Zbornik radova Prirodno-matematičkog fakulteta — Univerzitet u Novom Sadu, knjiga 6 (1976)
Review of Research of Science — University of Novi Sad, Volume 6 (1976)

Mileva Prvanović

ON TWO TENSORS IN A LOCALLY DECOMPOSABLE RIEMANNIAN SPACE*

1. An n-dimensional manifold M_n is called a locally decomposable Riemannian space V_n [1] if in M_n a tensor field $F_j^i \neq \delta_j^i$ and a positive definite Riemannian metric $ds^2 = g_{ij}(x^k)\, dx^i dx^j$ are given, satisfying the conditions

$$F_k^i F_j^k = \delta_j^i, \quad g_{st} F_i^s F_j^t = g_{ij}, \quad F_{j,k}^i = 0,$$

where the index after the comma indicates the covariant derivative with respect to the Riemannian metric. If we put

$$F_j^t g_{ti} = F_{ji},$$

then

$$F_{ij} = F_{ji},$$

and the condition $F_{i,k}^j = 0$ is equivalent to the condition $F_{ij,k} = 0$.

A locally decomposable space can be covered by a separating coordinate system [1], *j. e.* by such a system of coordinate neighbourhoods (x^i) that in any intersection of two coordinate neighbourhoods (x^i) and $(x^{i'})$ we have

$$x^{a'} = x^{a'}(x^a), \quad x^{y'} = x^{y'}(x^y),$$

where the indices a, b, c, d run over the range $1, 2, \ldots, p$ and the indices x, y, z, t run over the range $p+1, p+2, \ldots, p+q = n$ by M_p we denote the system of subspaces defined by $x^y = 0$ and by M_q the system of subspaces defined by $x^a = 0$. Then our space M_n is locally the product $M_p \, X \, M_q$ of two spaces.

With respect to a separating coordinate system, the metric of the space has the form

$$ds^2 = g_{ab}(x^c)\, dx^a dx^b + g_{xy}(x^z)\, dx^x dx^y,$$

that is g_{ab} are the functions of x^x only, $g_{ax} = 0$, and g_{xy} are the functions of x^y only. The Christoffel symbols $\left\{ {i \atop jk} \right\}$ are all zero except $\left\{ {a \atop bc} \right\}$ and $\left\{ {x \atop yz} \right\}$ and besides $\left\{ {a \atop bc} \right\}$

* This research was supported by the Institute of Mathematics, Belgrade.

are functions of x^a only and $\left\{ {x \atop yz} \right\}$ are functions of x^y only. With respect to a separating coordinate system we have, too,

$$F_{ij}=\begin{pmatrix} g_{ab} & 0 \\ 0 & -g_{xy} \end{pmatrix}, \quad F_j^i=\begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_y^x \end{pmatrix}.$$

Therefore

$$\varphi=F_a^a=p-q. \tag{1.1}$$

In the following we suppose $p>2$, $q>2$.

Let S^i be a decomposable vector field in M_n and let us consider the connexion

$$\Gamma_{jk}^i=\left\{ {i \atop jk} \right\}+S_k\,\delta_j^i-g_{kj}\,S^i+S_p\,F_k^p\,F_j^i-S^p\,F_p^i\,F_{jk}. \tag{1.2}$$

With respect to **a** separating coordinate system all the Γ_{jk}^i are zero except

$$\Gamma_{bc}^{\;a}=\left\{ {a \atop bc} \right\}+2\,S_c\delta_b^a-2\,g_{bc}\,S^a$$

$$\Gamma_{yz}^{\;x}=\left\{ {x \atop yz} \right\}+2\,S_z\,\delta_y^z-2\,g_{yz}\,S^x,$$

that is the connexions $\Gamma_{bc}^{\;a}$ and $\Gamma_{yz}^{\;x}$ are the semi-symmetric metric connexions **[3]** of M_p and M_q respectively. For this reason the connexion (1.2) is called a *separately semi-symmetric connexion* of the locally decomposable Riemannian space.

