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POSITIVE SOLUTIONS OF NEUTRAL DELAY DIFFERENCE EQUATION

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Abstract. Neutral delay difference equations with variable delays are studied in this paper. Using the method of generalized characteristic equations, we give conditions for the existence of positive solutions.

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1. Introduction

The oscillatory and asymptotic behavior of solutions of neutral differential equations with constant delays have been studied in many papers (see, for example, [7] and references therein). A quite comprehensive treatment of such results is given in the monograph [1] by I. Győri and G. Ladas.

In the paper [7], the scalar nonautonomous neutral delay differential equation with variable delays

$$\frac{d}{dt}\left[x(t)+\sum_{j=1}^{\ell}p_j(t)x(t-\tau_j(t))\right]+\sum_{i=1}^{m}q_i(t)x(t-\sigma_i(t))=0,$$

for $t_0 \le t < T \le \infty$, was considered and, using the method of characteristic equations, some conditions for the existence of positive solutions were given.

The problem of oscillation and nonoscillation of all solutions of the neutral difference equation with constant delays

$$\Delta(a_n + c a_{n-m}) + p_n a_{n-k} = 0, \quad n = 0, 1, 2, ...$$

has been investigated in many papers (see [2], [3], [4], [8] and references therein), where Δ denotes the forward difference operator: $\Delta a_n = a_{n+1} - a_n$.

Consider now a discrete analogue of the above neutral differential equation, the linear neutral difference equation with variable delays

$$(1) \qquad \Delta\left[a_n+\sum_{j=1}^{\ell}P_j(n)a_{n-k_j(n)}\right]+\sum_{i=1}^{m}Q_i(n)a_{n-s_i(n)}=0, \quad n\in\mathbf{N}^*$$

where $\mathbf{N}^* = \{n \in \mathbf{N} : n_0 \le n < M,\ n_0 < M \le \infty\}$, $\mathbf{N}$ is the set of positive integers, and the next hypotheses are satisfied:

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(H_1) $\{P_j(n)\}$ and $\{Q_i(n)\}$ are sequences of real numbers for $i = 1, 2, ..., m,\ j = 1, 2, ..., \ell$;

(H_2) $\{k_j(n)\}$ and $\{s_i(n)\}$ are sequences of positive integers for $i = 1, 2, ..., m,\ j = 1, 2, ..., \ell$.

We cite two characteristic results of the recent investigations. In the papers [5] and [6], the particular case of equation (1) is investigated, the linear delay difference equation of the form

$$a_{n+1} - a_n + \sum_{i=1}^{m} P_i(n) a_{n-k_i(n)} = 0, \quad n \in N^* \ , \tag{2}$$

and the following results are proved. Let

$$n_{-1} = \min_{1 \le i \le m} \left\{ \inf_{n_0 \le n < M} \{n - k_i(n)\} \right\}.$$

Theorem A. (Péics [6]) *Assume that $\{P_i(n)\}$ is a sequence of real numbers and $\{k_i(n)\}$ is a sequence of positive integers for $i = 1, 2, ..., m,\ n \in N^*$, and there exists a real number $\mu \in (0,1)$ such that*

$$\sum_{i=1}^{m} |P_i(n)|(1-\mu)^{-k_i(n)} \le \mu \quad \text{for n} \in \text{N}^* \ .$$

Then, for every

$$\{\phi_n\} \in \{\{\phi_n\} : \phi_{n_0} > 0,\ 0 < \phi_n \le \phi_{n_0} \ \text{for n} = \text{n}_{-1}, \text{n}_{-1}+1, ..., \text{n}_0\}\,,$$

the solution $a(\phi)_n$ of (2) remains positive for $n \in N^*$.

Theorem B. (Péics [6]) *Assume that $\{P_i(n)\}$ is a sequence of real numbers and $\{k_i(n)\}$ is a sequence of positive integers for $i = 1, 2, ..., m,\ n \in N^*$, and that*

$$0 \le k_1(n) \le k_2(n) \le ... \le k_m(n) \quad \text{for n} \in \text{N}^*,$$

$$\sum_{i=1}^{s} P_i(n) \le 0 \quad \text{for s} = 1, 2, ..., \text{m},\ \text{n} \in \text{N}^* \ .$$

Then, the equation (2) has a positive increasing solution for $n \in N^*$.

