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n-CONVEX FUNCTIONS ON A SEMIGROUP WITH A ROOT FUNCTION*

Introduction

In real analysis a (Jensen) convex function is defined on an interval $[a, b]$ by

$$(1) \qquad f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$$

for $x, y \in [a, b]$ (see D. S. Mitrinović [4]). It is well known that for convex functions so defined the following inequality holds for all $n \in N$

$$(2) \qquad f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

for $x_i \in [a, b]$ and $i=1, 2, \ldots, n$.

We shall consider the connection of the inequalities (1) and (2) in the case of the functions in a semigroup. We take a root function in the place of real division. The root function on a group is defined by S. S. Jou and S. Kurepa [1] and S. S. Jou [2].

1. A root function and a n-convex function

Let $(X, *)$ be a commutative semigroup. We take

$$\mathop{*}_{i=1}^{n} x_i \overset{\text{def}}{=} x_1 * x_2 * \ldots * x_n$$

and

$$n\,[x] \overset{\text{def}}{=} \underbrace{x * x * \ldots * x}_{n}.$$

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Now we have

Definition 1. *A root function of the order n is a function γ_n: $X \to X$, for fixed $n \in N$, such that*

$$(A_1) \qquad \gamma_n(x) * \gamma_n(y) = \gamma_n(x * y),$$

$$(A_2) \qquad n[\gamma_n(x)] = x$$

holds for all $x, y \in X$.

It is evident that a root function of the order n does not exist for all semigroups. An example: If $(S, .)$ is a semigroup with a neutral element e and $x^2 = e$ for all $x \in S$, then a root function of the order 2 does not exist. For every function h: $X \to X$ we have $h(x)\ h(x) = e$ and so (A_2) does not hold for $x \neq e$.

However, we have

Lemma 1. *If a root function of an order pq exists on X (in the notation γ_{pq}), then there are root functions of orders p and q (γ_p and γ_q) such that*

$$(3) \qquad \gamma_{pq} = \gamma_p \circ \gamma_q = \gamma_q \circ \gamma_p,$$

where $\circ$ denotes the composition of the functions.

Conversely, if there exist root functions of orders p and q (γ'_p and γ'_q) then there is a root function of the order pq given with

$$\gamma_{pq} = \gamma'_p \circ \gamma'_q.$$

Lemma 2. *A root function γ_n of the order n on a semigroup X is unique.*

Now, we can give

Definition 2. *Suppose $(X_1, *)$ is a commutative semigroup with a root function γ_n and $(X_2, \oplus)$ is a partially ordered commutative semigroup with a root function δ_n. A function f: $A \to X_2$ is called the n-convex function on a set $A \subset X_1$ (with the property that for all $x_i \in A$ and $i = 1, 2, \ldots, n$ is $\gamma_n(\overset{n}{\underset{i=1}{*}} x_i) \in A$) if it satisfies either the following inequality*

$$(4) \qquad f\left(\gamma_n\left(\overset{n}{\underset{i=1}{*}} x_i\right)\right) \leqslant \delta_n\left(\bigoplus_{i=1}^{n} f(x_i)\right)$$

or if the left-hand and the right -hand sides of the inequality (4) *are incomparable, for all $x_i \in A$ $(i = 1, 2, \ldots, n)$.*

For a partially ordered semigroup see L. Fuchs [3].

A root function δ_n has the following property: If $x \leqslant y$ then either $\delta_n(x) \leqslant \delta_n(y)$ or $\delta_n(x)$ and $\delta_n(y)$ are incomparable. If it were false we would have $\delta_n(y) \leqslant \delta_n(x)$ and then $n[\delta_n(y)] \leqslant n[\delta_n(x)]$ and so $y \leqslant x$, which is impossible for $x \neq y$.

2. Conditions for the n-convexity

From the p-convexity of the function f does not follow, in general, its n-convexity for all $n \in N$, as we have in real analysis. An example: If $(T,.)$ is a semigroup with a neutral element e and $x^3=e$ for all $x \in T$, then there does not exist a root function of the order 3. While a root function of the order 2 exists, $\gamma_2 : x \to x^2$ for $x \in T$.