The connexion (1.2) being non-symmetric, we can consider different kinds of covariant derivatives. These covariant derivatives of a tensor a_j^i for example, are given by

$$\underset{1}{\nabla}_k\,a_j^i=\frac{\partial\,a_j^i}{\partial x^k}+a_j^p\,\Gamma_{pk}^i-a_p^i\,\Gamma_{jk}^p$$

$$\underset{2}{\nabla}_k\,a_j^i=\frac{\partial\,a_j^i}{\partial\,x^k}+a_j^p\,\Gamma_{kp}^i-a_p^i\,\Gamma_{kj}^p$$

$$\underset{3}{\nabla}_k\,a_j^i=\frac{\partial\,a_j^i}{\partial\,x^k}+a_j^p\,\Gamma_{pk}^i-a_p^i\,\Gamma_{kj}^p$$

$$\underset{4}{\nabla}_k\,a_j^i=\frac{\partial\,a_j^i}{\partial\,x^k}+a_j^p\,\Gamma_{kp}^i-a_p^i\,\Gamma_{jk}^p.$$

Following the treatment used in [2], [3] and [4], we can prove the existence of four curvature tensors. In fact, considering a contravariant vector field a^i, we have

$$\underset{1}{\nabla}_k \underset{1}{\nabla}_j a_i - \underset{1}{\nabla}_j \underset{1}{\nabla}_k a^i = a^r \underset{1}{R}{}^i_{rkj} + \underset{1}{\nabla}_p a^i (T^p_{kj} - T^p_{jk})$$

$$\underset{2}{\nabla}_k \underset{2}{\nabla}_j a^i - \underset{2}{\nabla}_j \underset{2}{\nabla}_k a^i = a^r \underset{2}{R}{}^i_{rkj} + \underset{2}{\nabla}_s a^i (T^s_{jk} - T^s_{kj})$$

$$\underset{2}{\nabla}_k \underset{1}{\nabla}_j a^i - \underset{1}{\nabla}_j \underset{2}{\nabla}_k a^i = a^r \underset{3}{R}{}^i_{rkj}$$

$$\underset{4}{\nabla}_k \underset{3}{\nabla}_j a^i - \underset{3}{\nabla}_j \underset{4}{\nabla}_k a^i = a^r \underset{4}{R}{}^i_{rkj}$$

where

$$\underset{1}{R}{}^i_{rkj} = K^i_{rkj} + T^i_{rj,k} - T^i_{rk,j} + T^p_{rj} T^i_{pk} - T^p_{rk} T^i_{pj},$$

$$\underset{2}{R}{}^i_{rkj} = K^i_{rkj} + T^i_{jr,k} - T^i_{kr,j} + T^p_{jr} T^i_{kp} - T^p_{kr} T^i_{jp},$$

$$\underset{3}{R}{}^i_{rkj} = K^i_{rkj} + T^i_{rj,k} - T^i_{kr,j} + T^p_{rj} T^i_{kp} - T^p_{kr} T^i_{pj} + (T^i_{pr} - T^i_{rp}) T^p_{kj},$$

$$\underset{4}{R}{}^r_{rkj} = K^i_{rkj} + T^i_{rj,k} - T^i_{kr,j} + T^p_{rj} T^i_{kp} - T^p_{kr} T^i_{pj} + (T^i_{pr} - T^i_{rp}) T^p_{jk},$$

and

$$T^i_{rj} = S_j \delta^i_r - g_{rj} S^i + S_a F^a_j F^i_r - S^a F^i_a F_{jr}.$$

$$K^i_{rkj} = \frac{\partial}{\partial x^k}\left\{ {i \atop rj} \right\} - \frac{\partial}{\partial x^j}\left\{ {i \atop rk} \right\} + \left\{ {s \atop rj} \right\}\left\{ {i \atop sk} \right\} - \left\{ {s \atop rk} \right\}\left\{ {i \atop sj} \right\}.$$

In other words, K^i_{rkj} is the curvature tensor of a Riemannian space V_n having g_{ij} as a metric tensor.

