

## SEMILATTICES WITH SECTIONALLY ANTITONE BIJECTIONS

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**Abstract.** We study  $\vee$ -semilattices with the greatest element 1 where on each interval  $[a, 1]$  an antitone bijection is defined. We characterize these semilattices by means of two induced binary operations proving that the resulting algebras form a variety. The congruence properties of this variety and the properties of the underlying semilattices are investigated. We show that this variety contains a single minimal subquasivariety.

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Join-semilattices, whose principal filters are Boolean lattices, were used by J. C. Abbott [1] for a characterization of the logic connective implication in the classical propositional logic. These semilattices also have the property that on each principal filter of them an antitone involution is defined. Motivated by this observation, the notion of a  $\vee$ -semilattice with sectionally antitone involutions was defined in [3] and [5]. In this paper we introduce a further generalization of this concept, defining the notion of a semilattice with sectionally antitone bijections. Our aim is to obtain by means of these semilattices "nice" algebraic structures, i.e. a variety of algebras characterized by nice congruence properties.

Let  $\mathcal{S} = (S, \vee, 1)$  be a  $\vee$ -semilattice with the greatest element 1. For each  $a \in S$  the interval  $[a, 1]$  (with respect to the induced order) will be called a section. We say that  $\mathcal{S}$  is a semilattice with sectionally antitone bijections if for each  $a \in S$  there exists a bijection  $f_a$  of  $[a, 1]$  into itself such that

$$x \leq y \Leftrightarrow f_a(y) \leq f_a(x), \text{ for all } x, y \in [a, 1].$$

Of course, the inverse  $f_a^{-1}$  of  $f_a$  is also an antitone bijection on  $[a, 1]$ . If each  $f_a$  is an involution, i.e.  $f_a^2(x) = x$ , for all  $x \in [a, 1]$ , then  $\mathcal{S}$  is called a semilattice with sectionally antitone involutions (see [3]).

Given a semilattice  $\mathcal{S}$  with sectionally antitone bijections, we can introduce two new binary operations on  $S$  as follows:

$$x \circ y = f_y(x \vee y) \text{ and } x * y = f_y^{-1}(x \vee y) \quad (P)$$

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Since  $x \vee y \in [y, 1]$ ,  $\circ$  and  $*$  are everywhere defined operations on the set  $S$ . Conversely, one can check immediately that for any  $a \in S$  and  $x \in [a, 1]$

$$f_a(x) = x \circ a \text{ and } f_a^{-1}(x) = x * a \quad (A)$$

Clearly, if all the mappings  $f_a$  are involutions, then  $x \circ y = x * y$ , for all  $x, y \in S$  (since  $f_a = f_a^{-1}$  for each  $a \in S$ ).

**Lemma 1.** *Let  $S$  be a  $\vee$ -semilattice with sectionally antitone bijections and  $\circ, *$  be operations defined by (P). Then*

- (1)  $x \circ x = x * x = 1$ ,  $x \circ 1 = x * 1 = 1$ ,  $1 \circ x = 1 * x = x$
- (2)  $(x \circ y) * y = (x * y) \circ y = (y \circ x) * x = (y * x) \circ x$
- (3)  $((x \circ y) * y) \circ z = ((x \circ y) * y) * z = (x * z) \circ z = 1$

*Proof.* Suppose  $a, b \in S$  and  $a \leq b$ . Then

$$\left\{ \begin{array}{l} a \circ b = f_b(a \vee b) = f_b(b) = 1 \text{ and} \\ a * b = f_b^{-1}(a \vee b) = f_b^{-1}(b) = 1 \end{array} \right\} (Q)$$

Hence  $x \circ x = x * x = 1$  and  $x \circ 1 = x * 1 = 1$ . We also obtain  $1 \circ x = f_x(1) = x$  and  $1 * x = f_x^{-1}(1) = x$ . Thus (1) is satisfied.