So we have

Theorem 1. *If a function f is p_i — convex $(i=1, 2, \ldots, k)$ on $A \subset X_1$ (the set A is from definition 2) then f is n-convex on A for every natural number of the form $n=p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ where $\alpha_i \in N'$ $(i=1, 2, \ldots, k)$. N' denotes $N \cup \{0\}$.*

Proof. Let γ_{p_i} be the root function on X_1 and δ_{p_i} be the root function on X_2. Applying the lemma 1 we obtain that $\gamma_{p_i}^k$ (the superscript k denotes the k-iterated composition of the function γ_{p_i}) is a root function of the order p_i^k, in the notation $\gamma_{p_i^k}$ The same holds for $\delta_{p_i}^k$. We have for all $x_t \in A$ and $t=1, 2, \ldots, p_i^k$

$$\gamma_{p_i^k}\left(\mathop{*}_{t=1}^{p_i^k} x_t\right) \in A \quad (k \in N)$$

because

$$\gamma_{p_i}\left(\mathop{*}_{t=(S-1)\,p_i^k+1}^{s \cdot p^k} x_t\right) \in A\,(S=1,2,\ldots,p_i).$$

If the terms are comparable then we must prove that the inequality (4) holds for $n=p_i^k$ and $k \in N$. We shall prove this by induction. For $k=1$ (4) holds by assumption. We assume that (4) is true for k. Then we shall have for $k+1$

$$f\left(\gamma_{p_i^{k+1}}\left(\mathop{*}_{t=1}^{p_i^{k+1}} x_t\right)\right) = f\left(\gamma_{p_i}\left(\gamma_{p_i^k}\left(\mathop{*}_{t=1}^{p_i^k} x_t * \mathop{*}_{t=p_i^k+1}^{2p_i^k} x_t * \ldots * \mathop{*}_{t=(p_i-1)p_i^k+1}^{p_i \cdot p_i^k} x_t\right)\right)\right) \leqslant$$

$$\leqslant \delta_{p_i}\left(\bigoplus_{s=1}^{p_i} f\left(\gamma_{p_i^k}\left(\mathop{*}_{t=(s-1)p_i^k+1}^{s \cdot p_i^k} x_t\right)\right)\right) \leqslant \delta_{p_i}\left(\bigoplus_{s=1}^{p_i} \delta_{p_i^k}\left(\bigoplus_{t=(s-1)p_i^k+1}^{s \cdot p_i^k} f(x_t)\right)\right) =$$

$$= \delta_{p_i^{k+1}}\left(\bigoplus_{t=1}^{p_i^{k+1}} f(x_t)\right).$$

Let $n=p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ for $\alpha_i \in N'$ and $i=1, 2, \ldots, k$. Using the lemma 1 we can construct root functions of the order n on semigroups X_1 and X_2

$$\gamma_n = \gamma_{p_1}^{\alpha_1} \circ \ldots \circ \gamma_{p_k}^{\alpha_k}$$

$$\delta_n = \delta_{p_1}^{\alpha_1} \circ \ldots \circ \delta_{p_k}^{\alpha_k},$$

where $\gamma_{p_i}^{\alpha_i}$ is the α_i-iterated composition of the function γ_{p_i}. Applying the preceding result it is easy to obtain the n-convexity of the function f on A. If the terms are incomparable, we use theorem 3. (Contradiction.)

Corrolary 1. *If the function f is p-convex on $A \subset X_1$ (the set A is from definition 2) then f is also p^k-convex on A, for all $k \in N$.*

If we add some new assumptions we obtain that the 2-convexity of a functional on a semigroup implies its n-convexity for all $n \in N$. So we have

Theorem 2. *Let $(X_1, *)$ of the definition 2 be a commutative semigroup with root functions γ_n for all $n \in N$ and $(X_2, \oplus)$ be the set R of real numbers (or the set R^+ of nonnegative real numbers) with usual addition (multiplication). Then from the 2-convexity of the functional $f: A \to R$ (R^+) on $A \subset X_1$ (the set A is from definition 2 for all $n \in N$) follows its n-convexity on A for all $n \in N$.*

Proof. The root function on $(R, +)$ is $\delta_n(x) = \frac{x}{n}$ $(\delta_n(x) = \sqrt[n]{x}$ on $(R^+, .))$. Applying the corrolary 1 we obtain the 2^k-convexity of f, for all $k \in N$. We assume that (4) holds for n. We shall prove (4) for $n-1$.

$$f(\gamma_n(x_1 * \ldots * x_{n-1} * \gamma_{n-1}(x_1 * \ldots * x_{n-1}))) \leqslant \tag{5}$$

$$\leqslant \frac{1}{n}(f(x_1) + \ldots + f(x_{n-1}) + f(\gamma_{n-1}(x_1 * \ldots * x_{n-1}))).$$

On the other hand $(X_1, *)$ is the commutative semigroup and $x_i = (n-1)[\gamma_{n-1}(x)]$. Then

$$f(\gamma_n(x_1 * \ldots * \gamma_{n-1}(x_1 * \ldots * x_{n-1}))) =$$

$$= f\left(\gamma_n\left(\overset{n-1}{\underset{i=1}{*}} n[\gamma_{n-1}(x_i)]\right)\right) = f\left(\gamma_{n-1}\left(\overset{n-1}{\underset{i=1}{*}} x_i\right)\right)$$

holds. Putting this in (5) we obtain (4) for $n-1$.

