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INTEGRATED C-SEMIGROUPS OF UNBOUNDED LINEAR OPERATORS IN BANACH SPACES

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Abstract. A family of unbounded linear operators $(S(t))_{t\geq 0}$ in the Banach space $(E, \|\cdot\|)$ which satisfies the composition law for an integrated $C-$semigroup on a domain $D \subset E$ is introduced and investigated. The Banach spaces $(E_\omega, \|\cdot\|_\omega)$, $\omega > 0$, are used for the construction of a family of infinitesimal generators A^ω, $\omega > 0$ which determine an operator A called the infinitesimal generator of $(S(t))_{t\geq 0}$.

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1. Introduction

Integrated semigroups of unbounded linear operator in Banach spaces have been studied in [7], [8]. This paper is a continuation of these studies. Here we use also some results of [9], [15], for $n-$times integrated $C-$semigroups and mild integrated $C-$existence families of bounded operators.

We proved in [7] that any integrated semigroup of unbounded linear operators under additional conditions is an exponentially bounded integrated semigroup on a subspace with a possibly stronger norm. We obtain this result for the integrated $C-$semigroups of unbounded operators with additional condition for the operator C.

2. Structural properties

Let $(S(t))_{t\geq 0}$ be a family of unbounded linear operators in a Banach space $(E, \|\cdot\|)$ and let $C : D(C) \to E$ be an unbounded liner operator. Denote by $D(S(t))$ the domain of $S(t)$ and set

(1)
$$\mathbf{D} = \left\{ x \in \bigcap_{s,t\geq 0} D(S(s)S(t)) \;\middle|\; \begin{array}{l} S(0)x = 0 \\ S(t)x \text{ is strongly continuous for } t \geq 0, \\ S(t)Cx = CS(t)x \text{ for } t \geq 0, \\ S(s)S(t)x = \int_0^s (S(r+t) - S(r))Cx\,dr \\ = S(t)S(s)x \text{ for } t \geq 0. \end{array} \right\}$$

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If $\mathbf{D} \neq \{0\}$, then $(S(t))_{t\geq 0}$ is said to be *an integrated $C-$semigroup of unbounded linear opetarors* in E. Note that $\mathbf{D} \subset S(C)$.

The set

$$\mathcal{N} = \{x \in \mathbf{D};\ S(t)x = 0,\ \ t \geq 0\}$$

is called *a degeneration space* of an integrated $C-$semigroup of unbounded linear operators $(S(t))_{t\geq 0}$. A semigroup $(S(t))_{t\geq 0}$ is called *nondegenerate* if $\mathcal{N} = \{0\}$ and it is called *degenerate* otherwise.

Lemma 1. *If an integrated $C-$semigroup of bounded linear operators $(S(t))_{t\geq 0}$ is nondegenerate, then C is injective (cf. [15], the proof of Lemma 2.2).*

Definition 1. *For $\omega \in \mathbb{R}^+ = (0,\infty)$, $x \in \bigcap_{t\geq 0} D(S(t))$, let*

$$\|x\|_\omega := \sup_{t\geq 0} e^{-\omega t}\|S(t)x\| \tag{2}$$

and set

$$E_\omega := \{x \in \mathbf{D};\ \|x\|_\omega < \infty\}\,. \tag{3}$$

Then, $\|\cdot\|_\omega$ is a norm on E_ω.

Let $\overline{E}_\omega$ denote the closure of the set E_ω under the norm $\|\cdot\|$ and $S(t)|\overline{E}_\omega$ is the part of $S(t)$ in $\overline{E}_\omega$ i.e.

$$D(S(t)|\overline{E}_\omega) = \{x \in \overline{E}_\omega;\ x \in D(S(t)) \text{ and } S(t)x \in \overline{E}_\omega\}\,. \tag{4}$$

In this paper we assume that for all $\omega > 0$, C is bounded linear operator under the norm $\|\cdot\|$ and $\|C\|_\omega = M_\omega$.

Proposition 1.

a) If $\omega_1 \leq \omega_2$ and $x \in \mathbf{D}$, then $\|x\|_{\omega_2} \leq \|x\|_{\omega_1}$. Hence, if $\omega_1 \leq \omega_2$ then $E_{\omega_1} \subset E_{\omega_2}$.

b) If $x \in E_\omega$ then $S(t)x \in E_\omega$ and

$$\|S(t)x\|_\omega \leq \frac{2}{\omega} M_\omega e^{\omega t}\|x\|_\omega\,. \tag{5}$$

Proof.

a) Let $\omega_1 \leq \omega_2$ and $x \in \mathbf{D}$. Then, we have

$$\begin{aligned}\|x\|_{\omega_2} &= \sup_{t\geq 0} e^{-\omega_2 t}\|S(t)x\| \\ &= \sup_{t\geq 0} e^{-\omega_1 t}\cdot e^{(\omega_1-\omega_2)t}\|S(t)x\| \leq \sup_{t\geq 0} e^{\omega_1 t}\|S(t)x\| = \|x\|_{\omega_1}\,.\end{aligned}$$

Thus, $E_{\omega_1} \subset E_{\omega_2}$ if $\omega_1 \leq \omega_2$.

b) Let $x \in E_\omega$. Then

$$\|S(t)x\|_\omega = \sup_{s\geq 0} e^{-\omega s}\|S(s)S(t)x\| = e^{\omega t}\sup_{s\geq 0} e^{-\omega(t+s)}\|S(s)S(t)x\|$$

$$= e^{\omega t}\sup_{s\geq 0} e^{-\omega(s+t)}\Big\|\int_0^s (S(r+t)-S(r))Cxdr\Big\|$$

$$\leq e^{\omega t}\sup_{s\geq 0} e^{-\omega s}\Big(\int_0^s e^{\omega r}e^{-\omega(r+t)}\|S(r+t)Cx\|dr + e^{-\omega t}\int_0^s e^{\omega r}e^{-\omega r}\|S(r)Cx\|dr\Big)$$

$$\leq e^{\omega t}\|Cx\|_\omega \sup_{s\geq 0} e^{-\omega s}\Big(\int_0^s e^{\omega r}dr + e^{-\omega t}\int_0^s e^{\omega r}dr\Big)$$

$$\leq M_\omega e^{\omega t}\|x\|_\omega \sup_{s\geq 0} e^{-\omega s}(1+e^{-\omega t})\int_0^s e^{\omega r}dr$$

$$= M_\omega e^{\omega t}\|x\|_\omega \sup_{s\geq 0}\frac{1}{\omega}e^{-\omega s}(1+e^{-\omega t})(e^{\omega s}-1)$$

$$= M_\omega e^{\omega t}\|x\|_\omega \sup_{s\geq 0}\frac{1}{\omega}(1+e^{-\omega t})(1-e^{-\omega s}) \leq \frac{2}{\omega}M_\omega e^{\omega t}\|x\|_\omega\,.$$