(2)  $(x \circ y) * y = f_y^{-1}(f_y(x \vee y) \vee y) = f_y^{-1}(f_y(x \vee y)) = x \vee y$  since  $f_y(x \vee y) \geq y$  and hence  $f_y(x \vee y) \vee y = f_y(x \vee y)$ . Analogously, we can check  $(x * y) \circ y = x \vee y$ ,  $(y \circ x) * x = x \vee y$ , and  $(y * x) \circ x = x \vee y$ .

(3) As  $(x \circ y) * y = x \vee y$ , we get  $((x \circ y) * y) \circ z = f_z(x \vee y \vee z)$ . Further,  $x \circ z = f_z(x \vee z)$ . However,  $x \vee z \leq x \vee y \vee z$  and  $f_z$  is antitone, thus  $((x \circ y) * y) \circ z = f_z(x \vee y \vee z) \leq f_z(x \vee z) = x \circ z$ . Analogously, we prove  $((x \circ y) * y) * z = f_z^{-1}(x \vee y \vee z) \leq f_z^{-1}(x \vee z) = x * z$

By (Q) we obtain (3) immediately.  $\square$

**Theorem 1.** *Let  $\mathcal{A} = (A, \circ, *, 1)$  be an algebra of type  $(2, 2, 0)$  satisfying the identities (1) and (2). Define a binary relation  $\leq$  on  $A$  as follows:*

$$a \leq b \text{ if and only if } a \circ b = 1. \quad (R)$$

*Then the following assertions are equivalent:*

- (i) *The algebra  $\mathcal{A}$  satisfies identity (3).*
- (ii) *For any  $x, y, z \in A$  the implications*
  - (4)  $x \leq y \Rightarrow y \circ z \leq x \circ z$  and  $x \leq y \Rightarrow y * z \leq x * z$  *are satisfied.*
- (iii)  $\leq$  *is a partial order on  $A$  and  $(A, \leq)$  is a  $\vee$ -semilattice with the greatest element 1, where  $a \vee b = (a \circ b) * b$  and for any  $a \in A$  the maps  $f_a(x) = x \circ a$ ,  $f_a^{-1}(x) = x * a$  are antitone bijections on  $[a, 1]$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Suppose  $x \leq y$ . Then using (1), (R) and (3) we obtain:

$(y \circ z) \circ (x \circ z) = ((1 * y) \circ z) \circ (x \circ z) = (((x \circ y) * y) \circ z) \circ (x \circ z) = 1$ , and hence  $y \circ z \leq x \circ z$ .

Analogously, we obtain:

$(y * z) \circ (x * z) = (((1 * y) * z) \circ (x * z) = (((x \circ y) * y) * z) \circ (x * z) = 1$ , whence  $y * z \leq x * z$ .

(ii)  $\Rightarrow$  (iii). Assume that (1) (2) and (4) are satisfied. First we prove that the relation  $\leq$  defined by (R) is a partial order.

Due to (1),  $\leq$  is reflexive. Suppose  $x \leq y$  and  $y \leq x$ . Then  $x \circ y = 1$  and  $y \circ x = 1$  hence by (1) and (2),

$$x = 1 * x = (y \circ x) * x = (x \circ y) * y = 1 * y = y,$$

thus  $\leq$  is antisymmetrical.

Suppose  $x \leq y$  and  $y \leq z$ . Then we get  $y \circ z = 1$  by (R), and  $x \circ z \geq y \circ z$ , by (4). Hence we obtain  $x \circ z = 1$ , i.e.  $x \leq z$ . Thus  $\leq$  is transitive, i.e. it is a partial order.

As for any  $x \in A$  we have  $x \circ 1 = 1$ , we get  $x \leq 1$  for all  $x \in A$ . Therefore,  $z = 1 \circ z \leq x \circ z$ , for all  $x, z \in A$  and hence

$$z \circ (x \circ z) = 1, \text{ for all } x, z \in A \quad (S)$$

Define  $a \vee b = (a \circ b) * b$  for all  $a, b \in A$ . Then (2) and (S) implies

$$a \circ ((a \circ b) * b) = a \circ ((b * a) \circ a) = 1 \text{ and}$$

$$b \circ ((a \circ b) * b) = b \circ ((a * b) \circ b) = 1,$$

thus  $a \leq a \vee b$  and  $b \leq a \vee b$ .