The main purpose of this paper is to formulate the generalized characteristic equation associated to equation (1), to emphasize their importance at oscillation of all solutions of considered difference equation, and to establish conditions for the existence of positive solutions presented as the applications of the main result related to the generalized characteristic equation.

2. Notations, definitions

Define

$$M_1 := \min_{1\le j\le \ell}\left\{\inf_{n_0\le n<M}\{n-k_j(n)\}\right\},\quad M_2 := \min_{1\le i\le m}\left\{\inf_{n_0\le n<M}\{n-s_i(n)\}\right\},$$

$$n_{-1} := \min\{M_1,\ M_2\}\ . \tag{3}$$

By a solution of equation (1) we mean a sequence of real numbers $\{a_n\}$ defined for $n \ge n_{-1}$, where n_{-1} is defined by (3), and satisfies equation (1) for $n \in \mathbf{N}^*$.

If the values

$$a_n = \phi_n, \quad \text{for} \quad n = n_{-1}, n_{-1}+1, ..., n_0, \quad \phi_n \in R\ . \tag{4}$$

are given, then equation (1) has a unique solution satisfying the initial conditions (4).

The unique solution of the initial value problem (1) and (4) is denoted by $\{a_n^{\phi}\}$.

A nontrivial solution $\{a_n\}$ of equation (1) is said to be oscillatory if for every $N > n_0$ there exists an $n \ge N$ such that $x_n x_{n+1} \le 0$. Otherwise, it is nonoscillatory.

Define for $j = 1,2,...,\ell$, $i = 1,2,...,m$ and $n \in \mathbf{N}^*$ sequences

$$h_j(n) := min\{n_0, n-k_j(n)\},\quad H_j(n) := max\{n_0, n-k_j(n)\}\ ,$$

$$g_i(n) := min\{n_0, n-s_i(n)\},\quad G_i(n) := max\{n_0, n-s_i(n)\}\ .$$

The generalized characteristic equation associated with the initial value problem (1) and (4) is the equation

$$\begin{aligned} \lambda_n - 1 \quad &+ \quad \sum_{j=1}^{\ell}\Bigg[P_j(n+1)(1-\Delta k_j(n))\frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}}\lambda_{H_j(n)} + \\ &+ \quad (P_j(n+1)\Delta k_j(n) - P_j(n))\frac{\phi_{h_j(n)}}{\phi_{n_0}}\Bigg]\prod_{\nu=H_j(n)}^{n-1}\frac{1}{\lambda_\nu} + \\ &+ \quad \sum_{i=1}^{m} Q_i(n)\frac{\phi_{g_i(n)}}{\phi_{n_0}}\prod_{\nu=G_i(n)}^{n-1}\frac{1}{\lambda_\nu} = 0 \end{aligned} \tag{5}$$

for $n \in \mathbf{N}^*$.

This equation is obtained by looking for solutions of neutral difference equation (1) of the form

$$a_n = \prod_{j=n_0}^{n-1}\lambda_j\ ,$$

where $\{\lambda_n\}$ is a sequence of real numbers.

3. Main Results

The next result is very important in this topic, and it is applicable for the existence of positive solutions of the considered equation. It generalizes the main theorem proved in [5] for linear delay difference equations and it is a discrete analogue of the main result established in [7] for neutral differential equations with variable delays.

Theorem 1. *Assume that conditions* (H_1) *and* (H_2) *hold and let* $\phi_{n_0} > 0$. *Then the following statements are equivalent:*

a) the solution of the initial value problem (1) and (4) is positive for $n \in \mathbf{N}^*$,

b) the generalized characteristic equation (5) has a positive solution for $n \in \mathbf{N}^*$,

c) there exist real sequences $\{\beta_n\}$ *and* $\{\gamma_n\}$ *such that* $0 < \beta_n \le \gamma_n$ *for* $n \in \mathbf{N}^*$ *and such that*

