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A COMMENT ON $(n,m)-$GROUPS

Janez Ušan[1]

Abstract. This paper describes the $(n,m)-$groups for $n > 2m$ and $n \neq km$ with an additional condition.

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1. Preliminaries

Definition 1.1. **([1])** *Let $n \geq m+1$ $(n,m \in N)$ and $(Q;A)$ be an $(n,m)-$groupoid $(A : Q^n \to Q^m)$. We say that $(Q;A)$ is an $(n,m)-$group iff the following statements hold:*

$(|)$ *For every $i,j \in \{1,\ldots,n-m+1\}$, $i<j$, the following law holds*

$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$

[: $<i,j>-$associative law][2]*; and*

$(||)$ *For every $i \in \{1,\ldots,n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds*

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

Remark: For $m = 1$ $(Q;A)$ is an $n-$group [4]. Cf. Chapter I in [9].

Definition 1.2. **([7])** *Let $n \geq 2m$ and let $(Q;A)$ be an $(n,m)-$groupoid. Also, let* $\mathbf{e}$ *be a mapping of the set Q^{n-2m} into the set Q^m. Then, we say that* $\mathbf{e}$ *is a $\{1, n-m+1\}-$*neutral operation *of the $(n,m)-$groupoid $(Q;A)$ iff for every sequence a_1^{n-2m} over Q and for every $x_1^m \in Q^m$ the following equalities hold*

$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m \text{ and } A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m.$$

Remark: For $m = 1$ $\mathbf{e}$ is a $\{1,n\}-$neutral operation of the $n-$groupoid $(Q;A)$ [6]. Cf. Chapter II in [9].

Proposition 1.3. **([7])** *Let $n \geq 2m$ and let $(Q;A)$ be an $(n,m)-$groupoid. Then there is at most one $\{1, n-m+1\}-$neutral operation of $(Q;A)$.*

[1] Department of Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia and Montenegro

[2] $(Q;A)$ is an $(n,m)-$semigroup.

Proposition 1.4. (**[7]**) *Every $(n,m)-$group $(n \geq 2m)$ has a $\{1, n-m+1\}-$neutral operation.*

See, also [8].

2. Auxiliary part

Proposition 2.1. (**[2]**) *Let $(Q;A)$ be an $(n,m)-$groupoid and $n \geq 2m$. Also, let the following statements hold:*

(i) $(Q;A)$ is an $(n,m)-$semigroup;

(ii) For every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n;\ \text{and}$$

(iii) For every $a_1^n \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Then $(Q;A)$ is an $\{n,m\}-$group.

See, also 2.3 in [11].

Definition 2.2. *Let $(Q;A)$ be an $(n,m)-$groupoid; $n \geq m+1$. Then:*

(α) $\overset{1}{A} \overset{def}{=} A$; and

(β) For every $s \in N$ and for every $x_1^{(s+1)(n-m)+m} \in Q$

$$\overset{s+1}{A}(x_1^{(s+1)(n-m)+m}) \overset{def}{=} A(\overset{s}{A}(x_1^{s(n-m)+m}), x_{s(n-m)+m+1}^{(s+1)(n-m)+m}).$$

Proposition 2.3. *Let $(Q;A)$ be an $(n,m)-$semigroup and $s \in N$. Then, for every $x_1^{(s+1)(n-m)+m} \in Q$ and for every $t \in \{1,\ldots,s(n-m)+1\}$ the following equality holds*

$$\overset{s+1}{A}(x_1^{(s+1)(n-m)+m}) = \overset{s}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(s+1)(n-m)+m}).$$

Sketch of the proof. 1) $s=1$: By Def. 1.1 – (|) and by Def. 4.3-(α), we have

$$\overset{1+1}{A}(x_1^{2(n-m)+m}) = \overset{1}{A}(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2(n-m)+m})$$

for every $x_1^{2(n-m)+m} \in Q$ and for every $i \in \{1,\ldots,n-m+1\}$.