The results obtained in [4] will be generalized, in the present paper, for a locally decomposable Riemannian space and for a connexion (1.2). We shall show the existence of the tensors formed with $\underset{3}{R}{}^i_{rkj}$ and $\underset{4}{R}{}^i_{rkj}$ depending only on the locally decomposable Riemannian space V_n, that is independent of the vector field S^i of the connexion (1.2). We consider, in §2, the tensor formed with $\underset{3}{R}{}^i_{rkj}$, and in §3 the tensor formed with $\underset{4}{R}{}^i_{rkj}$. We shall see that this last tensor is the product-projective curvature tensor [5] of the space V_n.

2. The tensor $\underset{3}{R}{}^i_{rkj}$ can be expressed in the form:

$$\underset{3}{R}{}^i_{rkj} = K^i_{rkj}$$

$$\begin{aligned} &+ \delta^i_r \psi_{jk} - g_{rj} \psi^i_k + F^i_r \psi_{pk} F^p_j - F_{jr} \psi^p_k F^i_p \\ (2.1) \qquad &- \delta^i_k \psi_{rj} + g_{rk} \psi^i_j - F^i_k \psi_{pj} F^p_r + F_{rk} \psi^p_j F^i_p \\ &+ S(\delta^i_k g_{rj} - \delta^i_j g_{rk} + F^i_k F_{rj} - F^i_j F_{rk}) \\ &+ S^*(\delta^i_r g_{pj} - \delta^i_j g_{pk} + F^i_k F_{pj} - F^i_j F_{pk}) F^p_r, \end{aligned}$$

where

$$\psi_{jk}=S_{j,k}-S_j S_k+S_p S^p g_{jk}-S_a S_b F_j^a F_k^b+S^a S_b F_a^b F_{jk}$$

$$S_p S^p=S, \quad S^p S_k F_p^k=S^*.$$

We now introduce the notations:

$$\underset{3}{R}{}^p_{rkp}={}'\underset{3}{R}_{rk}, \quad \underset{3}{R}{}^p_{rpj}={}''\underset{3}{R}_{rj}, \quad \underset{3}{R}{}^p_{pkj}={}'''\underset{3}{R}_{kj},$$

$$'\underset{3}{R}=g^{jk}\,'\underset{3}{R}_{jk}, \quad ''\underset{3}{R}=g^{jk}\,''\underset{3}{R}_{jk}, \quad '''\underset{3}{R}=g^{jk}\,'''\underset{3}{R}_{jk},$$

(2.2) $$K^p_{rkp}=K_{rk}, \quad g^{rk} K_{rk}=K.$$

Then contracting (2.1) with respect to i and j and taking into account (2.2), we obtain:

$$'\underset{3}{R}_{rk}=K_{rk}-\psi_{rk}-\psi_{pq} F_r^p F_k^q+g_{rk}\psi_p^p+\psi_j^i F_i^j F_{rk}$$
$$+S[(n-2) g_{rk}-\varphi F_{rk}]+S^*[(n-2) g_{pk}-\varphi F_{pk}] F_r^p.$$

The vector field S_i being decomposable, we have ([1], p. 225):

(2.3) $$S_{j,k}=S_{i,p} F_j^i F_k^p.$$

Consequently

$$\psi_{pq} F_j^p F_k^q=S_{j,k}-S_p S_q F_j^p F_k^q+S_p S^p g_{jk}-S_j S_k+S^p S_q F_p^q F_{jk},$$

that is

$$\psi_{pq} F_j^p F_k^q=\psi_{jk},$$

and $'\underset{3}{R}_{rk}$ can be expressed in the form:

$$'\underset{3}{R}_{rk}=K_{rk}-2\,\psi_{rk}+g_{rk}\,\psi_p^p+\psi_q^p F_p^q F_{rk}$$
$$+S\,[-(n-2) g_{rk}-\varphi F_{rk}]+S^*\,[-(n-2) g_{pk}-\varphi F_{pk}] F_r^p,$$

from which we see that:

$$'\underset{3}{R}_{pk} F_r^p=K_{pk} F_r^p-2\psi_{pk} F_r^p+F_{rk}\,\psi_p^p+\psi_q^p F_p^q g_{rk}$$
$$+S\,[-(n-2) g_{pk}-\varphi F_{pk}] F_r^p+S^*\,[-(n-2) g_{rk}-\varphi F_{rk}]$$

and

$$'\underset{3}{R}-K-(n-2)\,\psi_i^i-\varphi\,\psi_j^i F_i^j=[n(2-n)-\varphi^2]\,S+2\varphi\,(1-n)\,S^*$$

$$'\underset{3}{R}{}_j^i F_i^j-K_j^i F_i^j-\varphi\,\psi_i^i-(n-2)\,\psi_j^i F_i^j=2\varphi\,(1-n)\,S+[(2-n)\,n-\varphi^2]S^*$$

The last two relations give:

$$
(2.4)\quad \left\{
\begin{aligned}
S&=\alpha[{}'\underset{3}{R}-K-(n-2)\,\psi^i_i-\varphi\,\psi^i_j F^j_i]\\
&\quad+\beta[{}'R^j_i F^i_j-K^j_i F^i_j-\varphi\,\psi^i_i-(n-2)\,\psi^i_j F^j_i]\\
S^*&=\beta[{}'\underset{3}{R}-K-(n-2)\,\psi^i_i-\varphi\psi^i_j F^j_i]\\
&\quad+\alpha\,[{}'\underset{3}{R^j_i} F^i_j-K^j_i F^i_j-\varphi\,\psi^i_i-(n-2)\,\psi^i_j F^j_i],
\end{aligned}
\right.
$$

where

$$
\alpha=\frac{n(2-n)\,\varphi^2}{[n(2-n)-\varphi^2]^2-4\varphi^2(1-n)^2},\quad \beta=\frac{-2\varphi(1-n)}{[n(2-n)-\varphi^2]^2-4\varphi^2(1-n)^2}
$$

On the other hand, contracting (2.1) with respect to i and r, we have

$$
{}'''\underset{3}{R_{kj}}=(n-2)\psi_{jk}+\varphi\,\psi_{ik}\,F^i_j,
$$

and consequently

$$
{}'''\underset{3}{R_{ki}}\,F^i_j=\varphi\psi_{jk}+(n-2)\psi_{ik}\,F^i_j.
$$

Putting

$$
(2.5)\qquad \alpha_1=\frac{n-2}{(n-2)^2-\rho^2},\quad \beta_1=\frac{-\varphi}{(n-2)^2-\varphi^2},
$$

we deduce from the last two equations:

$$
(2.6)\qquad \left\{
\begin{aligned}
\psi_{jk}&=\alpha_1{}'''\underset{3}{R_{kj}}+\beta_1\,{}'''\underset{3}{R_{kp}}\,F^p_j,\\
\psi_{ik}\,F^i_j&=\beta_1\,{}'''\underset{3}{R_{kj}}+\alpha_1\,{}'''\underset{3}{R_{kp}}\,F^p_j.
\end{aligned}
\right.
$$

Consequently, the relations

$$
(2.7)\qquad \left\{
\begin{aligned}
\psi^i_i&=\alpha_1\,{}'''\underset{3}{R^i_i}+\beta_1\,{}'''\underset{3}{R^j_i}\,F^i_j,\\
\psi^i_j\,F^j_i&=\beta_1\,{}'''\underset{3}{R^i_i}+\alpha_1\,{}'''\underset{3}{R^j_i}\,F^i_j
\end{aligned}
\right.
$$

are satisfied too. Substituting (2.7) into (2.4), we find

$$
S=\alpha\,({}'\underset{3}{R}-K-{}'''\underset{3}{R^i_i})+\beta\,({}'\underset{3}{R^j_i}F^i_j-K^j_i F^i_j-{}'''\underset{3}{R^j_i}F^i_j),
$$

$$
S^*=\beta\,({}'\underset{3}{R}-K-{}'''\underset{3}{R^i_i})+\alpha\,({}'\underset{3}{R^j_i}F^i_j-K^j_i F^i_j-{}'''\underset{3}{R^j_i}F^i_j).
$$