□

Remark 1. By the proof of Proposition 1 b), we have

$$e^{-\omega(t+s)}\|S(s)S(t)x\| \leq \frac{2M_\omega}{\omega}\|x\|_\omega$$

and

$$\|S(s)S(t)x\| \leq \frac{2M_\omega e^{\omega(t+s)}}{\omega}\|x\|_\omega\,.$$

The following additional assumption will be needed throughout the paper.
(6) For every $\omega > 0$ and for every $x \in \mathbf{D}$, there exists $K_\omega > 0$ such that $\|x\|_\omega \geq K_\omega\|x\|$.

Remark 2. If for an integrated $C-$semigroup of unbounded linear operators $(S(t))_{t\geq 0}$ there exist $t_0 \geq 0$ and $K_{t_0} > 0$ such that

$$\|S(t_0)x\| \geq K_{t_0}\|x\|,\ \ x \in \mathbf{D}\,, \tag{7}$$

then, for every $\omega > 0$

$$\|x\|_\omega = \sup_{t\geq 0} e^{-\omega t}\|S(t)x\| \geq e^{-\omega t_0}\|S(t_0)x\| \geq K_\omega\|x\|,\ x \in \mathbf{D}\,,$$

where $K_\omega = e^{-\omega t_0}K_{t_0}$.

Theorem 1. *Let $(S(t))_{t\geq 0}$ be an integrated $C-$semigroup of unbounded linear operators in E such that:*
(i) $(S(t))_{t\geq 0}$ is nondegenerate,
(ii) C is the bounded linear operator under the norm $\|\cdot\|_\omega$ in E_ω,
(iii) condition (6) holds.
Then:

a) *Let $\omega > 0$ be fixed. Suppose that for every $t \geq 0$, $S(t)|\overline{E}_\omega$ is a closed operator in $\overline{E}_\omega$.*
Then $(E_\omega, \|\cdot\|_\omega)$ is a Banach space.

b) *If $S(t)$ is a closed operator in E, then $S(t)|\overline{E}_\omega$ is a closed operator in $\overline{E}_\omega$ for $t \geq 0$ and $\omega > 0$.*

Proof.
a) Recall the assumption:
If $\{x_n\} \subset D(S(t)|\overline{E}_\omega)$, $\|x_n - x\| \to 0$ and $\|S(t)x_n - y\| \to 0$ as $n \to \infty$, then $x \in D(S(t)|\overline{E}_\omega)$ and $S(t)x = y$.

Suppose $\{x_n\} \subset E_\omega$ is a Cauchy sequence with respect to the norm $\|\cdot\|_\omega$. For every $\varepsilon > 0$ there exists a number $N > 0$ such that

$$\|x_m - x_n\|_\omega = \sup_{t\geq 0} e^{-\omega t}\|S(t)x_m - S(t)x_n\| < \varepsilon,\ m, n > N\,. \tag{8}$$

By (6) we have $\|x_m - x_n\| < \dfrac{\varepsilon}{K_\omega}$, $m, n > N$. Hence, there exists $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. By (8)

$$e^{-\omega t}\|S(t)x_m - S(t)x_n\| < \varepsilon,\ \ t \geq 0,\ m, n > N\,, \tag{9}$$

that is, for $t \geq 0$, $\{e^{-\omega t}S(t)x_n\}_{t\geq 0}$ is a Cauchy sequence in the norm of E. Therefore, for every $t \geq 0$ there exists $y_t \in E$ such that $\|e^{-\omega t}S(t)x_n - y_t\| \to 0$ as $n \to \infty$. Fix $n > N$ in (9) and let $m \to \infty$. Then,

$$\|e^{-\omega t}S(t)x_n - y_t\| \leq \varepsilon\,. \tag{10}$$

In (10) N is independent of t.

Since $S(t)|\overline{E}_\omega$ is closed for $t \geq 0$, the same holds for $e^{-\omega t}S(t)|\overline{E}_\omega,\ t \geq 0$. This implies $x \in D(e^{-\omega t}S(t)|\overline{E}_\omega) = D(S(t)|\overline{E}_\omega)$ and $y_t = e^{-\omega t}S(t)x$. Now, by (10)

$$e^{-\omega t}\|S(t)x_n - S(t)x\| \leq \varepsilon,\ \ n > N,\ t \geq 0\,. \tag{11}$$

This implies $\|x_n - x\|_\omega \leq \varepsilon$, for $n > N$ and $\|x_n - x\|_\omega \to 0$ as $n \to \infty$. Consequently, $\|x\|_\omega < \infty$.

It remains to prove that $x \in \mathbf{D}$. Since $x_n \in \mathbf{D}$ by (1), we have

$$S(s)S(t)x_n = \int_0^s (S(r+t) - S(r))Cx_n dr\,, \tag{12}$$

By Remark 1 we have

$$\|S(s)S(t)x_n - S(s)S(t)x\| \leq \frac{2M_\omega e^{\omega(t+s)}}{\omega}\|x_n - x\|_\omega \,.$$

Now, fix $s, t \geq 0$. Then by (5)

$$\|S(t)Cx_n - S(t)Cx\| \leq M_\omega e^{\omega t}\|x_n - x\|_\omega \,.$$

It implies

$$\Big\| \int_0^s (S(r+t) - S(r))Cx_n dr - \int_0^s (S(r+t) - S(r))Cx dr \Big\|$$

$$\leq \int_0^s \|(S(r+t) - S(r))C(x_n - x)\| dr$$

$$\leq \frac{M_\omega}{\omega}(e^{\omega(s+t)} + e^{\omega s})\|x_n - x\|_\omega \to 0 \text{ as } n \to \infty \,.$$

Further

$$\|S(s)S(t)x - \int_0^s (S(r+t) - S(r))Cx dr\| \to 0 \text{ as } n \to \infty \,. \tag{13}$$