2) $s=v$: Let for every $x_1^{(v+1)(n-m)+m} \in Q$ and for all $t \in \{1,\ldots,v(n-m)+1\}$ the following equality holds

$$\overset{v+1}{A}(x_1^{(v+1)(n-m)+m}) = \overset{v}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(v+1)(n-m)+m}).$$

3) $v \to v+1$:

$$\overset{(v+1)+1}{A}(x_1^{(v+2)(n-m)+m}) \overset{(\beta)}{=} A(\overset{v+1}{A}(x_1^{(v+1)(n-m)+m}), x_{(v+1)(n-m)+m+1}^{(v+2)(n-m)+m}) \overset{2)}{=}$$

$$A(\overset{v}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(v+1)(n-m)+m}), x_{(v+1)(n-m)+m+1}^{(v+2)(n-m)+m}) \overset{(\beta)}{=}$$

$$\overset{v+1}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(v+1)(n-m)+m}, x_{(v+1)(n-m)+m+1}^{(v+2)(n-m)+m}) \overset{2)}{=}$$

$$\overset{v}{A}(x_1^{t-1}, A(A(x_t^{t+n-1}), x_{t+n}^{t+2(n-m)+m-1}), x_{t+2(n-m)+m}^{(v+2)(n-m)+m}) \overset{1.1(|)}{=\!=}$$

$$\overset{v}{A}(x_1^{t-1}, A(x_t^{t+i-2}, A(x_{t+i-1}^{t+i+n-2}), x_{t+i+n-1}^{t+2(n-m)+m-1}), x_{t+2(n-m)+m}^{(v+2)(n-m)+m}) \overset{2)}{=}$$

$$\overset{v+1}{A}(x_1^{t-1}, x_t^{t+i-2}, A(x_{t+i-1}^{t+i+n-2}), x_{t+i+n-1}^{t+2(n-m)+m-1}, x_{t+2(n-m)+m}^{(v+2)(n-m)+m}) =$$

$$\overset{v+1}{A}(x_1^{t+i-2}, A(x_{t+i-1}^{t+i+n-2}), x_{t+i+n-1}^{(v+2)(n-m)+m}). \qquad \square$$

By Def. 1.1 – (|), Def. 2.2 and by Prop. 2.3, we obtain:

Proposition 2.4. **([1])** *Let $(Q;A)$ be an $(n,m)-$semigroup and $(i,j) \in N^2$. Then, for every $x_1^{(i+j)(n-m)+m} \in Q$ and for all $t \in \{1,\ldots,i(n-m)+1\}$ the following equality holds*

$$\overset{i+j}{A}(x_1^{(i+j)(n-m)+m}) = \overset{i}{A}(x_1^{t-1}, \overset{j}{A}(x_t^{t+j(n-m)+m-1}), x_{t+j(n-m)+m}^{(i+j)(n-m)+m}).$$

By Prop. 2.4 and by Def. 1.1 – (|), we have:

Proposition 2.5. **([1])** *: Let $(Q;A)$ be an $(n,m)-$semigroup and let $s \in N$. Then $(Q;\overset{s}{A})$ is an $(s(n-m)+m,m)-$semigroup.*

Remark: In [1] $\overset{s}{A}$ is written as $[\ ]_s$.

Proposition 2.6. **([1])** *: Let $(Q;A)$ be an $(n,m)-$group, $n \geq 2m$ and let $s \in N$. Then $(Q;\overset{s}{A})$ is an $(s(n-m)+m,m)-$group.*

Sketch of the proof. Firstly we prove the following statements:

$\circ 1$ $(Q;\overset{s}{A})$ is an $(s(n-m)+m,m)-$semigroup.

$\circ 2$ For every $a_1^{s(n-m)+m} \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$\overset{s}{A}(a_1^{s(n-m)}, x_1^m) = a_{s(n-m)+1}^{s(n-m)+m}$.

$\circ 3$ For every $a_1^{s(n-m)+m} \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$\overset{s}{A}(y_1^m, a_1^{s(n-m)}) = a_{s(n-m)+1}^{s(n-m)+m}$.

The proof of $\circ 1$: By Prop. 2.5.

