< \delta(z, Gz), \end{aligned}$$

a contradiction and so $Gz = \{z\} = Az$.

Thus we have shown that $Bz = Az = Fz = Gz = \{z\}$. Hence z is a common fixed point of F, G, A and B.

Now suppose that F is continuous, then we have $\{Fy_{2n}\} = \{FBx_{2n}\} \to \{Fz\}$. Since $By_{2n+1} \in BFx_{2n}$, the inequality (2) yields

$$\begin{aligned} \delta(Fy_{2n+1}, Gx_{2n+1}) \leq\ & \{\phi([\delta^p(BFx_{2n}, FBx_{2n}) + \delta^p(FBx_{2n}, Ax_{2n+1})]^2, \\ & [\delta^q(BFx_{2n}, FBx_{2n}) + \delta^q(FBx_{2n}, Fy_{2n+1})] \times \\ & \delta^{q^\star}(Ax_{2n+1}, Gx_{2n+1}), \\ & [\delta^r(BFx_{2n}, FBx_{2n}) + \delta^r(FBx_{2n}, Gx_{2n+1})] \times \\ & \delta^{r^\star}(Ax_{2n+1}, Fy_{2n+1}), \\ & [\delta^s(BFx_{2n}, FBx_{2n}) + \delta^s(FBx_{2n}, z) + \delta^s(z, Fy_{2n+1})] \times \\ & \delta^{s^\star}(Ax_{2n+1}, Fy_{2n+1}), \\ & [\delta^l(BFx_{2n}, FBx_{2n}) + \delta^l(FBx_{2n}, Gx_{2n+1})] \times \\ & \delta^{l^\star}(Ax_{2n+1}, Gx_{2n+1}))\}^{\frac{1}{2p}}. \end{aligned}$$

Suppose $Fz \neq z$. Then letting $n \to \infty$, we obtain

$$\begin{aligned} \delta(z, Fz) &\leq \{\phi(\delta^{2p}(z, Fz), 0, \delta^{2p}(z, Fz), 2\delta^{2p}(z, Fz), 0)\}^{\frac{1}{2p}} \\ &\leq \{\gamma(\delta^{2p}(z, Fz))\}^{\frac{1}{2p}} \\ &< \delta(z, Fz) \end{aligned}$$

a contradiction and so $Fz = \{z\}$. Since $F(X) \subseteq A(X)$, there exists a point z' in X such that $Az' = z$. Similarly, using (2) on $\delta(Gz', Fx_{2n})$ and letting $n \to \infty$ one can prove that $Gz' = \{z\}$. Now, by the slight commutativity of (G, A) we find

$$\begin{aligned} \delta(Gz, Az) &= \delta(GAz', AGz') \\ &\leq 0 \end{aligned}$$

which gives that $Gz = Az$. Further, applying (2) to $\delta(Fx_{2n}, Gz)$ and letting $n \to \infty$, we can show that $Gz = \{z\} = Az$.

Since $G(X) \subseteq B(X)$ there exists a point z'' in X such that $Bz'' = z$. Suppose that $Fz \neq z$. Then

$$\begin{aligned}\delta(Fz'', z) &= \delta(Fz'', Gz)\\ &\leq \{\phi(0, 0, 0, \delta^{2p}(z, Fz''), 0)\}^{\frac{1}{2p}}\\ &\leq \{\gamma(\delta^{2p}(z, Fz''))\}^{\frac{1}{2p}}\\ &< \delta(z, Fz''),\end{aligned}$$

a contradiction, implying that $Fz'' = \{z\}$.

By the slight commutativity of (F, B), we have

$$\begin{aligned}\delta(Fz, Bz) &= \delta(FBz'', BFz'')\\ &\leq 0,\end{aligned}$$

which gives that $Fz = Bz$. Thus we have shown that $Fz = Gz = Bz = Az = \{z\}$.

The other cases, A is continuous and G is continuous, can be disposed of a similar argument as above.