$$\beta_n \le \delta_n \le \gamma_n \quad \textit{implies} \quad \beta_n \le S\delta_n \le \gamma_n \quad \textit{for } n \in \mathbf{N}^* \tag{6}$$

for any real sequence $\{\delta_n\}$, *where*

$$\begin{aligned} S\delta_n \quad \equiv \quad & 1 - \sum_{j=1}^{\ell} \Big[P_j(n+1)(1-\Delta k_j(n)) \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \delta_{H_j(n)} + \\ & + (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{\phi_{h_j(n)}}{\phi_{n_0}} \Big] \prod_{\nu=H_j(n)}^{n-1} \frac{1}{\delta_\nu} - \\ & - \sum_{i=1}^{m} Q_i(n) \frac{\phi_{g_i(n)}}{\phi_{n_0}} \prod_{\nu=G_i(n)}^{n-1} \frac{1}{\delta_\nu} \,. \end{aligned}$$

Proof. $(a) \Rightarrow (b)$: Let $a_n = a(\phi)_n$ be the solution of the initial value problem (1) and (4) and suppose that $a_n > 0$ for $n \in \mathbf{N}^*$. It will be shown that the sequence $\{\lambda_n\}$ defined by

$$\lambda_n = \frac{a_{n+1}}{a_n}, \quad , n \in \mathbf{N}^*$$

is a solution of (5) for $n \in \mathbf{N}^*$. Equation (1) is equivalent to the form

$$\begin{aligned} a_{n+1} - a_n \quad + \quad & \sum_{j=1}^{\ell} \big[P_j(n+1)(1-\Delta k_j(n)) a_{n+1-k_j(n)} + \\ + \quad & (P_j(n+1)\Delta k_j(n) - P_j(n))\, a_{n-k_j(n)} \big] + \sum_{i=1}^{m} Q_i(n) a_{n-s_i(n)} = 0. \end{aligned}$$

By dividing both sides of the above equation by a_n, we obtain that

$$\frac{a_{n+1}}{a_n} - 1 + \sum_{j=1}^{\ell} \Big[P_j(n+1)(1-\Delta k_j(n)) \frac{a_{n+1-k_j(n)}}{a_n} + $$
$$+ (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{a_{n-k_j(n)}}{a_n} \Big] + \sum_{i=1}^{m} Q_i(n) \frac{a_{n-s_i(n)}}{a_n} = 0.$$

It follows from the definition of $\{\lambda_n\}$ that

$$a_n = \phi_{n_0} \prod_{j=n_0}^{n-1} \lambda_j, \quad \text{for} \quad n \in \mathbf{N}^*,$$

and hence

$$\frac{a_{H_j(n)}}{a_n} = \prod_{\nu=H_j(n)}^{n-1} \frac{1}{\lambda_\nu}, \quad \frac{a_{G_i(n)}}{a_n} = \prod_{\nu=G_i(n)}^{n-1} \frac{1}{\lambda_\nu},$$

where $j = 1, 2, ..., \ell$, $i = 1, 2, ..., m$, and $n \in \mathbf{N}^*$. It is obvious for the same values of j, i and n that

$$\frac{a_{n-k_j(n)}}{a_{H_j(n)}} = \frac{\phi_{h_j(n)}}{\phi_{n_0}}, \quad \frac{a_{n-s_i(n)}}{a_{G_i(n)}} = \frac{\phi_{g_i(n)}}{\phi_{n_0}}.$$

It remains to prove that

$$\frac{a_{n+1-k_j(n)}}{a_{H_j(n)+1}} = \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}}, \quad n \in \mathbf{N}^*, \quad j = 1, 2, ..., \ell.$$

Observe that $n - k_j(n) \geq n_0$ implies $h_j(n) = n_0$ and $H_j(n) = n - k_j(n)$, and hence

$$\frac{a_{n+1-k_j(n)}}{a_{H_j(n)+1}} = \frac{a_{H_j(n)+1}}{a_{H_j(n)+1}} = 1 = \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}}.$$

On the other hand, $n - k_j(n) < n_0$ implies $h_j(n) = n - k_j(n)$ and $H_j(n) = n_0$, and hence

$$\frac{a_{n+1-k_j(n)}}{a_{H_j(n)+1}} = \frac{a_{h_j(n)+1}}{a_{n_0+1}} = \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}}.$$

Using these equalities and the definition of $\{\lambda_n\}$, we obtain that equality (5) holds, and hence the proof of the part "$(a) \Rightarrow (b)$" is complete.