Since $\|S(t)x_n - S(t)x\| \leq e^{\omega t}\|x_n - x\|_\omega$ and $\|x_n - x\|_\omega \to 0$ as $n \to \infty$, we have $\|S(t)x_n - S(t)x\| \to 0$ as $n \to \infty$. On the other hand $S(s)|\overline{E}_\omega$ is closed in $\overline{E}_\omega$, by (13), we obtain

$$S(t)x \in D(S(s)) \text{ and } S(s)S(t)x = \int_0^s (S(r+t) - S(r))Cx dr \,.$$

Let $t_1 \geq 0$. Then,

$$\begin{aligned} \|S(t)x - S(t_1)x\| &\leq \|S(t)x - S(t)x_n\| + \|S(t)x_n - S(t_1)x_n\| \\ &+\|S(t_1)x_n - S(t_1)x\| \leq e^{\omega t}\|x_n - x\|_\omega \\ &+\|S(t)x_n - S(t_1)x_n\| + e^{\omega t_1}\|x_n - x\|_\omega \,. \end{aligned} \tag{14}$$

For $\varepsilon > 0$ and n sufficiently large choose $\delta > 0$ such that $e^{\omega t} < e^{\omega t_1} + \varepsilon$ and

$$\|S(t)x_n - S(t_1)x_n\| < \varepsilon, \quad \text{for } 0 < |t - t_1| < \delta \,.$$

Then (14) follows that $S(t)x$ is strongly continuous for $t \geq 0$.

Clearly, it holds

$$\|S(t)Cx - CS(t)x\| \leq \|S(t)Cx - S(t)Cx_n\| + \|CS(t)x_n - CS(t)x\|$$

$$\le M_\omega e^{\omega t}\|x_n - x\|_\omega + \frac{2M_\omega^2 e^{\omega t}}{\omega K_\omega}\|x_n - x\|_\omega = M_\omega e^{\omega t}\Big(1 + \frac{2M_\omega}{\omega K_\omega}\Big)\|x_n - x\|_\omega < \varepsilon$$

for n sufficiently large.

It is easy to see that

$$\|S(0)x\| = \|S(0)x - S(0)x_n\| \le \|x - x_n\|_\omega < \varepsilon$$

for n sufficiently large. Hence $S(0)x = 0$ and $x \in \mathbf{D}$.

b) We have $S(t)|\overline{E}_\omega \subset S(t),\ t \ge 0$, so if $S(t)$ is closed, then $S(t)|\overline{E}_\omega$ is closable, with the closure $\overline{S(t)|\overline{E}_\omega}$. If $x \in D(\overline{S(t)|\overline{E}_\omega})$, then there is a sequence $\{x_n\} \subset D(\overline{S(t)|\overline{E}_\omega})$, and $y \in \overline{E}_\omega$ such that $\|x_n - x\| \to 0$ and $\|S(t)x_n - y\| \to 0$ as $n \to \infty$. Then $x \in \overline{E}_\omega$ and since $S(t)$ is closed, $x \in D(S(t))$ and $S(t)x = y \in \overline{E}_\omega$. Thus $x \in D(\overline{S(t)|\overline{E}_\omega})$, and $S(t)|\overline{E}_\omega$ is a closed operator in $\overline{E}_\omega$. □

3. Family of $C-$pseudoresolvents

In this section we suppose that for a nondegenerate integrated $C-$semigroups $(S(t))_{t\ge 0}$ of unbounded linear operators for every $\omega > 0$ hold:

(i) The operator C is bounded under the norm $\|\cdot\|_\omega$ in E_ω.

(ii) There exists $K_\omega > 0$ such that

$$\|x\| \le \frac{1}{K_\omega}\|x\|_\omega .$$

(iii) The operator $S(t)|\overline{E}_\omega$ is closed in $\overline{E}_\omega$ for $t \ge 0$ and $\omega > 0$.

Then, we have $\overline{E}_\omega^{\|\cdot\|_\omega} = E_\omega$.

Definition 2. *For fixed $\omega > 0$ and $\lambda \in \mathbb{C},\ Re\lambda > \omega$ define*

$$R^\omega(\lambda)x = \lambda\int_0^\infty e^{-\lambda t}S(t)x\,dt,\quad x \in E_\omega .$$

Observe that

$$\Big\|\lambda\int_0^\infty e^{-\lambda t}S(t)x\,dt\Big\| \le |\lambda|\int_0^\infty e^{-tRe\lambda}\|S(t)x\|dt \le \frac{|\lambda|}{K_\omega}\int_0^\infty e^{-tRe\lambda}\|S(t)x\|_\omega dt$$

$$\le \frac{2M_\omega|\lambda|}{\omega K_\omega}\|x\|_\omega\int_0^\infty e^{(\omega - Re\lambda)t}dt = \frac{2M_\omega|\lambda|}{\omega K_\omega(Re\lambda - \omega)}\|x\|_\omega .$$

Thus, the integral is an improper Riemann integral converging absolutely in the norm of E. Observe that $R^\omega(\lambda)$ is in general unbounded in $(E, \|\cdot\|)$ and that its domain is E_ω.

Theorem 2. *Fix $\omega > 0$ and $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$.*

a) *(i) $R^\omega(\lambda)(E_\omega) \subset E_\omega$. Moreover,*

$$\frac{\omega(Re\lambda - \omega)}{2M_\omega|\lambda|} \|R^\omega(\lambda)x\|_\omega \leq \|x\|_\omega, \quad x \in E_\omega .$$

(ii) $R^\omega(\lambda)x \in D(S(t)C)$ and

$$S(t)CR^\omega(\lambda)x = R^\omega(\lambda)S(t)Cx = R^\omega(\lambda)CS(t)x, \quad t \geq 0, \; x \in E_\omega .$$

b) *(i) For every $x \in E_\omega$, $\|x\|_{R^\omega} < \infty$, where*

$$\|x\|_{R^\omega} := \sup_{n\in\mathbb{N}_0} \sup_{\lambda>0} \frac{(\lambda-\omega)^{n+1}}{n!} \Big\| \Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)} Cx \Big\|, \; \lambda > \omega . \tag{15}$$

The norm $\|\cdot\|_{R^\omega}$ is equivalent to the norm $\|\cdot\|_\omega$.

(ii) If $\omega_1 \leq \omega_2$ and $Re\lambda > \omega_2$, then $R^{\omega_1}(\lambda)x = R^{\omega_2}(\lambda)x$, $x \in E_\omega$. Thus, as operators in E, $R^{\omega_1}(\lambda) \subset R^{\omega_2}(\lambda)$ if $Re\lambda > \omega_2$.