For uniqueness, suppose that w is a second distinct fixed point of (F, B). Then

$$\begin{aligned}d(w, z) &= \delta(Fw, Gz)\\ &\leq \{\phi(0, 0, d^{2p}(w, z), 0, d^{2p}(w, z))\}^{\frac{1}{2p}}\\ &\leq \{\gamma(d^{2p}(w, z))\}^{\frac{1}{2p}}\\ &< d(w, z),\end{aligned}$$

a contradiction and so the fixed point z is unique. Similarly, one can show that z is the unique common fixed point of G and A. □

Theorem 2. *Let F, G be two set-valued mappings of a complete metric space X into $B(X)$, and A, B two self-mappings of (X, d) satisfying conditions* (1), (2)*, B is continuous or* (1), (2)*, A is continuous. If (F, B) and (G, A) are weakly commuting, then F, G, B and A have a unique common fixed point in X.*

Theorem 3. *Let F, G be two set-valued mappings of a complete metric space X into $B(X)$, and A, B two self-mappings of (X, d) satisfying conditions* (1), (2)*, F is continuous, (F, B) and (G, A) are quasi-commuting or* (1), (2)*, G is G is continuous, (F, B) and (G, A) are quasi-commuting, then F, G, B and A have a unique common fixed point in X.*

Remark 1. *The conclusion of Theorems 1-3 remains valid if the condition* (2) *is replaced by*

$$\begin{aligned}\delta^{2p}(Fx,Gy) \leq\ & \alpha d^{2p}(Bx,Ay) + \\ & \beta \max\{\delta^{q}(Bx,Fx)\ \delta^{q^\star}(Ay,Gy), \\ & \delta^{r}(Bx,Gy)\ \delta^{r^\star}(Ay,Fx), \\ & \delta^{s}(Bx,Fx)\ \delta^{s^\star}(Ay,Fx), \\ & \delta^{l}(Bx,Gy)\ \delta^{l^\star}(Ay,Gy)\} \qquad (2^\star)\end{aligned}$$

for all $x, y \in X$*, where* $\alpha > 0$, $\beta \geq 0$ *with* $\alpha + 4\beta < 1$ *and* $0 < p, q, q^\star, r, r^\star, s, s^\star, l, l^\star \leq 1$ *with* $2p = q + q^\star = r + r^\star = s + s^\star = l + l^\star$.

Theorem 4. *Let* F, G, A *and* B *be self-mappings of a complete metric space* (X, d) *satisfying* (1) *and*

$$\begin{aligned}d^{2p}(Fx,Gy) \leq\ & \phi(d^{2p}(Bx,Ay), \\ & d^{q}(Bx,Fx)\ d^{q^\star}(Ay,Gy), \\ & d^{r}(Bx,Gy)\ d^{r^\star}(Ay,Fx), \\ & d^{s}(Bx,Fx)\ d^{s^\star}(Ay,Fx), \\ & d^{l}(Bx,Gy)\ d^{l^\star}(Ay,Gy)) \qquad (2^{\star\star})\end{aligned}$$

for all $x, y \in X$*, where* $\phi \in \Phi$, $0 < p, q, q^\star, r, r^\star, s, s^\star, l, l^\star \leq 1$ *such that* $2p = q + q^\star = r + r^\star = s + s^\star = l + l^\star$ *and any one of these four mappings is continuous. If* (F, B) *and* (G, A) *are weakly commuting, then* F, G, B *and* A *have a unique common fixed point in* X.

Remark 2. *By Theorem 4, we get the improved version of Theorem 3.1 of Pathak-Mishra-Kalinde* [5].

We now give an example in which is used Theorem 1.