$(b) \Rightarrow (c)$: If $\{\lambda_n\}$ is a positive solution of (5), then take $\beta_n \equiv \gamma_n \equiv \lambda_n$, $n \in \mathbf{N}^*$ and the proof is obvious because of the fact that $\lambda_n = S\lambda_n$.

$(c) \Rightarrow (a)$: First it must be shown that, under hypothesis (c), equation (5) has a positive solution λ_n for $n \in \mathbf{N}^*$, and the sequence $\{a_n\}$ defined by

$$a_n = \begin{cases} \phi_n, & n = n_{-1}, n_{-1}+1, ..., n_0; \\ \phi_{n_0} \prod_{j=n_0}^{n-1} \lambda_j, & n \in \mathbf{N}^* \end{cases} \tag{7}$$

is a positive solution of the initial value problem (1) and (4).

The positive solution of equation (5) will be constructed as the limit of a sequence $\{\lambda_n(r)\}_r$ for $n \in \mathbf{N}^*$ defined by the following successive approximation. Take any numerical sequence $\{\lambda_n(0)\}$ such that

$$\beta_n \leq \lambda_n(0) \leq \gamma_n, \quad n \in \mathbf{N}^*$$

and set

$$\lambda_n(r+1) = S\lambda_n(r), \quad n \in \mathbf{N}^* \quad \text{for} \quad r = 0, 1, 2, ...$$

It follows from the assumption (6) that

$$\beta_n \leq \lambda_n(r) \leq \gamma_n, \quad n \in \mathbf{N}^*, \quad r = 0, 1, 2, ..., \tag{8}$$

and clearly the sequence $\{\lambda_n(r)\}_r$ exists and has positive members. We show that the sequence $\{\lambda_n(r)\}_r$ converges uniformly for $n = n_0, n_0+1, ..., n_1$, where $n_1 \in \mathbf{N}$, $n_0 \leq n_1 < M$. Set

$$K_1 := \max_{n_0 \leq n \leq n_1} \left\{ \sum_{j=1}^{\ell} \left| P_j(n+1)(1 - \Delta k_j(n)) \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \right| \prod_{\nu=H_j(n)+1}^{n-1} \frac{1}{(\beta_\nu)^2} \right\},$$

$$K_2 := \max_{n_0 \leq n \leq n_1} \left\{ \sum_{j=1}^{\ell} \left| (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{\phi_{h_j(n)}}{\phi_{n_0}} \right| \prod_{\nu=H_j(n)}^{n-1} \frac{1}{(\beta_\nu)^2} \right\},$$

$$K_3 := \max_{n_0 \leq n \leq n_1} \sum_{i=1}^{m} \left| Q_i(n) \frac{\phi_{g_i(n)}}{\phi_{n_0}} \right| \prod_{\nu=G_i(n)}^{n-1} \frac{1}{(\beta_\nu)^2},$$

$$L := \max_{n_0 \leq n \leq n_1} \{\gamma_n\}, \quad K_0 := \max\{K_1, K_2, K_3\}, \quad K := 3K_0 L^{n_1 - n_0}.$$

Then from (8) we obtain that

$$\max_{n_0 \leq n \leq n_1} \{\lambda_n(r)\} \leq L, \quad r = 0, 1, 2, ...$$

Using elementary transformations we have

$$\begin{aligned} & |\lambda_n(r+1) - \lambda_n(r)| \leq \\ \leq \; & \sum_{j=1}^{\ell} \left| P_j(n+1)(1 - \Delta k_j(n)) \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \right| \cdot \\ & \cdot \left| \prod_{\nu=H_j(n)+1}^{n-1} \frac{1}{\lambda_\nu(r-1)} - \prod_{\nu=H_j(n)+1}^{n-1} \frac{1}{\lambda_\nu(r)} \right| + \\ + \; & \sum_{j=1}^{\ell} \left| (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{\phi_{h_j(n)}}{\phi_{n_0}} \right| \cdot \end{aligned}$$