Sketch of the proof of $\circ 2$:

$s \geq 2$:

$\overset{s}{A}(a_1^{s(n-m)}, x_1^m) = a_{s(n-m)+1}^{s(n-m)+m} \overset{2.2}{\Longleftrightarrow}$
$A(\overset{s-1}{A}(a_1^{(s-1)(n-m)+m}), a_{(s-1)(n-m)+m+1}^{s(n-m)}, x_1^m) = a_{s(n-m)+1}^{s(n-m)+m}.$

Sketch of the proof of $°3$:
$s \geq 2$:
$\overset{s}{A}(y_1^m, a_1^{s(n-m)}) = a_{s(n-m)+1}^{s(n-m)+m} \overset{2.4}{\Longleftrightarrow}$
$A(y_1^m, a_1^{n-2m}, \overset{s-1}{A}(a_{n-2m+1}^{s(n-m)})) = a_{s(n-m)+1}^{s(n-m)+m}.$

Finally, by $°1 -° 3$ and by Prop. 2.1, we conclude that Prop. 2.6 holds. □

Proposition 2.7. **([10])** *Let $k > 2,\ m \geq 2, n = k{\cdot}m,\ (Q; A)$ be an $(n, m)-$group and $\mathbf{e}$ its $\{1, n-m+1\}-$neutral operation. Also, let there exist a sequence a_1^{n-2m} over Q such that for all $i \in \{0, 1, \ldots, 2m-1\}$, and for every $x_1^{2m} \in Q$ the following equality holds*
(0) $A(x_1^i, a_1^{n-2m}, x_{i+1}^{2m}) = A(x_1^{2m}, a_1^{n-2m})$.
Further on, let
(1) $B(x_1^{2m}) \overset{def}{=} A(x_1^m, a_1^{n-2m}, x_{m+1}^{2m})$ *and*
(2) $c_1^m \overset{def}{=} A(\overset{k}{\overline{\mathbf{e}(a_1^{n-2m})}\big|})$
for all $x_1^{2m} \in Q$. Then the following statements hold
(i) $(Q; B)$ *is a* $(2m, m)-$*group;*
(ii) *For all* $x_1^{k\cdot m} \in Q$

$$A(x_1^{k\cdot m}) = \overset{k}{B}(x_1^{k\cdot m}, c_1^m);\ \text{and}$$

(iii) *For all $j \in \{0, \ldots, m-1\}$ and for every $x_1^m \in Q$ the following equality holds*

$$B(x_1^j, c_1^m, x_{j+1}^m) = B(x_1^m, c_1^m).$$

Proposition 2.8. **([5])** *: Let $n > 2m,\ m > 1,\ (Q; A)$ be an $(n, m)-$group and $\mathbf{e}$ its $\{1, n-m+1\}-$neutral operation. Then for all $i \in \{0, 1, \ldots, m\}$, for every $t \in \{1, \ldots, n-2m+1\}$, for every $x_1^m \in Q^m$ and for all $a_1^{n-2m} \in Q$ the following equality holds*
$A(x_1^i, a_t^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = x_1^m.$

Remark: Prop. 2.8 for $n = 2m$ is proved in [2]. See, also [3].

Proposition 2.9. **([8])** *: Let $n > m+1$ and let $(Q; A)$ be an $(n, m)-$groupoid. Also, let*
(a) The $< 1, 2 > -$associative law holds in $(Q; A)$; and
(b) For every $a_1^{n-m} \in Q$ and for each $x_1^m, y_1^m \in Q^m$ the following implication holds
$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m.$
Then $(Q; A)$ is an $(n, m)-$semigroup.

3. Main part

Theorem 3.1. *Let $m \geq 2,\ s \geq 2,\ 0 < r < m,\ n = s \cdot m + r$ and let $(Q;A)$ be an $(n,m)-$group. Also, let there exist a sequence $a_1^{k\cdot m-2m}$, where $k = r - m + 1$, such that for all $i \in \{0, 1, \ldots, 2m-1\}$, and for every $x_1^{2m} \in Q$ the following equality holds*

(0) $\overset{m}{A}(x_1^i, a_1^{k\cdot m-2m}, x_{i+1}^{2m}) = \overset{m}{A}(x_1^{2m}, a_1^{k\cdot m-2m})$.