Example 1. *Let* X *be reals with* δ *induced by the Euclidean metric* d *and we define*

$$Fx = \begin{cases} \{0\} & \text{if } x \leq 0 \\ [0, \frac{x}{1+3x}] & \text{if } 0 < x \leq 1 \\ [0, \frac{1}{4}] & \text{if } x > 1 \end{cases}, \qquad Ax = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$$Gx = \begin{cases} \{0\} & \text{if } x \leq 0 \\ [0, \frac{x}{1+2x}] & \text{if } 0 < x \leq 1 \\ [0, \frac{1}{3}] & \text{if } x > 1 \end{cases}, \qquad Bx = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

for all x *in* X *and let* $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *be given by*

$$\gamma(t) < t$$

and let $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ *be given by*

$$\phi(t_1, t_2, a_1 t_3, a_2 t_4, a_3 t_5) = \begin{cases} 0 & \text{if } t_i = 0 \\ \gamma(t) & \text{if } t_i = t \text{ and } a_1 + a_2 + a_3 = 8 \\ \beta \max\{t_i\} & \text{otherwise} \end{cases}$$

for some $0 < \beta < 1, i = 1, 2, 3, 4, 5$. *Then for all* x *in* X. *Hence* $F(X) \subseteq A(X)$ *and* $G(X) \subseteq B(X)$.

Now we examine the following cases

case 1 : *if* $x \le 0$ *and* $y \le 0$, *then*

$$\delta^2(Fx, Gy) = 0 \le 0 = \phi(0, 0, 0, 0, 0)$$

case 2 : *if* $x \le 0$ *and* $0 < y \le 1$, *then*

$$\delta^2(Fx, Gy) = \left(\frac{y}{1+2y}\right)^2 \le \beta y^2 = \phi(y^2, 0, \frac{y^2}{1+2y}, 0, \frac{y^2}{1+2y})$$

case 3 : *if* $x \le 0$ *and* $y > 1$, *then*

$$\delta^2(Fx, Gy) = \left(\frac{1}{3}\right)^2 \le \beta = \phi(1, 0, \frac{1}{3}, 0, \frac{1}{3})$$

case 4 : *if* $0 < x \le 1$ *and* $y \le 0$, *then*

$$\delta^2(Fx, Gy) = \left(\frac{x}{1+3x}\right)^2 \le \beta x^2 = \phi(x^2, 0, \frac{x^2}{1+3x}, \frac{x^2}{1+3x}, 0)$$

case 5 : *if* $0 < x \le 1$ *and* $y > 1$, *then*

$$\begin{aligned} \delta^2(Fx, Gy) &= \left(\frac{1}{3}\right)^2 \le \begin{cases} \beta(1-x)^2 & \text{if } x \le \frac{1}{3} \\ \beta(1-x)^2 & \text{if } \frac{1}{3} \le x < \frac{3-\sqrt{5}}{2} \\ \beta x & \text{if } \frac{3-\sqrt{5}}{2} \le x \end{cases} \\ &= \begin{cases} \phi((1-x)^2, x, \frac{1}{3}, x, \frac{1}{3}) & \text{if } x \le \frac{1}{3} \\ \phi((1-x)^2, x, x, x, x) & \text{if } \frac{1}{3} < x \end{cases} \end{aligned}$$

case 6 : *if* $x > 1$ *and* $y \le 0$, *then*

$$\delta^2(Fx, Gy) = \left(\frac{1}{4}\right)^2 \le \beta = \phi(1, 0, \frac{1}{4}, \frac{1}{4}, 0)$$

case 7 : *if* $x > 1$ *and* $0 < y \le 1$, *then*

$$\begin{aligned} \delta^2(Fx, Gy) &= \begin{cases} \left(\frac{1}{4}\right)^2 & \text{if } y \le \frac{1}{2} \\ \left(\frac{y}{1+2y}\right)^2 & \text{if } \frac{1}{2} < y \end{cases} \\ &\le \begin{cases} \beta(1-y)^2 & \text{if } y \le \frac{1}{4} \\ \beta(1-y)^2 & \text{if } \frac{1}{4} \le y < \frac{3-\sqrt{5}}{2} \\ \beta y & \text{if } \frac{3-\sqrt{5}}{2} \le y \end{cases} \\ &= \begin{cases} \phi((1-y)^2, y, \frac{1}{4}, \frac{1}{4}, y) & \text{if } y \le \frac{1}{4}, \\ \phi((1-y)^2, y, y, y, y) & \text{if } \frac{1}{4} < y, \end{cases} \end{aligned}$$