$$
\begin{aligned}
&\cdot\left|\prod_{\nu=H_j(n)}^{n-1}\frac{1}{\lambda_\nu(r-1)}-\prod_{\nu=H_j(n)}^{n-1}\frac{1}{\lambda_\nu(r)}\right|+\\
+&\sum_{i=1}^{m}\left|Q_i(n)\frac{\phi_{g_i(n)}}{\phi_{n_0}}\right|\cdot\left|\prod_{\nu=G_i(n)}^{n-1}\frac{1}{\lambda_\nu(r-1)}-\prod_{\nu=G_i(n)}^{n-1}\frac{1}{\lambda_\nu(r)}\right|\le\\
\le&\sum_{j=1}^{\ell}\left|P_j(n+1)(1-\Delta k_j(n))\frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}}\right|\cdot\\
&\cdot\frac{L^{n_1-n_0}\sum_{\nu=H_j(n)+1}^{n-1}|\lambda_\nu(r)-\lambda_\nu(r-1)|}{\prod_{\nu=H_j(n)+1}^{n-1}(\beta_\nu)^2}+\\
+&\sum_{j=1}^{\ell}\left|(P_j(n+1)\Delta k_j(n)-P_j(n))\,\frac{\phi_{h_j(n)}}{\phi_{n_0}}\right|\cdot\\
&\cdot\frac{L^{n_1-n_0}\sum_{\nu=H_j(n)}^{n-1}|\lambda_\nu(r)-\lambda_\nu(r-1)|}{\prod_{\nu=H_j(n)}^{n-1}(\beta_\nu)^2}+\\
+&\sum_{i=1}^{m}\left|Q_i(n)\frac{\phi_{g_i(n)}}{\phi_{n_0}}\right|\frac{L^{n_1-n_0}\sum_{\nu=G_i(n)}^{n-1}|\lambda_\nu(r)-\lambda_\nu(r-1)|}{\prod_{\nu=G_i(n)}^{n-1}(\beta_\nu)^2}\le\\
\le&\ 3K_0L^{n_1-n_0}\sum_{\nu=n_0}^{n-1}|\lambda_\nu(r)-\lambda_\nu(r-1)|\le\\
\le&\ K\sum_{\nu=n_0}^{n-1}|\lambda_\nu(r)-\lambda_\nu(r-1)|.
\end{aligned}
$$

It can be shown by induction that

$$|\lambda_n(r+1)-\lambda_n(r)|\le LK^r\frac{(n-n_0)^{(r)}}{r!}.$$

for $r=0,1,2,...$ and $n_0\le n\le n_1$. Since

$$\lim_{r\to\infty}\frac{K^r(n-n_0)^{(r)}}{r!}=0\quad\text{for}\quad n_0\le n\le n_1,$$

it follows from the Weierstrass criterion that the series

$$\sum_{r=0}^{\infty}|\lambda_n(r+1)-\lambda_n(r)|$$

converges uniformly for $n=n_0,n_0+1,...,n_1$. Therefore, the sequence defined by

$$\lambda_n(r)=\lambda_n(0)+\sum_{j=0}^{r-1}[\lambda_n(j+1)-\lambda_n(j)]\ \ (r=0,1,2,...,\ n_0\le n\le n_1)$$

converges uniformly, and hence the limit sequence

(9) $$\lambda_n = \lim_{r\to\infty} \lambda_n(r)$$

exists and it is positive for $n = n_0, n_0+1, ..., n_1$. Because of the convergence,