Then there is a mapping B of the set Q^{2m} into the set Q^m, $c_1^m \in Q^m$ and the sequence $\varepsilon_1^{(m-1)(n-m)}$ over Q such that the following statements hold

(1) *$(Q;B)$ is a $(2m,m)-$group;*

(2) *For all $j \in \{0, \ldots, m-1\}$ and for every $x_1^m \in Q$ the following equality holds*

$$B(x_1^j, c_1^m, x_{j+1}^m) = B(x_1^m, c_1^m);$$

(3) *For all $x_1^m \in Q$ the following equality holds*

$$A(x_1^m) = B(\overset{n-m}{B}(x_1^n, \varepsilon_1^{(m-1)(n-m)}), c_1^m).$$

(4) *For all $t \in \{0, \ldots, m-1\}$ and for every $y_1^r, z_1^m \in Q$ the following equality holds*

$$\overset{n-m-s+1}{B}(y_1^r, z_1^t, \varepsilon_1^{(m-1)(n-m)}, z_{t+1}^m) = \overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}).$$

Proof. Firstly we prove the following statements:

1° $(Q, \overset{m}{A})$ is a $(km,m)-$group, where $k = n - m + 1$.

2° Let E be a $\{1, km-m+1\}-$neutral operation of $(km,m)-$group $(Q;\overset{m}{A})$. Also let

a) $B(x_1^m, y_1^m) \overset{def}{=} \overset{m}{A}(x_1^m, a_1^{km-2m}, y_1^m)$

for all $x_1^m, y_1^m \in Q^m$, where a_1^{km-2m} from (0); and

b) $c_1^m \overset{def}{=} \overset{m}{A}(\overset{k}{\overline{\mathsf{E}(a_1^{km-2m})}}\Big|)$.

Then:

1) $(Q;B)$ is a $(2m,m)-$group;

2) For all $x_1^m \in Q^m$ and for all $j \in \{0, \ldots, m-1\}$ the following equality holds

$$B(x_1^i, c_1^m, x_{i+1}^m) = B(x_1^m, c_1^m);\ \text{and}$$

3) For all $x_1^{km} \in Q$ the following equality holds

$$\overset{m}{A}(x_1^{km}) = \overset{k}{B}(x_1^{km}, c_1^m).$$

3° Let $\mathbf{e}$ be a $\{1, n-m+1\}-$neutral operation of $(n,m)-$group $(Q;A)$. Then for all $x_1^m \in Q$ and for every $\overset{(i)}{b}{}_1^{n-2m}, i \in \{1, \ldots, m-1\}$, the following equality holds

$$A(x_1^n) = \overset{m}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-1}).$$

4° Let $\overset{(i)}{b}{}_1^{n-2m}$, $i \in \{1, \ldots, m-1\}$, be an arbitrary sequence over Q. Also, let

$$\varepsilon_1^{(m-1)(n-m)} \overset{def}{=} \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-1}.$$

Then for all $x_1^{(s-1)m}, y_1^r, z_1^m \in Q$ and for all $j \in \{0, \ldots, m-1\}$ the following equality holds

$\overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}) = \overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)})$.

The proof of 1° : By Prop. 2.6.

The proof of 2° : By Prop. 2.7.

Sketch of the proof of 3° :

a) $m = 2$:

$$\overset{2}{A}(x_1^n, b_1^{n-2m}, \mathbf{e}(b_1^{n-2m})) \overset{2.2}{=}$$
$$A(A(x_1^n), b_1^{n-2m}, \mathbf{e}(b_1^{n-2m})) \overset{1.2}{=} A(x_1^n)$$

b) $m > 2$:

$$\overset{m}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-2}, \overset{(m-1)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(m-1)}{b}{}_1^{n-2m}) \overset{2.2}{=}$$
$$A(\overset{m-1}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-2}), \overset{(m-1)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(m-1)}{b}{}_1^{n-2m}) \overset{1.2}{=}$$
$$\overset{m-1}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-2}) = \ldots \overset{1.2}{=} A(x_1^n).$$