Suppose now  $a \leq c$  and  $b \leq c$  for some  $c \in A$ . Then  $b \circ c = 1$  and  $c = 1 * c = (b \circ c) * c = (c \circ b) * b$  by (2). This gets  $((a \circ b) * b) \circ c = ((a \circ b) * b) \circ ((c \circ b) * b)$ . Due to (4) we infer  $a \leq c \Rightarrow c \circ b \leq a \circ b \Rightarrow (a \circ b) * b \leq (c \circ b) * b$  and hence  $((a \circ b) * b) \circ ((c \circ b) * b) = 1$ , i.e.  $((a \circ b) * b) \circ c = 1$  proving  $a \vee b \leq c$ . Thus  $a \vee b$  is  $\sup\{a, b\}$  w.r.t.  $\leq$ .

Now consider  $a \in A$ ,  $f_a, f_a^{-1}$  defined by (A) and  $x \in [a, 1]$ . Then

$$f_a^{-1}(f_a(x)) = (x \circ a) * a = x \vee a = x \text{ and}$$

$$f_a(f_a^{-1}(x)) = (x * a) \circ a = x \vee a = x,$$

thus  $f_a$  and  $f_a^{-1}$  are bijections on  $[a, 1]$  (and inverses each of other).

For  $x, y \in [a, 1]$  with  $x \leq y$  we have by (4)

$$f_a(y) = y \circ a \leq x \circ a = f_a(x) \text{ and}$$

$$f_a^{-1}(y) = y * a \leq x * a = f_a^{-1}(x),$$

therefore  $f_a$  and  $f_a^{-1}$  are antitone bijections.

(iii)  $\Rightarrow$  (i). By the assumptions of (iii)  $(A, \leq)$  is a  $\vee$ -semilattice with sectionally antitone bijections. Take any  $x, y \in A$ . Since

$$f_y(x \vee y) = (x \vee y) \circ y = ((x \circ y) * y) \circ y = f_y(f_y^{-1}(x \circ y)) = x \circ y \text{ and}$$

$$f_y^{-1}(x \vee y) = (x \vee y) * y = ((x * y) \circ y) * y = f_y^{-1}(f_y(x * y)) = x * y, \circ \text{ and } *$$

can be also defined using relation (P). By applying Lemma 1, we obtain that the algebra  $(A, \circ, *, 1)$  satisfies the identity (3).  $\square$

**Corollary 1.** Let  $\mathcal{A} = (A, \circ, *, 1)$  be an algebra of type  $(2, 2, 0)$  satisfying the identities (1), (2) and (3) and let  $\leq$  be the induced order (defined by (R)). Then

(i)  $(A, \leq)$  is a  $\vee$ -semilattice with 1, where for each  $a \in A$  the section  $[a, 1]$  is a lattice and the maps  $f_a, f_a^{-1}$  (defined by (A)) are dual lattice automorphisms of  $[a, 1]$ . Moreover, for any  $x, y \in [a, 1]$  we have:

$$x \wedge y = \{[(x \circ a) \circ (y \circ a)] * (y \circ a)\} * a$$

(ii) The algebra  $\mathcal{A}' = (A, *, \circ, 1)$  also satisfies the identities (1), (2) and (3).