Proof.

a) *(i)* Let $t \geq 0$ and $x \in E_\omega$. Then,

$$\|S(s)x\|_\omega \leq \frac{2M_\omega e^{\omega s}}{\omega} \|x\|_\omega < \infty$$

which implies

$$\Big\|\lambda \int_0^\infty e^{-\lambda t} S(t)x dt\Big\|_\omega \leq |\lambda| \int_0^\infty e^{-tRe\lambda} \|S(t)x\|_\omega dt$$

$$\leq |\lambda| \int_0^\infty e^{(\omega - Re\lambda)t} \frac{2M_\omega}{\omega} \|x\|_\omega dt \leq \frac{2M_\omega|\lambda|}{\omega(Re\lambda-\omega)} \|x\|_\omega < \infty .$$

(ii) We obtain $R^\omega(\lambda)x \in E_\omega \subset D(S(t)C)$. Since $S(t)$ is closed under the norm $\|\cdot\|$ and $S(t)C = CS(t)$, then holds

$$S(t)CR^\omega(\lambda)x = R^\omega(\lambda)S(t)Cx = R^\omega(\lambda)CS(t)x, \quad x \in E_\omega .$$

b) *(i)* We will show that, for every $x \in E_\omega$, $R^\omega(\lambda)x \in \mathbf{D}$. Theorem 2a) implies that $R^\omega(\lambda)x \in \bigcap_{t\geq 0} D(S(t))$ and $S(t)R^\omega(\lambda)x = R^\omega(\lambda)S(t)x$, $t \geq 0$.

It follows $R^\omega(\lambda)S(t)x \in \bigcap_{s\geq 0} D(S(s))$ and also $S(t)R^\omega(\lambda)x \in \bigcap_{s\geq 0} D(S(s))$. Thus, $R^\omega(\lambda)x \in \bigcap_{s,t\geq 0} D(S(s)S(t))$.

Therefore

$$S(s)S(t)R^\omega(\lambda)x = R^\omega(\lambda)S(s)S(t)x$$

$$= \lambda \int_0^\infty e^{-\lambda p} S(p)S(s)S(t)x dp = \lambda \int_0^\infty e^{-\lambda p} S(p) \int_0^s (S(r+t) - S(r))Cx dr\, dp$$

$$= \int_0^s (S(r+t) - S(r))CR^\omega(\lambda)x\, dr\,.$$

Moreover, $S(t)R^\omega(\lambda)x = R^\omega(\lambda)S(t)x$ implies

$$S(0)R^\omega(\lambda)x = \int_0^\infty e^{-\lambda s} S(0)S(s)x\, ds = 0\,.$$

We will prove $\lim_{t \to t_1} S(t)R^\omega(\lambda)x = S(t_1)R^\omega(\lambda)x,\ x \in E_\omega$. For $x \in E_\omega$ and $s \geq 0$, using strong continuity, we have

$$\|S(t)S(s)x - S(t_1)S(s)x\| \to 0 \text{ as } t \to t_1\,.$$

Remark 1 implies

$$\|S(t)S(s)x\| \leq \frac{2M_\omega e^{\omega s}}{\omega} e^{\omega t}\|x\|_\omega \leq \frac{2M_\omega e^{\omega s}}{\omega}(e^{\omega t_1} + \varepsilon)\|x\|_\omega\,,$$

for sufficiently small $|t - t_1|$. The dominated convergence theorem for vector valued integrals implies

$$\lim_{t \to t_1} S(t)R^\omega(\lambda)x = \lim_{t \to t_1} \lambda \int_0^\infty e^{-\lambda s} S(t)S(s)x ds = \lambda \int_0^\infty e^{-\lambda s} \lim_{t \to t_1} S(t)S(s)x ds$$

$$= \lambda \int_0^\infty e^{-\lambda s} S(t_1)S(s)x ds = \lambda S(t_1) \int_0^\infty e^{-\lambda s} S(s)x ds = S(t_1)R^\omega(\lambda)x.$$

By *(i)*, $\|R^\omega(\lambda)x\|_\omega < \infty$ and $\frac{\omega(Re\lambda - \omega)}{2|\lambda|M_\omega}\|R^\omega(\lambda)x\|_\omega \leq \|x\|_\omega$. Thus $R^\omega(\lambda)$ is a bounded linear operator with respect to the norm $\|\cdot\|_\omega$.

(ii) Let $x \in E_\omega$ and $\lambda > \omega$. Then, for $n \in \mathbb{N}_0$,

$$\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)} Cx = (-1)^n \int_0^\infty t^n e^{-\lambda t} S(t)Cx dt\,, \tag{16}$$

and

$$\Big\|\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)} Cx\Big\| \leq \int_0^\infty t^n e^{-\lambda t}\|S(t)Cx\| dt \leq \int_0^\infty t^n e^{-\lambda t} \frac{1}{K_\omega}\|S(t)Cx\|_\omega dt$$

$$\leq \frac{2M_\omega}{\omega K_\omega}\int_0^\infty t^n e^{-(\lambda-\omega)t}\|x\|_\omega dt = \frac{2M_\omega}{\omega K_\omega}\,\frac{n!}{(\lambda-\omega)^{n+1}}\,\|x\|_\omega .$$

This implies

$$\frac{\omega K_\omega}{2M_\omega}\sup_{n\in\mathbb{N}_0}\sup_{\lambda>\omega}\frac{(\lambda-\omega)^{n+1}}{n!}\Big\|\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)}Cx\Big\| \leq \|x\|_\omega\,.$$

We will use the following assertion (cf. [3]):

Let $f(t)$ be continuous and bounded. If $\lambda\to\infty, ; n\to\infty$ so that $\dfrac{n}{\lambda-\omega}\to t$, then,

$$\frac{(\lambda-\omega)^{n+1}}{n!}\int_0^\infty e^{-(\lambda-\omega)s}s^n f(s)ds \to f(t)\,.$$

By (16)

$$\frac{(\lambda-\omega)^{n+1}}{n!}\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)}Cx = (-1)^n\frac{(\lambda-\omega)^{n+1}}{n!}\int_0^\infty e^{-(\lambda-\omega)s}s^n e^{-\omega s}S(s)Cxds$$

and by using the preceding statement, we obtain

$$e^{-\omega t}S(t)Cx = \lim_{\substack{\lambda\to\infty\\ n\to\infty}}(-1)^n\frac{(\lambda-\omega)^{n+1}}{n!}\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)}Cx\,.$$