case 8 : *if* $x > 1$ *and* $y > 1$, *then*

$$\delta^2(Fx, Gy) = \left(\frac{1}{3}\right)^2 \leq \beta = \phi(0, 1, 1, 1, 1)$$

case 9 : *if* $0 < x \leq 1$ *and* $0 < y \leq 1$, *then*

$$\delta^2(Fx, Gy) = \begin{cases} \left(\frac{x}{1+3x}\right)^2 & \textit{if } \left(\frac{y}{1+2y}\right)^2 \leq \left(\frac{x}{1+3x}\right)^2 \\ \left(\frac{y}{1+2y}\right)^2 & \textit{if } \left(\frac{x}{1+3x}\right)^2 < \left(\frac{y}{1+2y}\right)^2 \end{cases}$$

subcase 9_1 : *if* $\frac{y}{1+2y} < y < \frac{x}{1+3x} < x$, *then*

$$\phi((x-y)^2, xy, \frac{x^2}{1+3x}, \frac{x^2}{1+3x}, xy) = \begin{cases} \beta\,(x-y)^2 & \textit{if } \frac{x^2}{1+3x} \leq (x-y)^2 \\ \beta\frac{x^2}{1+3x} & \textit{if } (x-y)^2 < \frac{x^2}{1+3x} \end{cases}$$

subcase 9_2 : *if* $\frac{y}{1+2y} < \frac{x}{1+3x} < y < x$, *then*

$$\phi((x-y)^2, xy, xy, xy, xy) = \begin{cases} \beta\,(x-y)^2 & \textit{if } xy \leq (x-y)^2 \\ \beta xy & \textit{if } (x-y)^2 < xy \end{cases}$$

subcase 9_3 : *if* $\frac{y}{1+2y} < \frac{x}{1+3x} < x < y$, *then*

$$\phi((x-y)^2, xy, xy, xy, xy) = \begin{cases} \beta\,(x-y)^2 & \textit{if } xy \leq (x-y)^2, \\ \beta xy & \textit{if } (x-y)^2 < xy, \end{cases}$$

subcase 9_4 : *if* $\frac{x}{1+3x} < \frac{y}{1+2y} < x < y$, *then*

$$\phi((x-y)^2, xy, xy, xy, xy) = \begin{cases} \beta\,(x-y)^2 & \textit{if } xy \leq (x-y)^2 \\ \beta xy & \textit{if } (x-y)^2 < xy \end{cases}$$

subcase 9_5 : *if* $\frac{x}{1+3x} < x < \frac{y}{1+2y} < y$, *then*

$$\phi((x-y)^2, xy, \frac{y^2}{1+2y}, xy, \frac{y^2}{1+2y}) = \begin{cases} \beta\,(x-y)^2 & \textit{if } \frac{y^2}{1+2y} \leq (x-y)^2 \\ \beta\frac{y^2}{1+2y} & \textit{if } (x-y)^2 < \frac{y^2}{1+2y} \end{cases}$$

subcase 9_6 : *if* $\frac{x}{1+3x} < \frac{y}{1+2y} < y < x$, *then*

$$\phi((x-y)^2, xy, xy, xy, xy) = \begin{cases} \beta\,(x-y)^2 & \textit{if } xy \leq (x-y)^2 \\ \beta xy & \textit{if } (x-y)^2 < xy \end{cases}$$

and

$$\begin{aligned} \delta^{2p}(Fx, Gy) \leq\ & \phi(d^{2p}(Bx, Ay), \\ & \delta^q(Bx, Fx)\ \delta^{q^\star}(Ay, Gy), \\ & \delta^r(Bx, Gy)\ \delta^{r^\star}(Ay, Fx), \\ & \delta^s(Bx, Fx)\ \delta^{s^\star}(Ay, Fx), \\ & \delta^l(Bx, Gy)\ \delta^{l^\star}(Ay, Gy)) \end{aligned}$$