Substituting this and (2.6) into (2.1), by straightforward calculation we get:

$$\begin{aligned}
&\underset{3}{R}{}^{i}_{rkj} + \alpha_1 \,{}'''\underset{3}{R}{}^{i}_{rkj} + \beta_1 \,{}'''\underset{3}{R}{}^{i}_{pkj} F^p_r \\
(2.8)\qquad &- [\alpha \underset{3}{R} + \beta \underset{3}{R}{}^q_p F^p_q]\, \rho^i_{rkj} - [\beta \underset{3}{R} + \alpha \underset{3}{R}{}^q_p F^p_q]\, \rho^i_{tkj} F^t_r = \\
&K^i_{rkj} - (\alpha K + \beta K^q_p F^p_q)\, \rho^i_{rkj} - (\beta K + \alpha K^q_p F^p_q)\, \rho^i_{tkj} F^t_r,
\end{aligned}$$

where we have put:

$$\begin{aligned}
&{}'''\underset{3}{R}{}^{i}_{rkj} = \delta^i_k \,{}'''\underset{3}{R}{}_{jr} - g_{rk} \,{}'''\underset{3}{R}{}^{i}_{j} + F^i_k \,{}'''\underset{3}{R}{}_{jp} F^p_r - F_{rk} \,{}'''\underset{3}{R}{}^{p}_{j} F^i_p \\
&\qquad - \delta^i_r \,{}'''\underset{3}{R}{}_{kj} + g_{rj} \,{}'''\underset{3}{R}{}^{i}_{k} - F^i_r \,{}'''\underset{3}{R}{}_{kp} F^p_j + F_{rj} \,{}'''\underset{3}{R}{}^{p}_{k} F^i_p, \\
(2.9)\qquad &\rho^i_{rkj} = \delta^i_k g_{rj} - \delta^i_j g_{rk} + F^i_k F_{rj} - F^i_{j\,rk} \\
&\underset{3}{R} = {}'\underset{3}{R} - {}'''\underset{3}{R}, \quad \underset{3}{R}{}^j_i F^i_j = ({}'\underset{3}{R}{}^j_i - {}'''\underset{3}{R}{}^j_i) F^i_j.
\end{aligned}$$

We see that the tensor on the right — hand side of the equation (2.8) depends only on the locally decomposable Riemannian space V_n. Therefore the tensor on the left-hand side of the equation (2.8), although it is formed with the curvature tensor $\underset{3}{R}{}^i_{rkj}$ of the connexion (1.2), depends only on the Riemannian space too, i. e. it is independent of the vector field S^i of the connexion (1.2).

3. We transcribe the tensor $\underset{4}{R}{}^i_{rkj}$ in the following form:

$$\begin{aligned}
\underset{4}{R}{}^i_{rkj} = {} & K^i_{rkj} + \delta^i_r S_{j,k} - g_{rj} S^i_{,k} + F^i_r S_{p,k} F^p_j - F_{jr} S^p_{,k} F^i_p - \\
& - \delta^i_k S_{r,j} + g_{rk} S^i_{,j} - F^i_k S_{p,j} F^p_r + F_{rk} S^p_{,j} F^i_p - \\
& - \delta^i_r (S_j S_k - S^p S_p g_{jk} + S_p S_q F^p_j F^q_k - S^p S_q F^q_p F_{jk}) + \\
& + g_{rj} (S^i S_k - S^p S_p \delta^i_k + S^p F^i_p S^q F_{qk} - S^p S_q F^q_p F^i_k) - \\
& - F^i_r (S_p F^p_j S_k - S^p S_p F_{kj} + S_p F^p_k S_j - S^p S_q F^q_p g_{kj}) + \\
& + F_{jr} (S^i S^p F_{pk} - S^p S_p F^i_k + S^p F^i_p S_k - S^p S_q F^q_p \delta^i_k) - \\
& - g_{rk} (S^i S_j - S^p S_p \delta^i_j + S^p S_q F^i_p F^q_j - S^p S_q F^q_p F^i_j) - \\
& - F_{rk} (S^i S_p F^p_j - S^p S_p F^i_j + S_j S^p F^i_p - S^p S_q F^q_p \delta^i_j) + \\
& + \delta^i_j (S_r S_k - S^p S_p g_{rk} + S_p S_q F^p_r F^q_k - S^p S_q F^q_p F_{rk}) + \\
& + F^i_j (S_k S_p F^p_r - S^p S_p F_{rk} + S_p F^p_k S_r - S^p S_q F^q_p g_{rk}).
\end{aligned}$$