Sketch of the proof of 4° [to the case $m = 3,\ n = 7$]:

$$\overset{3}{A}(x_1^3, y, z_1^3, b, \mathbf{e}(b), c, \mathbf{e}(c)) \overset{2.3}{=}$$
$$\overset{2}{A}(x_1^3, y, A(z_1^3, b, \mathbf{e}(b)), c, \mathbf{e}(c)) \overset{1.2,2.8}{=}$$
$$\overset{2}{A}(x_1^3, y, A(z_1^i, b, \mathbf{e}(b), z_{i+1}^3), c, \mathbf{e}(c)) =$$
$$\overset{2}{A}(x_1^3, y, A(z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3}, z_{i+1}^3), c, \mathbf{e}(c)) =$$
$$\overset{2}{A}(x_1^3, y, A(z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, \overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, z_{i+1}^3), c, \mathbf{e}(c)) \overset{2.3}{=}$$
$$\overset{2}{A}(x_1^3, y, z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, A(\overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, z_{i+1}^3, c, \mathbf{e}(c))) \overset{1.2,2.8}{=}$$
$$\overset{2}{A}(x_1^3, y, z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, A(\overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, c, \mathbf{e}(c), z_{i+1}^3)) \overset{2.3}{=}$$
$$\overset{3}{A}(x_1^3, y, z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, \overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, c, \mathbf{e}(c), z_{i+1}^3) =$$
$$\overset{3}{A}(x_1^3, y, z_1^i, b, \mathbf{e}(b), c, \mathbf{e}(c), z_{i+1}^3).$$

By 1° and 2°, we have (1) and (2).

Sketch of the proof of (3): By 2° [:3)] and by 3°.

$$(k = n - m + 1, \varepsilon_1^{(m-1)(n-m)} \overset{def}{=} \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-1}.)$$

Sketch of the proof of (4):

$$\overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}) \overset{4^\circ}{=}$$
$$\overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m) \overset{4^\circ-3)}{\Longrightarrow}$$
$$\overset{k}{B}(x_1^{(s-1)m}, y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}, c_1^m) =$$
$$\overset{k}{B}(x_1^{(s-1)m}, y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m, c_1^m) \overset{1^\circ,2.4}{\Longrightarrow}$$
$$\overset{s}{B}(x_1^{(s-1)m}, \overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}), c_1^m) =$$
$$\overset{s}{B}(x_1^{(s-1)m}, \overset{n-m-s+1}{B}(y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m), c_1^m) \overset{1^\circ,2.6}{\Longrightarrow}$$
$$\overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}) = \overset{n-m-s+1}{B}(y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m).$$

The proof of Th. 3.1 is completed. $\square$

Theorem 3.2. *Let $(Q;B)$ be a $(2m,m)-$group and $m \geq 2$. Also let:*

(a) c_1^m be an element of the set Q^m such that for every $i \in \{0,\dots,m-1\}$, and for every $x_1^m \in Q$ the following equality holds

$B(x_1^i, c_1^m, x_{i+1}^m) = B(x_1^m, c_1^m)$; *and*

(b) $\varepsilon_1^{(m-1)(n-m)}$ be a sequence over Q such that for all $j \in \{0,\dots,m-1\}$, and for every $y_1^r, z_1^m \in Q$ the following equality holds

$$\overset{n-m-s+1}{B}(y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m) = \overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}),$$

where $s \geq 2$, $0 < r < m$ and $n = s \cdot m + r$.

Further on, let

(c) $A(x_1^m) \overset{def}{=} B(\overset{n-m}{B}(x_1^n, \varepsilon_1^{(m-1)(n-m)}), c_1^m)$

for all $x_1^n \in Q$.

Then $(Q;A)$ is an $(n,m)-$group.

Proof. Firstly we prove the following statements:

$\overset{\circ}{1}$ The $< 1,2 > -$associative law holds in $(Q;A)$.