$$\begin{aligned} \lambda_n &= \lim_{r\to\infty} \lambda_n(r+1) = \\ &= \lim_{r\to\infty} \Bigg\{ 1 - \sum_{j=1}^{\ell} \Bigg[P_j(n+1)(1-\Delta k_j(n)) \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \lambda_{H_j(n)}(r) \; + \\ &+ \; (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{\phi_{h_j(n)}}{\phi_{n_0}} \Bigg] \prod_{\nu=H_j(n)}^{n-1} \frac{1}{\lambda_\nu(r)} + \\ &+ \; \sum_{i=1}^{m} Q_i(n) \frac{\phi_{g_i(n)}}{\phi_{n_0}} \prod_{\nu=G_i(n)}^{n-1} \frac{1}{\lambda_\nu(r)} \Bigg\} = \\ &= 1 - \sum_{j=1}^{\ell} \Bigg[P_j(n+1)(1-\Delta k_j(n)) \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \lambda_{H_j(n)} \; + \\ &+ \; (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{\phi_{h_j(n)}}{\phi_{n_0}} \Bigg] \prod_{\nu=H_j(n)}^{n-1} \frac{1}{\lambda_\nu} + \\ &+ \; \sum_{i=1}^{m} Q_i(n) \frac{\phi_{g_i(n)}}{\phi_{n_0}} \prod_{\nu=G_i(n)}^{n-1} \frac{1}{\lambda_\nu}. \end{aligned}$$

Finally, the fact that $\{\lambda_n\}$ defined by (7) is the solution of the initial value problem (1) and (4) can be verified by direct substitution:

$$\begin{aligned} a_{n+1} &= a_n \lambda_n = \\ &= a_n - a_n \sum_{j=1}^{\ell} P_j(n+1)(1-\Delta k_j(n)) \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \lambda_{H_j(n)} \prod_{\nu=H_j(n)}^{n-1} \frac{1}{\lambda_\nu} - \\ &- \; a_n \sum_{j=1}^{\ell} (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{\phi_{h_j(n)}}{\phi_{n_0}} \prod_{\nu=H_j(n)}^{n-1} \frac{1}{\lambda_\nu(r)} - \\ &- \; \sum_{i=1}^{m} Q_i(n) \frac{\phi_{g_i(n)}}{\phi_{n_0}} \prod_{\nu=G_i(n)}^{n-1} \frac{1}{\lambda_\nu} = \\ &= a_n - a_n \sum_{j=1}^{\ell} P_j(n+1)(1-\Delta k_j(n)) \frac{a_{n+1-k_j(n)}}{a_{H_j(n)+1}} \cdot \frac{a_{H_j(n)+1}}{a_{H_j(n)}} \cdot \frac{a_{H_j(n)}}{a_n} - \\ &- \; a_n \sum_{j=1}^{\ell} (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{a_{n-k_j(n)}}{a_{H_j(n)}} \cdot \frac{a_{H_j(n)}}{a_n} - \end{aligned}$$

$$
\begin{aligned}
- & \sum_{i=1}^{m} Q_i(n) \frac{a_{n-s_i(n)}}{a_{G_i(n)}} \cdot \frac{a_{G_i(n)}}{a_n} = \\
= & \ a_n - \sum_{j=1}^{\ell} \left[P_j(n+1)(1-\Delta k_j(n)) a_{n+1-k_j(n)} + \right. \\
+ & \left. \left(P_j(n+1)\Delta k_j(n) - P_j(n) \right) a_{n-k_j(n)} \right] + \sum_{i=1}^{m} Q_i(n) a_{n-s_i(n)} = \\
= & \ a_n - \Delta \left[\sum_{j=1}^{\ell} P_j(n) a_{n-k_j(n)} \right] + \sum_{i=1}^{m} Q_i(n) a_{n-s_i(n)}
\end{aligned}
$$

for $n \in \mathbf{N}^*$. It completes the proof of Theorem 1. □

4. Existence of Positive Solutions

In the next two theorems we formulate statements for the existence of positive solutions of equation (1). The first result generalizes Theorem A given for linear delay difference equations and it is a discrete analogue of Theorem 2 in [7] given for neutral differential equations with variable delays.