If we put

$$\theta_{jk}=S_j S_k-S^p S_p g_{jk}+S_p S_q F_j^p F_k^q-S^p S_q F_p^q F_{jk},$$

$$\theta_k^i=g^{ip}\theta_{pk},$$

we see that

$$\theta_p^i F_j^p=\theta_j^p F_p^i,$$

and the tensor $\underset{4}{R}{}_{rkj}^{i}$ may be written as

$$\begin{aligned}\underset{4}{R}{}_{rkj}^{i}=&K_{rkj}^{i}+\delta_r^i S_{j,k}-g_{rj}S_{,k}^i+F_r^i S_{p,k}F_j^p-F_{rj}S_{,k}^p F_p^i-\\&-\delta_k^i S_{r,j}+g_{rk}S_{,j}^i-F_k^i S_{p,j}F_r^p+F_{rk}F_{,j}^p F_p^i-\\&-\delta_r^i\theta_{jk}+g_{rj}\theta_k^i-F_r^i\theta_{pk}F_j^p+F_{rj}\theta_k^p F_p^i-\\&-g_{rk}\theta_j^i-F_{rk}\theta_j^p F_p^i+\delta_j^i\theta_{rk}+F_j^i\theta_{pk}F_r^p.\end{aligned}\tag{3.1}$$

We now introduce the notations:

$$\underset{4}{R}{}_{rkp}^{p}={}'\underset{4}{R}{}_{rk},\quad \underset{4}{R}{}_{pkj}^{p}={}'''\underset{4}{R}{}_{kj}.$$

Taking this into account (2.2) and (2.3), we obtain from (3.1):

$$\begin{aligned}{}'\underset{4}{R}{}_{rk}=&K_{rk}-2S_{r,k}+g_{rk}(S_{,p}^p-\theta_p^p)+\\&+F_{rk}(S_{,q}^p-\theta_q^p)F_p^q+n\theta_{rk}+\varphi\theta_{pk}F_r^p,\end{aligned}$$

or

$$n\theta_{rk}+\varphi\theta_{pk}F_r^p-2S_{r,k}={}'\underset{4}{R}{}_{rk}-K_{rk}+p g_{rk}+q F_{rk},\tag{3.2}$$

where

$$p=\theta_i^i-S_{,i}^i,\quad q=(\theta_j^i-S_{,j}^i)F_i^j.$$

To get the quantities p and q, we contract (3.1) with respect to i and r and take into account (2.3). Thus we find:

$${}'''\underset{4}{R}{}_{kj}=(n-2)(S_{j,k}-\theta_{jk})+\varphi(S_{p,k}-\theta_{pk})F_j^p,$$

$${}'''\underset{4}{R}{}_{kp}F_j^p=\varphi(S_{j,k}-\theta_{jk})+(n-2)(S_{p,k}-\theta_{pk})F_j^p.$$

Consequently, taking account of (2.5) we find

$$\begin{aligned}&S_{j,k}-\theta_{jk}=\alpha_1\,{}'''\underset{4}{R}{}_{kj}+\beta_1\,{}'''\underset{4}{R}{}_{kp}F_j^p\\&(S_{p,k}-\theta_{pk})F_j^p=-\beta_1\,{}'''\underset{4}{R}{}_{kj}-\alpha_1\,{}'''\underset{4}{R}{}_{kp}F_j^p.\end{aligned}\tag{3.3}$$