For $t\geq 0$

$$e^{-\omega t}\|S(t)Cx\| \leq \lim_{n\to\infty}\sup_{\lambda>\omega}\Big\|\frac{(\lambda-\omega)^{n+1}}{n!}\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)}Cx\Big\|$$

$$\leq \sup_{n\in\mathbb{N}_0}\sup_{\lambda>\omega}\frac{(\lambda-\omega)^{n+1}}{n!}\Big\|\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)}Cx\Big\|$$

and

$$\|x\|_\omega \leq \sup_{n\in\mathbb{N}_0}\sup_{\lambda>\omega}\frac{(\lambda-\omega)^{n+1}}{n!}\Big\|\Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)}Cx\Big\|\,.$$

(ii) Obviously, $E_{\omega_1}\subseteq E_{\omega_2}$ if $\omega_1\leq\omega_2$. For $x\in E_{\omega_1}$ and $Re\lambda>\omega_2$ the operators $R^{\omega_1}(\lambda)$ and $R^{\omega_2}(\lambda)$ are defined and $R^{\omega_1}(\lambda)x = R^{\omega_2}(\lambda)x$. Thus $R^{\omega_1}(\lambda)\subset R^{\omega_2}(\lambda)$. □

4. Family of infinitesimal generators

Definition 3. *A function $R(\cdot)$ defined on a subset $D(R)$ of the complex plane with values in $L(E)$ is called $C-$pseudoresolvent if it comutes with C and satisfies the equation*

$$(\mu-\lambda)R(\lambda)R(\mu) = R(\lambda)C - R(\mu)C,\ (\lambda,\mu\in D(R))\,. \tag{17}$$

$R(\cdot)$ is said to be nondegenerate if $R(\lambda)x=0$ for all $\lambda\in D(R)$ implies $x=0$.

Theorem 3. *The family of operators* $(R^\omega(\lambda))_{Re\lambda>\omega}$ *on* E_ω, $\omega > 0$ *is the* $C-$*pseudoresolvent i.e.*

$$(\mu-\lambda)R^\omega(\lambda)R^\omega(\mu) = R^\omega(\lambda)C - R^\omega(\mu)C,\ Re\lambda > \omega,\ Re\mu > \omega\,.$$

Proof. Note that the operator C is bounded under the norm $\|\cdot\|_\omega$ and

$$CR^\omega(\lambda) = R^\omega(\lambda)C\,.$$

Fix $\omega > 0$. We will show that the family of operators $(R^\omega(\lambda))_{Re\lambda>\omega}$ satisfies equation (17). Let $\lambda, \mu \in \mathbb{C}$, $\lambda \neq \mu$, $Re\lambda, Re\mu > \omega$, and $x \in E_\omega$. Then $R^\omega(\lambda)R^\omega(\mu)$ is well defined because $((R^\omega(\mu))(E_\omega) \subset E_\omega$. We have

$$R^\omega(\lambda)R^\omega(\mu)x = \lambda\int_0^\infty e^{-\lambda s}S(s)R^\omega(\mu)xds \tag{18}$$

$$= \lambda\mu\int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}S(s)S(t)x\,dt\,ds$$

$$= \lambda\mu\int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}\int_0^s (S(r+t)-S(r))Cxdrdtds$$

$$= \frac{1}{\lambda-\mu}\Big[\lambda\mu(\lambda-\mu)\Big(\int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}\int_0^s S(r+t)Cxdrdtds$$

$$-\int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}\int_0^s S(r)Cxdrdtds\Big)\Big] = \frac{1}{\lambda-\mu}[\lambda\mu(\lambda-\mu)(I_1-I_2)]\,.$$

By using Theorem 2a) and the change of variables, we obtain

$$I_1 = \int_0^\infty e^{-\lambda s}\int_0^\infty e^{\mu t}\int_0^s S(r+t)Cxdrdtds = \int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}\int_t^{s+t} S(v)Cxdvdtds \tag{19}$$

$$= \int_0^\infty\int_0^v\int_{v-t}^\infty e^{-\lambda s}e^{-\mu t}S(v)Cxdsdtdv = \frac{1}{\lambda}\int_0^\infty\int_0^v e^{-\lambda(v-t)}e^{-\mu t}S(v)Cxdtdv$$

$$= \frac{1}{\lambda(\lambda-\mu)}\int_0^\infty e^{-\lambda v}(e^{(\lambda-\mu)v}-1)S(v)Cxdv = \frac{1}{\lambda(\lambda-\mu)}\Big(\frac{R^\omega(\mu)Cx}{\mu} - \frac{R^\omega(\lambda)Cx}{\lambda}\Big),$$

$$I_2 = \int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}\int_0^s S(r)Cxdrdtds = \frac{1}{\mu}\int_0^\infty e^{-\lambda s}\int_0^s S(r)Cxdrds \tag{20}$$

$$= \frac{1}{\mu} \int_0^\infty S(r)Cx \int_r^\infty e^{-\lambda s} ds dr = \frac{1}{\mu} \int_0^\infty e^{-\lambda r} S(r)Cx \int_r^\infty e^{-\lambda(s-r)} ds dr$$

$$= \frac{1}{\lambda\mu} \int_0^\infty e^{-\lambda r} S(r)Cx dr = \frac{R^\omega(\lambda)Cx}{\lambda^2\mu}\,.$$

Thus (18), (19) and (20) imply

$$R^\omega(\lambda)R^\omega(\mu)x \tag{21}$$

$$= \frac{1}{\lambda-\mu}\Big[\lambda\mu(\lambda-\mu)\Big(\frac{1}{\lambda(\lambda-\mu)}\Big(\frac{R^\omega(\mu)Cx}{\mu} - \frac{R^\omega(\lambda)Cx}{\lambda}\Big) - \frac{R^\omega(\lambda)Cx}{\lambda^2\mu}\Big)\Big]$$

$$= \frac{1}{\lambda-\mu}\Big[\mu\Big(\frac{R^\omega(\mu)Cx}{\mu} - \frac{R^\omega(\lambda)Cx}{\lambda}\Big) - \frac{\lambda-\mu}{\lambda} R^\omega(\lambda)Cx\Big]$$

$$= \frac{1}{\mu-\lambda}\Big(R^\omega(\lambda)Cx - R^\omega(\mu)Cx\Big)$$

and the family of operators $(R^\omega(\lambda))_{Re\lambda>\omega}$ satisfies equation (17). □

Lemma 2.