$\overset{\circ}{2}$ For every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$A(x_1^m, a_1^{n-m}) = a_{n-m+1}^n$.

$\overset{\circ}{3}$ $(Q;A)$ is an $(n,m)-$group.

$\overset{\circ}{4}$ For every $a_1^n \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$A(a_1^{n-m}, y_1^m) = a_{n-m+1}^n$.

Sketch of the proof of $\overset{\circ}{1}$:

$$a)\ A(A(x_1^n), x_{n+1}^{2n-m}) \overset{(c)}{=}$$

$$\overset{n-m+1}{B}(\overset{n-m+1}{B}(x_1^n, \varepsilon_1^{(m-1)(n-m)}, c_1^m), x_{n+1}, x_{n+2}^{2n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) \overset{2.4}{=}$$

$$\overset{n-m+1}{B}(x_1, \overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m)}, c_1^m, x_{n+1}), x_{n+2}^{2n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m).$$

$$b)\ \overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m)}, c_1^m, x_{n+1}) \overset{2.3}{=}$$

$$\overset{n-m}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, B(\varepsilon_{(m-1)(n-m+1)+1}^{(m-1)(n-m)}, c_1^m, x_{n+1})) \overset{(a)}{=}$$

$$\overset{n-m}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, B(\varepsilon_{(m-1)(n-m+1)+1}^{(m-1)(n-m)}, x_{n+1}, c_1^m)) \overset{2.3}{=}$$

$$\overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, \varepsilon_{(m-1)(n-m+1)+1}^{(m-1)(n-m)}, x_{n+1}, c_1^m) =$$

$$\overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, x_{n+1}, c_1^m) \overset{2.4}{=}$$

$$\overset{s}{B}(x_2^{(s-1)m+1}, \overset{n-m-s+1}{B}(x_{(s-1)m+2}^n, \text{[3]}\ \varepsilon_1^{(m-1)(n-m)}, x_{n+1}), c_1^m) \overset{(b)}{=}$$

$$\overset{s}{B}(x_2^{(s-1)m+1}, \overset{n-m-s+1}{B}(x_{(s-1)m+2}^n, x_{n+1}, \varepsilon_1^{(m-1)(n-m)}), c_1^m) =$$

---

[3] $n = sm + r$.

$\overset{s}{B}(x_2^{(s-1)m+1}, \overset{n-m-s+1}{B}(x_{(s-1)m+2}^{n+1}, \varepsilon_1^{(m-1)(n-m)}), c_1^m) \overset{2.4}{=}$
$\overset{n-m+1}{B}(x_2^{(s-1)m+2}, x_{(s-1)m+2}^{n+1}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) =$
$\overset{n-m+1}{B}(x_2^{n+1}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) \overset{(c)}{=} A(x_2^{n+1})$.
Finally, by $a)$, $b)$ and by (c), we obtain $\overset{\circ}{1}$.

Sketch of the proof of $\overset{\circ}{2}$:
$A(x_1^m, a_1^{n-m}) = a_{n-m+1}^n \overset{(c)}{\Longleftrightarrow}$
$\overset{n-m+1}{B}(x_1^m, a_1^{n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) = a_{n-m+1}^n \overset{2.4}{\Longleftrightarrow}$
$B(x_1^m, \overset{n-m}{B}(a_1^{n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m)) = a_{n-m+1}^n$.

The proof of $\overset{\circ}{3}$: By $\overset{\circ}{1}, \overset{\circ}{2}$ and Prop. 2.9.

Sketch of the proof of $\overset{\circ}{4}$:
$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \overset{(c)}{\Longleftrightarrow}$
$\overset{n-m+1}{B}(a_1^{n-m}, y_1^m, \varepsilon_1^{(m-1)(n-m)}, c_1^m) = a_{n-m+1}^n$.
Whence, by Prop. 2.6 and by Def. 1.1, we obtain $\overset{\circ}{4}$.

Finally, by $\overset{\circ}{2} - \overset{\circ}{4}$ and by Prop. 2.1, we conclude that Th. 3.2 holds. □

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