(In the same way we can obtain the theorem 2 for $(R^+, .)$ and $\delta_n(x) = \sqrt[n]{x}$).

Examples.

We take the following sets of real functions on R or R^+ or their subsets as examples of n-convex functions

$$K_1 = \left\{ f \mid f(\sqrt{xy}) \leqslant \frac{f(x) + f(y)}{2} \right\},$$

$$K_2 = \left\{ f \mid f(\sqrt{xy}) \leqslant \sqrt{f(x) f(y)} \right\},$$

$$K_3 = \left\{ f \mid f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x) + f(y)}{2} \right\},$$

$$K_4 = \left\{ f \mid f\left(\frac{x+y}{2}\right) \leqslant \sqrt{f(x) f(y)} \right\}.$$

Applying theorem 2 on the functions of some set K_i, i.e. one which satisfies the corresponding inequality for 2, we obtain that they satisfy the corresponding inequality for all $n \in N$.

We take a special case $f(x)=x$. This function belongs to K_1. So we obtain the known inequality between arithmetic and geometric means for all $n \in N$. Applying this inequality on the functions of sets K_i $(i=1, 2, 3, 4)$ we obtain

$$K_2 \subset K_1 \text{ and } K_4 \subset K_3.$$

If we take the other special case $\frac{1}{x}$ of K_1 we obtain the inequality between geometric and harmonic means for all $n \in N$.

The set K_3 is the usual set of the (Jensen) convex functions. Gamma-function $\Gamma(x)$ for $x>0$ belongs to K_4 and so satisfies the corresponding inequality for all $n \in N$ (see D. S. Mitrinović [4], p. 280). The function

$$\frac{\Gamma(x)}{(x-1)^x} \quad \text{for } x>1$$

belongs to K_4 (see [5]).

3. The inverse problem

In some sense we can obtain a inverse theorem of the theorem 1. We have seen that a semigroup $(S, .)$ is without a root function of the order 2. But this semigroup has the root function $\gamma_3(x)=x$ of the order 3.

So we have

Theorem 3. *If a function f is n-convex on $A \subset X_1$ (the set A is from definition 2) then it is also p_i-convex for $i=1, 2, \ldots, k$, where $p_1^{\alpha_1} \ldots p_k^{\alpha_k}=n$ for $\alpha_i \in N'$ $(i=1, \ldots, k)$.*

Proof. Let the inequality (4) hold if the terms are comparable. Applying the lemma 1 we obtain that

$$\gamma_{p_i}(x)=n_i\,[\gamma_n(x)]$$

are root functions of orders $p_i (i=1, 2, \ldots, k)$ on X_1, where $n_1=p_1^{\alpha_1} \ldots p_1^{\alpha_i-1} \ldots p_k^{\alpha_k}$. Also

$$\delta_{p_i}(x)=n_i\,[\delta_n(x)]$$

are root functions on X_2.

It is evident that from $x_s \in A$ for $s=1, 2, \ldots, p_i$ follows

$$\gamma_{p_i}\left(\mathop{*}_{s=1}^{p_i} x_s\right) \in A.$$

We have the following representations

$$\gamma_n=\gamma_p^{\alpha_1} \circ \ldots \circ \gamma_{p_k}^{\alpha_k}$$

$$\delta_n=\delta_{p_1}^{\alpha_1} \circ \ldots \circ \delta_{p_k}^{\alpha_k}.$$

The preceding equalities are independent of the order of functions.

Now we have, if the terms are comparable,

$$f\left(\gamma p_i\left(\overset{p_i}{\underset{s=1}{*}} x_s\right)\right)=f\left(\gamma p_i\left(\overset{p_i}{\underset{s=1}{*}} n_i\left[\gamma n_i(x_s)\right]\right)\right)=$$

$$=f\left(\gamma p_i\left(\gamma n_i\left(\overset{p_i}{\underset{s=1}{*}} n_i[x_s]\right)\right)\right)=f\left(\gamma n\left(\overset{p_i}{\underset{s=1}{*}} n_i[x_s]\right)\right)\leqslant$$

$$\leqslant \delta_n\left(\overset{p_i}{\underset{s=1}{\oplus}} n_i\left[f(x_s)\right]\right)=\delta_{p_i}\left(\delta_{n_i}\left(\overset{p_i}{\underset{s=1}{\oplus}} n_i\left[f(x_s)\right]\right)\right)=$$

$$=\delta_{p_i}\left(\overset{p_i}{\underset{s=1}{\oplus}} f(x_s)\right) \quad \text{for } i=1, 2, \ldots, k.$$

Thus the proof of theorem 3 is finished.