(i) The null space

$$\mathcal{N}(R^\omega(\lambda)) = \{x \in E_\omega;\ R^\omega(\lambda)x = 0\}$$

is independent of the choice of λ with $Re\lambda > \omega$.

(ii) The inverse $C^{-1}(Range(R^\omega(\lambda))$, $Re\lambda > \omega$, is independent of the choice of λ.

Proof.

(i) Let $x \in \mathcal{N}(R^\omega(\lambda))$. Then (17) implies

$$CR^\omega(\mu)x = CR^\omega(\lambda)x + (\lambda-\mu)R^\omega(\mu)R^\omega(\lambda)x = 0,\ x \in E_\omega,\ Re\lambda, Re\mu > \omega\,.$$

The operator C is injective and we have $R^\omega(\mu)x = 0$ for $Re\mu > \omega$. Then $\mathcal{N}(R^\omega(\lambda)) = \mathcal{N}(R^\omega(\mu))$.

(ii) Let $x \in C^{-1}(Range(R^\omega(\lambda)))$. Then there exists $y \in E_\omega$ such that $Cx = R^\omega(\lambda)y$ for $Re\lambda > \omega$. For $Re\mu > \omega$ $(\lambda \neq \mu)$ we have

$$C^2x = CR^\omega(\lambda)y = CR^\omega(\mu)y - (\lambda-\mu)R^\omega(\mu)R^\omega(\lambda)y$$

$$= R^\omega(\mu)(Cy-(\lambda-\mu)R^\omega(\lambda)y) = R^\omega(\mu)(Cy-(\lambda-\mu)Cx) = CR^\omega(\mu)(y-(\lambda-\mu)x)\,.$$

Since C is injective, we obtain

$$Cx = R^\omega(\mu)z \quad \text{for} \quad z = y - (\lambda-\mu)x\,.$$

Therefore,

$$x \in C^{-1}(Range(R^\omega(\mu)),\quad Re\mu > \omega\,.$$

□

Lemma 3.

(i) The null space $\mathcal{N}(C - \lambda R^{\omega}(\lambda))$ *is independent of* λ *with* $Re\lambda > \omega$.

(ii) The inverse $C^{-1}(Range(C - \lambda R^{\omega}(\lambda)))$ *is independent of* λ *with* $Re\lambda > \omega$.

Proof.

(i) For $\mathcal{N}(C - \lambda R^{\omega}(\lambda))$ we have $Cx - \lambda R^{\omega}(\lambda)x = 0$. Hence,

$$R^{\omega}(\mu)Cx - \lambda R^{\omega}(\mu)R^{\omega}(\lambda)x = 0$$

and

$$CR^{\omega}(\mu)x - \frac{\lambda}{\lambda - \mu}(CR^{\omega}(\mu)x - CR^{\omega}(\lambda)x) = 0\,.$$

Since C is injective, we have $R^{\omega}(\mu)x - \dfrac{\lambda}{\lambda - \mu}(R^{\omega}(\mu)x - R^{\omega}(\lambda)x = 0,\ \lambda \neq \mu$. By multiplying both sides of the equality by $\lambda - \mu$ it follows

$$\lambda R^{\omega}(\mu)x - \mu R^{\omega}(\mu)x - \lambda R^{\omega}(\mu)x + \lambda R^{\omega}(\lambda)x = 0$$

and

$$\lambda R^{\omega}(\lambda)x = \mu R^{\omega}(\mu)x\,.$$

Therefore,

$$Cx - \mu R^{\omega}(\mu)x = 0, \quad Re\mu > \omega\,.$$

(ii) Let $x \in C^{-1}(Range(C - \lambda R^{\omega}(\lambda)))$. Then

$$Cx = Cy - \lambda R^{\omega}(\lambda)y \tag{22}$$

for some $y \in E_{\omega}$ and $Re\lambda > \omega$. We will show that for $z = x + \dfrac{\mu}{\lambda}(y - x)$ the following holds

$$Cx = Cz - \mu R^{\omega}(\mu)z, \quad Re\mu > \omega\,.$$

By using (22) we obtain

$$C^2x = C^2y - \lambda R^{\omega}(\lambda)Cy$$

$$= C^2y - \lambda[R^{\omega}(\mu)Cy - (\lambda - \mu)R^{\omega}(\mu)R^{\omega}(\lambda)y]$$

$$= C^2y - \lambda R^{\omega}(\mu)(Cy - (\lambda - \mu)R^{\omega}(\lambda)y)$$

$$= C^2y - \lambda R^{\omega}(\mu)\Big(Cy - \frac{\lambda - \mu}{\lambda}C(y - x)\Big) = C\Big(Cy - \lambda R^{\omega}(\mu)\Big(x + \frac{\mu}{\lambda}\,(y - x)\Big)\Big)\,.$$

Since C is injective, we have

$$Cx = Cy - \lambda R^{\omega}(\mu)\Big(x + \frac{\mu}{\lambda}\,(y - x)\Big)$$

and after multiplication with $\dfrac{\mu}{\lambda}$ we obtain

$$0 = \frac{\mu}{\lambda}C(y - x) - \mu R^{\omega}(\mu)\Big(x + \frac{\mu}{\lambda}(y - x)\Big)\,.$$

Finally

$$Cx = C\Big(\lambda + \frac{\mu}{\lambda}(y-x) - \mu R^{\omega}(\mu)\Big(x + \frac{\mu}{\lambda}(y-x)\Big)\Big)$$

and therefore

$$Cx = Cz - \mu R^{\omega}\mu(z), \quad \text{for} \;\; z = x + \frac{\mu}{\lambda}(y-x),\; \mu \neq \lambda,\; Re\lambda > \omega\,.$$

Note that

$$\|R^{\omega}(\lambda)Cx\|_{\omega} \leq \frac{2|\lambda|M_{\omega}}{\omega(Re\lambda - \omega)}\|x\|_{\omega}, \;\; Re\lambda > \omega,\; \omega \in E_{\omega}\,.$$

and the operators $R^{\omega}(\lambda)C$ are bounded under the norm $\|\cdot\|_{\omega}$. □