for $0 < p = q = q^* = r = r^* = s = s^* = l = l^* = 1$ *and* $2 = q + q^* = r + r^* = s + s^* = l + l^*$. *Also* (F, B) *and* (G, A) *are slightly commuting. Really,*

$$FBx = \begin{cases} \{0\} & \text{if } x \leq 0 \\ [0, \frac{x}{1+3x}] & \text{if } 0 < x \leq 1 \\ [0, \frac{1}{4}] & \text{if } x > 1 \end{cases}, \qquad BFx = \begin{cases} 0 & \text{if } x \leq 0 \\ [0, \frac{x}{1+3x}] & \text{if } 0 < x \leq 1 \\ [0, \frac{1}{4}] & \text{if } x > 1 \end{cases}$$

$$GAx = \begin{cases} \{0\} & \text{if } x \leq 0 \\ [0, \frac{x}{1+2x}] & \text{if } 0 < x \leq 1 \\ [0, \frac{1}{3}] & \text{if } x > 1 \end{cases}, \qquad AGx = \begin{cases} 0 & \text{if } x \leq 0 \\ [0, \frac{x}{1+2x}] & \text{if } 0 < x \leq 1 \\ [0, \frac{1}{3}] & \text{if } x > 1 \end{cases}$$

and

i) if $x \leq 0$, *then*

$$\begin{aligned} \delta(FBx, BFx) &= 0 \leq 0 = \delta(Fx, Bx) \leq \max\{\delta(Fx, Bx), diamFx\}, \\ \delta(GAx, AGx) &= 0 \leq 0 = \delta(Gx, Ax) \leq \max\{\delta(Gx, Ax), diamGx\} \end{aligned}$$

ii) if $0 < x \leq 1$, *then*

$$\begin{aligned} \delta(FBx, BFx) &= \frac{x}{1+3x} \leq x = \delta(Fx, Bx) \leq \max\{\delta(Fx, Bx), diamFx\}, \\ \delta(GAx, AGx) &= \frac{x}{1+2x} \leq x = \delta(Gx, Ax) \leq \max\{\delta(Gx, Ax), diamGx\}. \end{aligned}$$

iii) if $x > 1$, *then*

$$\begin{aligned} \delta(FBx, BFx) &= \frac{1}{4} \leq 1 = \delta(Fx, Bx) \leq \max\{\delta(Fx, Bx), diamFx\}, \\ \delta(GAx, AGx) &= \frac{1}{3} \leq 1 = \delta(Gx, Ax) \leq \max\{\delta(Gx, Ax), diamGx\}. \end{aligned}$$

Further, B *and* T *are continuous. Then* F, G, B *and* A *have a unique common fixed point in* X *by Theorem 1.*

References

[1] Sessa, S., Khan, M. S., Imdad, M., A common fixed point theorem with a weak commutativity condition. Glas. Math. 21 (41) (1986), 225-235.

[2] Fisher, B., Common fixed point of mappings and set-valued mappings. Rostock Math Kolloq. 18 (1981), 69-77.

[3] Imdad, M., Khan, M. S., Sessa, S., On some weak conditions of commutativity in common fixed point theorems. Int. J. Math. Math. Sci. 11 (2) (1988), 289-296.

[4] Singh, S. P., Meade, B. A., A common fixed point theorems. Bull. Austral. Math. Soc. 16 (1977), 49-53.

[5] Pathak, H. K., Mishra, S. N., Kalinde, A. K., Common fixed point theorems with applications to nonlinear integral equations. Demonstratio Math. 3 (1999), 547-564.

[6] Imdad, M., Ahmad, A., On common fixed point of mappings and set-valued mappings with some weak conditions of commutativity. Publ. Math. Debrecen. 44 (1994), 105-114.

[7] Sessa, S., On a weak commutativity condition of mappings in fixed point considerations. Publ. Inst. Math. (Beograd) 32 (46) (1982), 149-153.

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