Set

$$
F = \{\{\phi_j\} : \phi_{n_0} > 0,\ 0 < \phi_j \le \phi_{n_0} \text{ for } j = n_{-1}, n_{-1}+1, ..., n_0\}.
$$

Theorem 2. *Assume that (H_1) and (H_2) hold and there exists a real number $\mu \in (0,1)$ such that*

$$
\sum_{j=1}^{\ell} \left[|P_j(n)(1-\Delta k_j(n))|(1+\mu) + |P_j(n+1)\Delta k_j(n) - P_j(n)| \right] (1-\mu)^{-k_j(n)} +
$$

$$
+ \sum_{i=1}^{m} |Q_i(n)|(1-\mu)^{-s_i(n)} \le \mu \quad \text{for} \quad n \in \mathbf{N}^*.
$$

Then, for every $\{\phi_n\} \in F$, the solution a_n^{ϕ} of Equation (1) remains positive for $n \in \mathbf{N}^*$.

Proof. It will be shown that the statement (c) of Theorem 1 is true with $\beta_n = 1-\mu$ and $\gamma_n = 1+\mu$ for $n \in \mathbf{N}^*$. For any real sequence $\{\delta_n\}$ between $\{\beta_n\}$ and $\{\gamma_n\}$ holds that

$$
\prod_{\nu=H_j(n)}^{n-1} \frac{1}{\delta_\nu} \le \prod_{\nu=H_j(n)}^{n-1} \frac{1}{1-\mu} \le (1-\mu)^{-k_j(n)} \quad \text{for} \quad j = 1,2,...,\ell \quad \text{and}
$$

$$
\prod_{\nu=G_i(n)}^{n-1} \frac{1}{\delta_\nu} \le \prod_{\nu=G_i(n)}^{n-1} \frac{1}{1-\mu} \le (1-\mu)^{-s_i(n)} \quad \text{for} \quad i = 1,2,...,m\ .
$$

Then it follows that

$$\begin{aligned}
1-\mu &\le 1-\sum_{j=1}^{\ell}\left[|P_j(n+1)(1-\Delta k_j(n))|(1+\mu)+\right.\\
&+ \left.|P_j(n+1)\Delta k_j(n)-P_j(n)|\right](1-\mu)^{-k_j(n)} -\\
&- \sum_{i=1}^{m}|Q_i(n)|(1-\mu)^{-s_i(n)}\\
&\le 1-\sum_{j=1}^{\ell}\left[P_j(n+1)(1-\Delta k_j(n))\frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}}\delta_{H_j(n)}+\right.\\
&+ \left.(P_j(n+1)\Delta k_j(n)-P_j(n))\frac{\phi_{h_j(n)}}{\phi_{n_0}}\right]\prod_{\nu=H_j(n)}^{n-1}\frac{1}{\delta_\nu} -\\
&- \sum_{i=1}^{m}Q_i(n)\frac{\phi_{g_i(n)}}{\phi_{n_0}}\prod_{\nu=G_i(n)}^{n-1}\frac{1}{\delta_\nu}\\
&\equiv S\delta_n\\
&\le 1+\sum_{j=1}^{\ell}\left[|P_j(n+1)(1-\Delta k_j(n))|(1+\mu)+\right.\\
&+ \left.|P_j(n+1)\Delta k_j(n)-P_j(n)|\right](1-\mu)^{-k_j(n)} -\\
&- \sum_{i=1}^{m}|Q_i(n)|(1-\mu)^{-s_i(n)}\\
&\le 1+\mu
\end{aligned}$$

for $n \in \mathbf{N}^*$, and the proof is complete. □

The following result generalizes Theorem B given for linear delay difference equations and it is a discrete analogue of Theorem 3 in [7] given for neutral differential equations with variable delays.

Theorem 3. *Assume that (H_1) and (H_2) hold, and that*

$$0 \le k_1(n) \le k_2(n) \le ... \le k_\ell(n) \quad \text{for} \quad n \in \mathbf{N}^*,$$

$$0 \le s_1(n) \le s_2(n) \le ... \le s_m(n) \quad \text{for} \quad n \in \mathbf{N}^*,$$

$$\sum_{j=1}^{\nu} P_j(n+1)(1-\Delta k_j(n)) \le 0 \quad \text{for} \quad \nu = 1,2,...,\ell, \quad n \in \mathbf{N}^*\ ,$$

$$\sum_{i=1}^{\nu}(P_j(n+1)\Delta k_j(n)-P_j(n)) \le 0 \quad \text{for} \quad \nu = 1,2,...,\ell, \quad n \in \mathbf{N}^*\ ,$$

$$\sum_{i=1}^{\nu} Q_i(n) \le 0 \quad \text{for} \quad \nu = 1,2,...,m, \quad n \in \mathbf{N}^*\ .$$

Then, for every $\{\phi_n\} \in F$, the solution a_n^{ϕ} of Equation (1) has a positive increasing solution for $n \in \mathbf{N}^*$.