Thus we have

$$(3.4)\qquad \begin{cases} p=-(S^p_{,p}-\theta^p_p)=-\alpha_1\,'''\underset{4}{R}{}^p_p-\beta_1\,'''\underset{4}{R}{}^p_q F^q_p, \\ q=-(S^p_{,q}-\theta^p_q)F^q_p=\beta_1\,'''\underset{4}{R}{}^p_p+\alpha_1\,'''\underset{4}{R}{}^p_q F^q_p. \end{cases}$$

We now define the tensor r_{jk} by

$$r_{jk}=\alpha_1\,'''\underset{4}{R}_{kj}+\beta_1\,'''\underset{4}{R}_{kp}F^p_j.$$

Then (3.3) gives

$$(3.5)\qquad S_{j,k}=\theta_{jk}+r_{jk}.$$

Substituting this in (3.2) we obtain:

$$(n-2)\theta_{rk}+\varphi\,\theta_{pk}F^p_r={}'\underset{4}{R}_{rk}-K_{rk}+p\,g_{rk}+q\,F_{rk}+2\,r_{rk},$$

from which we deduce

$$\varphi\,\theta_{rk}+(n-2)\,\theta_{pk}F^p_r={}'\underset{4}{R}_{pk}F^p_r-K_{pk}F^p_r+q\,g_{rk}+p\,F_{rk}+2\,r_{pk}F^p_r.$$

The last two relations give

$$(3.6)\qquad \begin{aligned} \theta_{rk}=\alpha_1\,'\underset{4}{R}_{rk}-\alpha_1 K_{rk}+\beta_1\,'\underset{4}{R}_{pk}F^p_r-\beta_1 K_{pk}F^p_r+ \\ +(p\,\alpha_1+q\,\beta_1)g_{rk}+(q\,\alpha_1+p\,\beta_1)F_{rk} \\ +2\,(\alpha_1 r_{rk}+\beta_1 r_{pk}F^p_r) \end{aligned}$$

$$(3.7)\qquad \begin{aligned} \theta_{pk}F^p_r=\beta_1\,'\underset{4}{R}_{rk}-\beta_1 K_{rk}+\alpha_1'\underset{4}{R}_{pk}F^p_r-\alpha_1 K_{pk}F^p_r+ \\ +(p\,\beta_1+q\,\alpha_1)g_{rk}+(q\,\beta_1+p\,\alpha_1)F_{rk} \\ +2\,(\beta_1 r_{rk}+\alpha_1 r_{pk}F^p_r). \end{aligned}$$

Substituting (3.6) and (3.7) into (3.1) and taking into consideration (3.5), we find:

$$(3.8)\qquad \begin{aligned} \underset{4}{R}{}^i_{rkj}+\alpha_1\,'\underset{4}{R}{}^i_{rkj}+\beta_1\,'\underset{4}{R}{}^i_{pkj}F^p_r+2\,\alpha_1\,'''\underset{4}{R}{}^i_{rkj}+2\,\beta_1\,'''\underset{4}{R}{}^i_{pkj}F^p_r+ \\ +(\alpha_1 p+\beta_1 q)\,\overset{i}{\rho}_{rkj}+(\alpha_1 q+\beta_1 p)\,\overset{i}{\rho}_{pkj}F^p_r= \\ =K^i_{rkj}+\alpha_1\{\delta^i_k K_{rj}+F^i_k K_{pj}F^p_r-\delta^i_j K_{rk}-F^i_j K_{pk}-F^p_r\}+ \\ +\beta_1\{\delta^i_k K_{pj}F^p_r+F^i_k K_{rj}-\delta^i_j K_{pk}F^p_r-F^i_j K_{rk}\}, \end{aligned}$$

where we have put

$$(3.9)\qquad {}'\underset{4}{R}{}^i_{rkj}=\delta^i_k\,'\underset{4}{R}_{rj}-\delta^i_j\,'\underset{4}{R}_{rk}+F^i_k\,'\underset{4}{R}_{pj}F^p_r-F^i_j\,'\underset{4}{R}_{pk}F^p_r,$$

$$(3.10)\qquad '''\underset{4}{R}{}^i_{rkj}=\delta^i_k r_{rj}-\delta^i_j r_{rk}+F^i_k r_{pj}F^p_r-F^i_j r_{pk}F^p_r,$$

and the tensor $\overset{i}{\rho}_{rkj}$ is defined by (2.9).