(iii) For each  $x, y, z \in A$  there exists  $(x \circ z) \wedge (y \circ z)$  and  $(x * z) \wedge (y * z)$  and it holds

$$\begin{aligned} (x \vee y) \circ z &= (x \circ z) \wedge (y \circ z), \\ (x \vee y) * z &= (x * z) \wedge (y * z). \end{aligned}$$

*Proof.* In view of Theorem 1, we have  $x \leq 1$ , for all  $x \in S$  and every section  $[a, 1]$  is a  $\vee$ -semilattice with respect to  $\leq$ . Since the poset  $([a, 1], \leq)$  and its dual  $([a, 1], \geq)$  are isomorphic via the mapping  $f_a$  (and  $f_a^{-1}$ ), the poset  $([a, 1], \geq)$  is a  $\vee$ -semilattice, too. However this means that  $([a, 1], \leq)$  is also  $\wedge$ -semilattice, and hence it is a lattice. Therefore,  $f_a$  and  $f_a^{-1}$  are dual lattice isomorphisms. Let  $x, y \in [a, 1]$ . Then  $x \wedge y \in [a, 1]$ , and by using Theorem 1(iii) we get  $x \wedge y = f_a^{-1}(f_a(x) \vee f_a(y)) = [(x \circ a) \vee (y \circ a)] * a = \{[(x \circ a) \circ (y \circ a)] * (y \circ a)\} * a$ .

(ii) Clearly, the identities (1) and (2) are also satisfied by  $\mathcal{A}' = (A, *, \circ, 1)$ . To prove that the algebra  $\mathcal{A}'$  satisfies (3), in view of Theorem 1(ii) it is enough to show that for any  $x, y \in A$ ,  $x \leq y \Leftrightarrow x * y = 1$ . Using the above (i) we get

$$x \leq y \Leftrightarrow y = x \vee y \Leftrightarrow f_y^{-1}(x \vee y) = f_y^{-1}(y) = 1 \Leftrightarrow x * y = 1.$$

(iii) Since  $x \leq 1$ , we have  $z = 1 \circ z \leq x \circ z$ , also  $z \leq y \circ z$ , thus  $x \circ z, y \circ z \in [z, 1]$ , and by (i), their meet exists. By the antitone property (4), we get  $(x \vee y) \circ z \leq x \circ z$  and  $(x \vee y) \circ z \leq y \circ z$ , thus  $(x \vee y) \circ z \leq (x \circ z) \wedge (y \circ z)$ .

Suppose  $a \in [z, 1]$  with  $a \leq (x \circ z) \wedge (y \circ z)$ . Then  $a \leq x \circ z, a \leq y \circ z$  imply

$$\begin{aligned} a * z &\geq (x \circ z) * z = x \vee z, \\ a * z &\geq (y \circ z) * z = y \vee z. \end{aligned}$$

Thus  $(x \vee y) \circ z \geq (x \vee y \vee z) \circ z \geq (a * z) \circ z = a \vee z = a$ .

Hence,  $(x \vee y) \circ z$  is the infimum of  $x \circ z$  and  $y \circ z$ . The second equality can be proven similarly.  $\square$

**Examples.** (1) A typical semilattice with sectionally antitone bijections is  $M_3$  (see Figure 1), where  $f_0(a) = b, f_0(b) = c, f_0(c) = a, f_0(1) = 0$  and for  $x \in \{a, b, c\}$   $f_x(x) = 1, f_x(1) = x$  and finally  $f_1(1) = 1$ . Clearly,  $f_0$  is not an involution.

(2) Another example, where the underlying semilattice is a distributive lattice, is presented in Figure 2.

We set  $f_0(a) = p, f_0(b) = r, f_0(c) = q, f_0(p) = b, f_0(q) = a, f_0(r) = c, f_0(0) = 1, f_0(1) = 0$ , and for  $x \in \{a, b, c, p, q, r\}$  we take  $f_x(y)$  the complement of  $y$  in  $[x, 1]$ . Of course,  $f_0$  is an antitone bijection which is neither an involution nor the complementation.