Theorem 4. *For the family of operators $(R^{\omega}(\lambda))_{Re\lambda > \omega}$ it holds:*

(i) There exists some linear operator B^{ω} such that $\lambda I - B^{\omega}$ is injective and

$$\left\{ \begin{array}{l} Range(R^{\omega}(\lambda)) \subset D(B^{\omega}), \\ R^{\omega}(\lambda)(\lambda I - B^{\omega}) \subset (\lambda I - B^{\omega})R^{\omega}(\lambda) = C, \\ \textit{for all } \lambda \textit{ with } Re\lambda > \omega, \end{array} \right. \tag{23}$$

if and only if

$$\mathcal{N}(R^{\omega}(\lambda)) = \{0\}\,.$$

(ii) The largest operator which satisfies (23) is the closed linear operator A^{ω} defined by

$$\left\{ \begin{array}{l} D(A^{\omega}) := C^{-1}[Range(R^{\omega}(\lambda)] = \{x \in E_{\omega}; \\ \quad Cx \in Range(R^{\omega}(\lambda))\}, \\ A^{\omega}x := (\lambda - ((R^{\omega}(\lambda))^{-1})Cx,\; x \in D(A^{\omega}), \\ \textit{which is independent of } \lambda, Re\lambda > \omega\,. \end{array} \right. \tag{24}$$

(iii) If B^{ω} satisfies (23) then $C^{-1}B^{\omega}C = A^{\omega}$, where

$$D(C^{-1}B^{\omega}C) = \{x \in E_{\omega};\; Cx \in D(B^{\omega}) \textit{ and } B^{\omega}Cx \in Range(C)\}.$$

In particulary $C^{-1}A^{\omega}C = A^{\omega}$.

Proof. (see [11] Theorem 3.4) □

Let $D(A) = \bigcup\limits_{\omega>0} D(A^{\omega})$, where A^{ω} is given in Theorem 4. For $x \in D(A)$ let $\omega > 0$ such that $x \in D(A^{\omega})$. There exists $y \in E_{\omega}$ such that $Cx = R^{\omega}(\lambda)y,\; Re\lambda > \omega$. We define

$$Ax := \lambda x - y\,. \tag{25}$$

We call A the *infinitesimal generator* of the once integrated $C-$semigroup of unbounded linear operators $(S(t))_{t\geq 0}$.

It is clear that $x \in D(A)$ implies $x \in D(A^\omega)$ for some $\omega > 0$ and

$$Ax = \lambda x - y = \lambda x - (R^\omega(\lambda))^{-1}Cx = A^\omega x\,.$$

For $y \in E_\omega$ we have $Cx = R^\omega(\lambda)y$. Thus, the operator A is well defined and

$$D(A|E_\omega) = A^\omega\,.$$

It is easy to show that $D(A)$ is a subspace of E and A is a linear operator.

Theorem 5.

(i) If $\omega_1 \le \omega_2$ then $A^{\omega_1} \subset A^{\omega_2}$.

(ii) For all $x \in E_\omega$ the resolvent equation

$$(\lambda I - A)y = x, \quad Re\lambda > \omega$$

has a unique solution belonging to E_ω and $y = C^{-1}R^\omega(\lambda)x$.

(iii) Let $\omega > 0$. Then for $t \ge 0$, $S(t)(D(A^\omega)) \subset D(A^\omega)$ and

$$S(t)A^\omega x = A^\omega S(t)x, \quad x \in D(A^\omega)\,.$$

(iv) The operator A is closed under the topology induced by the norm $\|\cdot\|_\omega$ and

$$CA^\omega \subset A^\omega C\,.$$

(v) For all $t \ge 0$ and $x \in D(A)$, $\int_0^t S(r)xdr \in D(A)$. The function $t \to S(t)x$ is differentiable of t for $t > 0$. It holds $S'(t)x - Cx = S(t)Ax$, or equivalenty, $S(t)x - tCx = \int_0^t S(r)Axdr$, $t > 0$.

Proof.

(i) Let $\omega_1 \le \omega_2$ and $x \in D(A^{\omega_1})$. Then we have $x \in Range(C^{-1}R^\omega(\lambda)) = C^{-1}Range(R^\omega(\lambda))$ and $x = C^{-1}R^\omega(\lambda)y$, for some $y \in E_{\omega_1}$ and $Re\lambda > \omega_1$. It is clear that $\omega_1 \le \omega_2$ implies $R^{\omega_1}(\lambda) < R^{\omega_2}(\lambda)$ and

$$C^{-1}R^{\omega_1}(\lambda)y = C^{-1}R^{\omega_2}(\lambda)y.$$

Hence

$$Cx = R^{\omega_2}(\lambda)y, \quad x \in D(A^{\omega_1}).$$

Then $A^{\omega_1}x = \lambda x - y = A^{\omega_2}(x)$, $Re\lambda > \omega_2$, $x \in D(A^{\omega_1})$. It implies $A^{\omega_1} \subset A^{\omega_2}$.

(ii) We will show that $y = C^{-1}R^\omega(\lambda)Cx \in E_\omega$ is the unique solution of the resolvent equation. For $x \in E_\omega$ and $Re\lambda > \omega$ we have

$$(\lambda I - A^\omega)C^{-1}R^\omega(\lambda)x = [\lambda I - (\lambda I - (R^\omega(\lambda)C)^{-1}C]C^{-1}R^\omega(\lambda)x = x.$$

Then $A|E_\omega = A^\omega$ implies (ii).

(iii) For $x \in D(A)$, let $\omega > 0$ such that $x \in D(A^\omega)$. Therefore we have

$$S(t)Ax = S(t)A^\omega x = S(t)(\lambda x - y)$$

$$= \lambda S(t)x - S(t)y = A^\omega S(t)x = AS(t)x,$$

where $Cx = R^\omega(\lambda)y$ for $Re\lambda > \omega$.