The tensor on the right — hand side of the equation (3.8) is the product-projective curvature tensor [5] of the locally decomposable space V_n. Therefore, the tensor on the left — hand side of (3.8), although it is formed with the curvature tensor $\underset{4}{R}{}^{i}_{rkj}$ of the connexion (1.2), is independent of this connexion.

If $\underset{4}{R}{}^{i}_{rkj}=0$, we see, using (3.4), (3.8), (3.9) and (3.10), that the product--projective curvature tensor of the space V_n vanishes. Tachibana [5] showed that in order for the locally decomposable Riemannian space to be of separately constant curvature, it is necessary and sufficient that the product-projective curvature tensor should vanish. Thus we have:

If $\underset{4}{R}{}^{i}_{rkj}=0$, *the space* V_n *is a space of separately constant curvature.*

Conversely,

If the space V_n *is a space of separately constant curvature, the tensor* $\underset{4}{R}{}^{i}_{rkj}$ *has the form*

$$\underset{4}{R}{}^{i}_{rkj}=-\alpha_1\,\underset{4}{R}{}^{i}_{rkj}-2\,\alpha_1\,{}'''\underset{4}{R}{}^{i}_{rkj}-(\alpha_1\,p+\beta_1\,q)\,\rho^{i}_{rkj}$$
$$-\left[\beta_1\,{}'\underset{4}{R}{}^{i}_{pkj}+2\,\beta_1\,{}'''\underset{4}{R}{}^{i}_{pkj}+(\alpha_1\,q+\beta_1\,p)\,\rho^{i}_{pkj}\right]F^{p}_{r}.$$

REFERENCES

[1] Yano K.: *Differential geometry on complex and almost complex spaces*, Pergamon Press, 1965.

[2] Singh U. P.: *On relative curvature tensors in the subspace of a Riemannian space*, Revue de la Faculté des Sciences de l'Université d'Istanbul, vol. 33 (1968).

[3] Prvanović M.: *Some tensors of metric semi-symmetric connexion*, Atti della Accademia della Scienze di Torino, vol 107 (1972—1973), pp. 303—316.

[4] Prvanović M.: *On pseudo-metric semi-symmetric connexions*, Publication de l'Institut Mathématique 18 (32), 1975, pp. 157—164.

[5] Tachibana S.: *Some theorems on locally product Riemannian spaces*, Tohoku Math. J., 12 (1960), pp. 281—292.

Mileva Prvanović

DVA TENZORA LOKALNO DEKOMPONOVANOG RIMANOVOG PROSTORA

Rezime

U lokalno dekompozibilnom Rimanovom prostoru V_n posmatra se koneksija (1.2). U odnosu na koordinatni sistem koji je adaptiran strukturi prostora, ta koneksija se svodi na semi--simetrične metričke koneksije prostora M_p i M_q, pri čemu su prostori M_p i M_q takvi da je lokalno $V_n=M_p\times M_q$.

Kako je koneksija (1.2) nesimetrična, mogu se posmatrati četiri vrste kovarijantnog diferencijanja i, u vezi s tim, četiri tenzora krivine. U ovom radu se ispituju dva od njih. Preciznije rečeno, pomoću svakog od njih konstruisan je nov tenzor i pokazano je da svaki od njih zavisi samo od prostora V_n a ne i od koneksije (1.2) (iako je pomoću nje konstruisan). U § 3 je pokazano da je jedan od novokonstruisanih tenzora u stvari tenzor produkt-projektivne krivine koje je za lokalno dekopozibilne Rimanove prostore pronašao Tachibana [5].