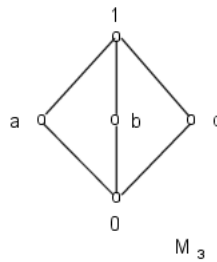


Figure 1:

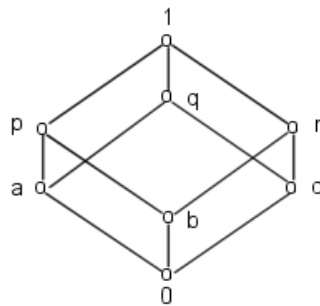


Figure 2:

(3) Let  $C : a_0 < a_1 < \dots < a_k$  be a finite chain (with  $k \geq 1$ ) and  $f : C \rightarrow C$  an antitone bijection. Then

$f(a_k) < f(a_{k-1}) < \dots < f(a_1) < f(a_0)$ , whence we get

$$f(a_i) = a_{k-i}, \text{ for all } i \in \{0, \dots, k\}. \quad (T)$$

Therefore,  $f$  is unique and  $f^2 = f \circ f$  is the identity mapping on  $C$ . Thus  $f$  is an involution and  $f = f^{-1}$ . It is also easy to see that the formula (T) defines an antitone involution on any finite chain  $C$ . From here it follows that any finite chain is a  $\vee$ -semilattice with antitone involutions, and these involutions can be defined in a unique way.

We say that a poset  $(P, \leq)$  with the greatest element 1 is a *tree*, if for any element  $p \in P$  the section  $[p, 1]$  is a chain. If every section  $[p, 1]$  is a finite chain, then  $(P, \leq)$  is called a *locally finite tree*. Clearly, any locally finite tree is a join-semilattice, too.

(4) Now let  $\mathcal{S} = (S, \leq)$  be a locally finite tree. As for any  $p \in S$   $[p, 1]$  is finite chain, an antitone involution  $f_p : [p, 1] \rightarrow [p, 1]$  can be defined by formula (T) (and by  $f_1(1) = 1$ ). Thus  $\mathcal{S}$  is a semilattice with antitone bijections.

By Theorem 1,  $(S, \leq)$  determines an algebra  $(S, \circ, *, 1)$  of type  $(2, 2, 0)$  which satisfies the identities (1), (2) and (3). As each  $f_p$  is an involution, we have  $x \circ y = x * y$  for all  $x, y \in S$ .

An element  $m \in P$  of a poset  $(P, \leq)$  is called *completely meet-irreducible*, if for any  $x_i \in P, i \in I$  whenever the meet  $\bigwedge_{i \in I} x_i$  there exists and equals to  $m$  then  $m = x_{i_0}$  for some  $i_0 \in I$ . (The completely join-irreducible elements of  $(P, \leq)$  are defined dually.) Let  $M(P)$  stand for the set of the completely meet-irreducible elements of  $(P, \leq)$ . If  $(P, \leq)$  has a greatest element 1 then we will consider  $1 \in M(P)$  (although the top of a poset usually is not considered to be meet irreducible).

A semilattice (lattice) is called *discrete* if any chain of it is finite. It is well-known that any discrete lattice is complete (see e.g. [6]).

**Proposition 1.** *Let  $(S, \vee, 1)$  be a discrete semilattice with sectional antitone bijections. Then for every  $m \in M(S)$  the section  $[m, 1]$  is a finite chain and  $(M(S), \leq)$  is a locally finite tree. Moreover, for any  $m \in M(S)$  and any  $x \in S$  we have  $x \circ m = x * m$  and  $(M(S), \circ, *, 1)$  is a subalgebra of  $(S, \circ, *, 1)$ .*

*Proof.* Take an  $m \in M(S) \setminus \{1\}$ . First, we prove that the section  $[m, 1]$  is a chain. As  $[m, 1]$  is a complete lattice and  $m$  is completely meet-irreducible we have  $m < \hat{m} = \bigwedge \{x \in S \mid m < x\}$ . Hence any  $x \in S$  with  $x > m$  satisfies  $x \geq \hat{m}$ . In view of (A) we obtain:

$$x \circ m \leq \hat{m} \circ m < m \circ m = 1. \quad (V)$$

Now, take any element  $z \in S$  with  $m < z < 1$ . Then  $f_m^{-1}(z) > f_m^{-1}(1) = m$  implies  $z * m > m$ . Substituting  $x = z * m$  in (V) we obtain  $(z * m) \circ m \leq \hat{m} \circ m$ .