Proof. Let $\phi_j = 1$ for $j = n_{-1}, n_{-1}+1, ..., n_0$. One can claim that the statement (c) of Theorem 1 will be true with $\beta_n = 1$ and

$$\gamma_n = 1+\sum_{j=1}^{\ell} |P_j(n+1)(1-\Delta k_j(n))|+\sum_{j=1}^{\ell} |P_j(n+1)\Delta k_j(n)-P_j(n)|+\sum_{i=1}^{m} |Q_i(n)| .$$

Let $\{\delta_n\}$ be an arbitrary sequence such that

$$\beta_n \equiv 1 \le \delta_n \le \gamma_n \quad \text{for} \quad \text{n} \in \mathbf{N}^* .$$

Then it holds that

$$\begin{aligned} S\delta_n &= 1 - \sum_{j=1}^{\ell} \left[P_j(n+1)(1-\Delta k_j(n)) \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \delta_{H_j(n)} + \right. \\ &+ \left. (P_j(n+1)\Delta k_j(n) - P_j(n)) \frac{\phi_{h_j(n)}}{\phi_{n_0}} \right] \prod_{\nu=H_j(n)}^{n-1} \frac{1}{\delta_\nu} - \\ &- \sum_{i=1}^{m} Q_i(n) \frac{\phi_{g_i(n)}}{\phi_{n_0}} \prod_{\nu=G_i(n)}^{n-1} \frac{1}{\delta_\nu} \\ &\le 1 + \sum_{j=1}^{\ell} |P_j(n+1)(1-\Delta k_j(n))| + \\ &+ \sum_{j=1}^{\ell} |P_j(n+1)\Delta k_j(n) - P_j(n)| + \sum_{i=1}^{m} |Q_i(n)| = \gamma_n . \end{aligned}$$

Because of the inequalities

$$H_1(n) \ge H_2(n) \ge ... \ge H_\ell(n) \quad \text{for} \quad n \in \mathbf{N}^* ,$$

$$G_1(n) \ge G_2(n) \ge ... \ge G_m(n) \quad \text{for} \quad n \in \mathbf{N}^* ,$$

it follows that

$$\begin{aligned} S\delta_n &\ge 1 + \left(-\sum_{j=1}^{\ell} P_j(n+1)(1-\Delta k_j(n)) \right) \min_{j=1}^{\ell} \frac{\phi_{h_j(n)+1}}{\phi_{n_0+1}} \prod_{\nu=H_\ell(n)}^{n-1} \frac{1}{\delta_\nu} + \\ &+ \left(-\sum_{j=1}^{\ell} (P_j(n+1)\Delta k_j(n) - P_j(n)) \right) \min_{j=1}^{\ell} \frac{\phi_{h_j(n)}}{\phi_{n_0}} \prod_{\nu=H_\ell(n)}^{n-1} \frac{1}{\delta_\nu} + \\ &+ \left(-\sum_{j=1}^{\ell} Q_i(n) \right) \min_{i=1}^{m} \frac{\phi_{h_j(n)}}{\phi_{n_0}} \prod_{\nu=G_m(n)}^{n-1} \frac{1}{\delta_\nu} \ge 1 \end{aligned}$$

Therefore, by Theorem 1, the solution a_n^ϕ of Equation (1) is positive for $n \in \mathbf{N}^*$. Moreover,

$$a_n = \prod_{j=n_0}^{n-1} \lambda_j \quad \text{for} \quad \mathrm{n} \in \mathbf{N}^* \ ,$$

where $\{\lambda_n\}$ is a positive solution of characteristic equation associated with Equation (1) between β_n and γ_n for $n \in \mathbf{N}^*$. Hence, $\{a_n\}$ is an increasing solution of Equation (1) and the proof is complete. □

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