(iv) The operator A^ω is closed under the norm $\|\cdot\|_\omega$ (Theorem 4) and

$$CA^\omega x = C(\lambda x - y) = \lambda Cx - Cy = A^\omega Cx.$$

(v) Let $\omega > 0$ and $t \geq 0$ be fixed. Then

$$e^{-\omega s}\Big\|S(s)C\int_0^t S(r)xdr\Big\| \leq e^{-\omega s}\int_0^t \Big\|S(s)S(r)Cx\Big\|dr$$

$$\leq e^{-\omega s}\,\frac{2M_\omega e^{\omega s}}{\omega}\,\|x\|_\omega \int_0^t dr \leq \frac{2M_\omega e^{\omega t}}{\omega^2}\,\|x\|_\omega < \infty\,.$$

Hence, $\int_0^t S(r)xdr \in E_\omega$. There exists

$$R^\omega(\lambda)\int_0^t S(r)xdr = \lambda\int_0^\infty e^{-\lambda s}S(s)\int_0^t S(r)xdrds\,.$$

Let $y \in E_\omega$ such that $Cx = R^\omega(\lambda)y$. The operator A^ω is closed and for $A^\omega x = \lambda x - y$ we have

$$\int_0^\infty S(r)A^\omega xdr = A^\omega\int_0^t S(r)xdr = \lambda\int_0^t S(r)xdr - \int_0^t S(r)ydr\,.$$

Therefore $\int_0^t S(r)xdr \in D(A^\omega)$ and $\int_0^t S(r)xdr \in D(A)$ because $A|E_\omega = A^\omega$.

We obtain, for $x \in D(A^\omega)$, $Cx = R^\omega(\lambda)y$ for some $y \in E_\omega$ with $Re\lambda > \omega$ and $A^\omega x = \lambda x - y$. By Fubini's theorem it holds (cf. [7])

$$\frac{S(t+h) - S(t)}{h}Cx = \frac{\lambda}{\mu}\Big(S(t+h)\int_0^\infty e^{-\lambda r}S(r)ydr - S(t)\int_0^\infty e^{-\lambda r}S(r)ydr\Big)$$

$$= \frac{e^{\lambda h}-1}{h}e^{\lambda t}\int_0^\infty e^{-\lambda v}S(v)Cydv - \frac{e^{\lambda(t+h)}}{h}\int_0^{t+h} e^{-\lambda v}S(v)Cydv + \frac{e^{\lambda t}}{h}\int_0^t e^{-\lambda v}S(v)Cydv.$$

Let $h \to 0$. We have

$$S'(t)x = e^{\lambda t}C^2x - f'(t) \qquad (26)$$

where

$$f(t)=e^{\lambda t}\int_0^t e^{-\lambda v}S(v)Cydv.$$

Differentiating, it follows

$$f'(t)=\lambda e^{\lambda t}\int_0^t e^{-\lambda v}S(v)Cydv+e^{\lambda t}\cdot e^{-\lambda t}S(t)Cy$$

and (26) implies

$$S'(t)Cx=e^{\lambda t}C^2x-\lambda e^{\lambda t}\int_0^t e^{-\lambda v}S(v)Cydv-S(t)Cy\,. \tag{27}$$

Therefore,

$$e^{\lambda t}C^2x-\lambda e^{\lambda t}\int_0^t e^{-\lambda v}S(v)Cydv=\lambda e^{\lambda t}\Big(\int_0^\infty e^{-\lambda v}S(v)Cydv-\int_0^t e^{-\lambda v}S(v)Cydv\Big)$$

$$\lambda\int_0^\infty e^{-\lambda p}S(p+t)Cydp=\lambda\int_0^\infty e^{-\lambda p}(S'(p)S(t)y+S(p)Cy)dv$$

$$=\lambda S(t)\int_0^\infty e^{-\lambda p}S'(p)ydp+\lambda\int_0^\infty e^{-\lambda p}S(p)Cydp=\lambda S(t)Cx+C^2x.$$

Since $S(t)C=CS(t)$ and by using (27) we obtain

$$CS'(t)x=C^2x+\lambda CS(t)x-CS(t)y\,.$$

The operator C is injective and we have

$$S'(t)x=Cx+\lambda S(t)x-S(t)y.$$

Therefore

$$S'(t)x=Cx+S(t)A^\omega x,\;\;\omega>0,$$

and

$$S(t)A^\omega x=S'(t)x-Cx\,.$$

Since $A=A^\omega$ on E_ω, it implies

$$\int_0^t S(r)Axdr=S(t)x-tCx,\;\;t>0.$$

□

References

[1] Arendt, W., Resolvent positive operators and integrated semigroups. Proc. London Math. Soc., (3) 54 (1987), 321-349.

[2] Arendt, W., Vector valued Laplace tranfsorms and Cauchy problems. Israel J. Math., 59 (1987), 327-352.

[3] Feller, W., On the generation of unbounded semigroups of bounded linear operators. Ann. Math., (2) 58 (1953), 166-174.

[4] Hille, E., Philips, R. S., Functional Analysis and Semigroups. Providence, R.I.: Amer. Math. Soc. Colloquium Publications, Vol 31, 1957.

[5] Hughes, R. J., Semigroups of unbounded linear operators in Banach space. Trans. of the Amer. Math. Soc., 230 (1977), 113-145.

[6] Kellermann, H., Hieber, M., Integrated semigroups. J. Func. Anal., 84 (1989), 160-180.

[7] Kravarušić, R., Mijatović, M., Pilipović, S., Integrated semigroups of unbounded linear operators in a Banach spaces, Part I. Bull. TCXVI Acad. Serb. Sci. Math., 23 (1998), 45-62.

[8] Kravarušić, R., Mijatović, M., Pilipović, S., Integrated semigroups of unbounded linear operators in a Banach spaces, Part II. Novi Sad J. Math., 28 (1998), 107-122.

[9] Li, Y.-C., Shaw, S.-Y., N-Times integrated $C-$semigroups and the abstract Cauchy problem. Taiwanese J. of Math., 26 (1997), 75-102.

[10] Mijatović, M., Pilipović, S., Vajzović, E., $\alpha-$times integrated semigroup ($\alpha \in \mathbb{R}^+$),. J. Math. Anal. Appl., 210 (1997), 790-803.

[11] Neubrander, F., Integrated semigroups and their applications to the abstract Cauchy problem. Pac. J. Math., 135 (1988), 111-155.

[12] Oharu, S., Semigroups of linear operators in Banavh spaces. Publ. Res. Inst. Math. Sci., 204 (1973), 189-198.

[13] Tanaka, N., Okazawa, N., Local $C-$semigroups and local integrated semigroups. Proc. London Math. Soc., 61 (1990), 63-90.

[14] Thieme, Integrated semigroups and integrated solutions to abstract Cauchy problems. J. Math. Anal. Appl., 152 (1990), 416-447.

[15] Wang, S. W., Mild integrated $C-$existence families. Stud. Math., 112 (1995), 251-266.

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