As by Theorem 1(iii)  $z = m \vee z = (z * m) \circ m$ , we deduce

$$z \leq \hat{m} \circ m, \text{ for all } z \in S \text{ with } m < z < 1. \quad (*)$$

Assume now by contradiction that  $u, v$  are incomparable elements in  $[m, 1]$ . Then clearly,  $m \leq u \wedge v < u, v < 1$  and hence

$$1 = f_{u \wedge v}(u \wedge v) > f_{u \wedge v}(u) > f_{u \wedge v}(1) = u \wedge v \geq m \text{ and}$$

$$1 = f_{u \wedge v}(u \wedge v) > f_{u \wedge v}(v) > f_{u \wedge v}(1) = u \wedge v \geq m.$$

In view of (\*)  $m < f_{u \wedge v}(u) < 1$  and  $m < f_{u \wedge v}(v) < 1$  imply

$$f_{u \wedge v}(u), f_{u \wedge v}(v) \leq \hat{m} \circ m.$$

From here it follows:

$1 = f_{u \wedge v}(u \wedge v) = f_{u \wedge v}(u) \vee f_{u \wedge v}(v) \leq \hat{m} \circ m < 1$ , a contradiction. Thus for any  $m \in M(S)$  the section  $[m, 1]$  is a chain. As  $(S, \leq)$  is discrete,  $[m, 1]$  is a finite chain. Hence  $(M(S), \leq)$  is a locally finite tree and any  $z \in [m, 1]$  is completely meet-irreducible, as well. Thus  $[m, 1] \subseteq M(S)$ . In view of Corollary 1(i)  $f_m^2 : [m, 1] \rightarrow [m, 1]$  is a bijection and  $x \leq y \Leftrightarrow f_m^2(x) \leq f_m^2(y)$ , i.e.  $f_m^2$  is an automorphism of the lattice  $([m, 1], \leq)$ . Since any finite chain admits as a

lattice automorphism only the identity map, we get  $f_m = f_m^{-1}$ . Hence for each  $x \in S$  we obtain

$$x \circ m = f_m(x \vee m) = f_m^{-1}(x \vee m) = x * m.$$

As for every  $x \in M(S)$   $x * m = x \circ m = f_m(x \vee m) \in [m, 1] \subseteq M(S)$ , the algebra  $(M(S), \circ, *, 1)$  is a subalgebra of  $(S, \circ, *, 1)$ .  $\square$

An *implication algebra* is an algebra  $(A, \circ)$  satisfying the identities

- (I1)  $(x \circ y) \circ x = x$ ,
- (I2)  $(x \circ y) \circ y = (y \circ x) \circ x$ ,
- (I3)  $x \circ (y \circ z) = y \circ (x \circ z)$ .

It is well-known that any implication algebra  $(A, \circ)$  contains an element  $1 \in A$  such that  $x \circ x = 1$ , for all  $x \in A$  (see e.g. [1]). The algebra  $(A, \circ, \circ, 1)$  of type  $(2, 2, 0)$  will be called a *double implication algebra*.

It is easy to see that any double implication algebra satisfies the identities (1), (2) and (3).

Indeed, for the algebra  $\mathcal{A} = (A, \circ, \circ, 1)$  the identity (2) is the same as (I2) and we have  $x \circ x = 1$ ,  $1 \circ x = x$  and  $x \circ 1 = (1 \circ x) \circ (1 \circ 1) = 1 \circ ((1 \circ x) \circ 1) = 1 \circ 1 = 1$ , i.e. (1) is also satisfied by  $\mathcal{A}$ . In view of [1], relation (R) defines a partial order  $\leq$  with the antitone property (4) on any implication algebra  $(A, \circ)$ . Hence, by Theorem 1, we obtain that  $(A, \circ, \circ, 1)$  satisfies the identity (3), as well.