It is easy to prove using theorems 1 and 3

Corrolary 2. If a function is n-convex ($n=p_1^{\alpha_1}\ldots p_k^{\alpha_k}$ for $\alpha_i \in N'$ and $i=1, 2, \ldots, k$) on $A \subset X_1$ (the set A is from definition 2) then it is also s-convex, where $s=p_1^{\beta_1}\ldots p_k^{\beta_k}$ for $\beta_i \in N'$ and $i=1, 2, \ldots, k$.

REFERENCES

[1] S. S. Jou, S. Kurepa, *Some properties of almost open sets in topological groups and application*, Glasnik Mat. Ser. III 7 (27) (1972), 189–200.

[2] S. S. Jou, *Convex functions on topological groups*, Glasnik Mat. Ser. III 8 (28) (1973), 175–178.

[3] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, 1963.

[4] D. S. Mitrinović, *Analitičke nejednakosti*, Beograd, 1970.

[5] *Problem* 549, Math. Monthly vol 79 N° 8, 1972, p. 917–918.

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n-KONVEKSNE FUNKCIJE NAD POLUGRUPOM SA FUNKCIJOM ANTISTEPENOVANJA

Rezime

U ovom radu je proučena mogućnost prenosa jedne osobine realnih Jensen konveksnih funkcija realne promenljive na funkcije definisane nad komutativnom polugrupom, a sa vrednostima u parcijalno uređenoj komutativnoj polugrupi.

Neka je $(X, *)$ komutativna polugrupa. Uzimamo da je

$$\overset{n}{\underset{i=1}{*}} x_i \overset{\text{def}}{=} x_1 * \ldots * x_n \quad \text{i} \quad n[x] \overset{\text{def}}{=} \underbrace{x * \ldots * x}_{n}.$$

Definicija 1. Funkcija γ_n: $X \to X$, za dato $n \in N$, jeste funkcija antistepenovanja reda n ako važi

$$(A_1) \quad \gamma_n(x) * \gamma_n(y) = \gamma_n(x * y),$$

$$(A_2) \quad n[\gamma_n(x)] = x \text{ za sve } x, y \in X$$

(ako γ_n postoji).

Definicija 2. Neka je $(X_1, *)$ komutativna polugrupa sa funkcijom antistepenovanja γ_n, a $(X_2, \oplus)$ je parcijalno uređena komutativna polugrupa sa funkcijom antistepenovanja δ_n. Funkcija f: $A \to X_2$ je n-konveksna funkcija nad skupom $A \subset X_1$ (sa osobinom da za sve $x_i \in A$ i $i = 1, 2, \ldots, n$ je $\gamma_n\left(\overset{n}{\underset{i=1}{*}} x_i\right) \in A$) ako:

ili zadovoljava sledeću nejednakost

$$f\left(\gamma_n\left(\overset{n}{\underset{i=1}{*}} x_i\right)\right) \leqslant \delta_n\left(\overset{n}{\underset{i=1}{\oplus}} f(x_i)\right)$$

ili leva i desna strana u prethodnoj nejednakosti nisu uporedljive, za sve $x_i \in A$ i $i = 1, 2, \ldots, n$.

Osnovni rezultati su sadržani u tri teoreme:

Teorema 1. Ako je funkcija f p_i-konveksna za $i = 1, 2, \ldots, k$ nad $A \subset X_1$ (skup A je iz definicije 2), tada je f i n-konveksna nad A za svaki prirodan broj oblika $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ gde $\alpha_i \in N'$ i $i = 1, 2, \ldots, k$. N' je skup $N \cup \{0\}$.

Teorema 2. Neka je $(X_1, *)$ sa funkcijama antistepenovanja γ_n za sve $n \in N$ i neka je $(X_2, \oplus)$ skup realnih brojeva R (ili skup R^+ nenegativnih realnih brojeva) sa uobičajenom operacijom sabiranja (množenja). Tada iz 2-konveksnosti funkcionele f: $A \to R$ (R^+) nad $A \subset X_1$ (skup A je iz definicije 2 za sve $n \in N$) sledi njena n-konveksnost nad A za sve $n \in N$.

Teorema 3. Ako je funkcija f n-konveksna nad $A \subset X_1$ (skup A je iz definicije 2) tada je i p_i-konveksna nad A za $i = 1, 2, \ldots, k$ gde je $p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n$ za $\alpha_i \in N'$ $(i = 1, 2, \ldots, k)$.

Kao specijalni slučajevi n-konveksnih funkcija navode se klase K_i $(i = 1, 2, 3, 4)$ realnih funkcija realne promenljive.