Now, observe that for a two-element chain  $(\{0, 1\}, \leq)$  the algebra  $\mathcal{S}_2 = (\{0, 1\}, \circ, *)$ , constructed as in Example 3 (or 4), is a double implication algebra. Indeed,  $x * y = x \circ y$ , for all  $x, y \in \{0, 1\}$ , i.e.  $\circ$  and  $*$  are the same and hence (I2) is satisfied. As  $0 \circ 0 = 1 \circ 1 = 0 \circ 1 = 1$  and  $1 \circ 0 = 0$ , one can easily check that (I1) and (I3) hold on  $\mathcal{S}_2$ .

In [9] is proved that the implication algebras form a minimal quasivariety which is generated by the two element implication algebra  $(\{0, 1\}, \circ)$ . Hence it is not hard to see that the variety generated by the algebra  $\mathcal{S}_2$  is also a minimal quasivariety and it coincides to the variety of all double implication algebras.

**Proposition 2.** *The variety  $\mathcal{V}$  of the algebras  $(A, \circ, *, 1)$  of type  $(2, 2, 0)$  satisfying the identities (1), (2) and (3) contains a single minimal quasivariety, namely the variety of double implication algebras.*

*Proof.* Let  $\mathcal{W}$  be a nontrivial subquasivariety of  $\mathcal{V}$  and  $\mathcal{A} = (A, \circ, *, 1)$  a nontrivial algebra in  $\mathcal{W}$ . In view of Theorem 1, the corresponding poset  $(A, \leq)$  is a  $\vee$ -semilattice with sectionally antitone bijections. As  $|A| \geq 2$ , there exists an element  $a \in A$  with  $a \neq 1$ . In view of Lemma 1 we have  $a \circ a = a * a = 1 \circ 1 = 1 * 1 = 1$ ,  $a \circ 1 = a * 1 = 1$  and  $1 \circ a = 1 * a = a$ . Hence  $(\{a, 1\}, \circ, *, 1)$  is a subalgebra of  $(A, \circ, *, 1)$ . Since  $(\{a, 1\}, \leq)$  is a two-element chain, the algebra  $\mathcal{S}_2 = (\{a, 1\}, \circ, *, 1)$  is a double implication algebra with two elements. Denote the variety generated by it as  $\mathcal{V}_2$ . We already shown that  $\mathcal{V}_2$  is the variety of all double implication algebras. Since  $\mathcal{V}_2$  is a minimal quasivariety as well, we have  $\mathcal{V}_2 = Q(\mathcal{S}_2)$ , where  $Q(\mathcal{S}_2)$  denotes the quasivariety generated by the algebra  $\mathcal{S}_2$ . As  $\mathcal{S}_2 \in \mathcal{W}$  we get  $\mathcal{V}_2 = Q(\mathcal{S}_2) \subseteq \mathcal{W}$ , and this proves that  $\mathcal{V}_2$  is the unique

minimal quasivariety contained in variety the  $\mathcal{V}$ .  $\square$

**Corollary 2.** *Any nontrivial algebra  $\mathcal{A} = (A, \circ, *, 1)$  satisfying the identities (1), (2) and (3) contains a nontrivial subalgebra which satisfies the identity  $x * y = x \circ y$ .*

Let  $\mathcal{A} = (A, \circ, *, 1)$  be an algebra of type  $(2, 2, 0)$ . A nonempty subset  $K \subseteq A$  is called a *congruence kernel* of  $\mathcal{A}$  if  $K = [1]_\theta = \{x \in A \mid (x, 1) \in \theta\}$  for some congruence  $\theta$  of  $\mathcal{A}$ . Recall that  $\mathcal{A}$  is called *3-permutable* if  $\theta_1 \circ \theta_2 \circ \theta_1 \subseteq \theta_2 \circ \theta_1 \circ \theta_2$  holds for every  $\theta_1, \theta_2 \in \text{Con } \mathcal{A}$  (see e.g. [2]). According to J. Hagemann and A. Mitschke [8], a variety  $\mathcal{V}$  of algebras is congruence 3-permutable if and only if there exists ternary terms  $p_0, p_1, p_2$  and  $p_3$  in  $\mathcal{V}$  such that the identities below hold in  $\mathcal{V}$ :

$$\left\{ \begin{array}{l} p_0(x, y, z) = x, \quad p_3(x, y, z) = z \\ p_i(x, x, y) = p_{i+1}(x, y, y) \text{ for } i \in \{0, 1, 2\}. \end{array} \right\} \quad (B)$$

An algebra  $\mathcal{A}$  with a constant 1 is called *weakly regular* if every  $\theta \in \text{Con } \mathcal{A}$  is determined by its kernel, i.e. if  $[1]_\theta = [1]_\phi$  implies  $\theta = \phi$  for every  $\phi, \theta \in \text{Con } \mathcal{A}$ . A variety  $\mathcal{V}$  is weakly regular if every algebra  $\mathcal{A} \in \mathcal{V}$  has this property. The following characterization of weakly regular varieties was given by B. Csákány in [7].

**Proposition 3.** *([7]) A variety  $\mathcal{V}$  with 1 is weakly regular if and only if there exist  $n \in \mathbb{N}$  and binary terms  $q_1(x, y), q_2(x, y), \dots, q_n(x, y)$  such that*

$$q_1(x, y) = q_2(x, y) = \dots = q_n(x, y) = 1 \Leftrightarrow x = y \quad (C)$$

*is satisfied for every algebra  $\mathcal{A} \in \mathcal{V}$ .*

**Theorem 2.** *The variety  $\mathcal{V}$  of the algebras  $(A, \circ, *, 1)$  of type  $(2, 2, 0)$  satisfying the identities (1), (2) and (3) is weakly regular, 3-permutable, arithmetical at 1 and congruence distributive.*

*Proof.* Consider the terms  $q_1(x, y) = x \circ y$  and  $q_2(x, y) = y \circ x$ . Then  $q_1(x, x) = q_2(x, x) = x \circ x = 1$ . If  $q_1(x, y) = 1$  and  $q_2(x, y) = 1$ , then by (R) we have  $x \leq y$  and  $y \leq x$  thus  $x = y$ . In view of Proposition 3, we conclude that  $\mathcal{V}$  is weakly regular (at 1).

Now, let  $p_0(x, y, z) = x$ ,  $p_3(x, y, z) = z$  and  $p_1(x, y, z) = (z * y) \circ x$ ,  $p_2(x, y, z) = (x * y) \circ z$ . It is easy to see that these terms satisfy the identities from (B). Consequently,  $\mathcal{V}$  is congruence 3-permutable.

In [4] is proved that a variety is arithmetical at 1 if and only if there exists a binary term  $b(x, y)$  of it such that  $b(x, x) = b(1, x) = 1$  and  $b(x, 1) = x$ . Obviously, in our case we can take  $b(x, y) = y \circ x$ .

Since  $\mathcal{V}$  is arithmetical at 1 it is congruence distributive at 1, i.e.

$$[1]_{\Theta \wedge (\Phi \vee \Psi)} = [1]_{(\Theta \wedge \Phi) \vee (\Theta \wedge \Psi)}, \text{ for all } \Theta, \Phi, \Psi \in \text{Con } \mathcal{A} \text{ for } \mathcal{A} \in \mathcal{V}.$$

As  $\mathcal{V}$  is weakly regular, this equality implies



$\Theta \wedge (\Phi \vee \Psi) = (\Theta \wedge \Phi) \vee (\Theta \wedge \Psi)$  (for all  $\Theta, \Phi, \Psi \in \text{Con}\mathcal{A}$  and  $\mathcal{A} \in \mathcal{V}$ ), thus the variety  $\mathcal{V}$  is congruence distributive.  $\square$

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