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STUDY OF THE GENERALIZED LAGRANGE SPACE $\mathbf{GL^{2n}\left(M, g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)\right)}$

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Abstract. The purpose of this paper is to study the fundamental equations for a generalized Lagrange space of order 2. The metric tensor $g_{ij}(x) = e^{2\sigma(x)}\gamma_{ij}(x)$ was introduced on the base manifold by Watanabe, Ykeda S. and Ykeda F. Einstein and Maxwell equations for the space $GL^n(M, g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x))$ were studied by R. Miron and R. Tavakol.

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1. The coefficients of the canonical metrical N-linear connection

Let M be an n-dimensional C^∞ differentiable manifold endowed with the metric tensor $g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)$ defined on Osc^2M and N the canonical nonlinear connection with the coefficients:

$$(1.1)\qquad \begin{cases} \underset{(1)}{N^i}_j = \gamma^i_{kj} y^{(1)k} \\ \underset{(2)}{N^i}_j = \frac{1}{2}\left(\dfrac{\partial \gamma^i_{kj}}{\partial x^r} - \gamma^i_{rh}\gamma^h_{kj}\right) y^{(1)k} y^{(2)r} + \gamma^i_{kj} y^{(2)k} \end{cases},$$

We know that on the total space E there exists a single N-linear connection depending only on the Lagrangian L and which satisfies Matsumoto's axioms

$$(1.2)\quad g_{ij|m} = 0;\quad g_{ij} \overset{(A)}{|}_{m} = 0;\quad T^m{}_{ij} = 0;\quad \underset{(A)}{S}{}^m_{ij} = 0 \quad A = 1,2$$

This is the N-linear canonical metrical connection $C\Gamma(N)$ and it has the coefficients $C\Gamma(N) = \left(L^i{}_{jk},\ \underset{(1)}{C^i_{jk}},\ \underset{(2)}{C^i_{jk}}\right)$ with the espressions:

$$(1.3)\qquad L^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\delta g_{ks}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s}\right)$$

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$$\underset{(\alpha)}{C^i_{jk}} = \frac{1}{2} g^{is} \left(\frac{\delta g_{ks}}{\delta y^{(\alpha)^j}} + \frac{\delta g_{sj}}{\delta y^{(\alpha)^k}} - \frac{\delta g_{jk}}{\delta y^{(\alpha)^s}} \right) \quad (\alpha = 1, 2)$$

Proposition 1.1. *For the considered metric, the coefficients of $C\Gamma(N)$ are:*

$$L^i_{jk} = \gamma^i_{jk} + \underset{0}{\Lambda^i_{jk}} \ ; \ \underset{(\alpha)}{C^i_{jk}} = \underset{\alpha}{\Lambda^i_{jk}} \qquad (\alpha = 1, 2) \tag{1.4}$$

where $\underset{\beta}{\Lambda^i_{jk}} = \delta^i_k \overset{\beta}{\sigma_j} + \delta^i_j \overset{\beta}{\sigma_k} - \gamma_{jk} \overset{\beta}{\sigma^i} \qquad (\beta = 0, 1, 2)$ *and*

$\overset{\beta}{\sigma_j} = \dfrac{\delta \sigma}{\delta y^{(\beta)^j}}, \quad \overset{\beta}{\sigma^i} = \gamma^{is} \overset{\beta}{\sigma_s}$; $y^{(0)} = x$; δ^i_j *is the Kroneker symbol.*

Corollary 1.1. $\underset{\alpha}{\Lambda^i_{jk}} = 0 \ \ \alpha = 0, 1, 2$ *if and only if σ is constant.*

Proposition 1.2. *With respect to $C\Gamma(N)$ the h-paths are given by the equations:*

$$\frac{d^2 x^i}{dt^2} + \left(\gamma^i_{jk} + \underset{0}{\Lambda^i_{jk}} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \tag{1.5}$$

$$\frac{dy^{(1)i}}{dt} + \gamma^i_{jk} y^{(1)j} \frac{dx^k}{dt} = 0$$

$$\frac{dy^{(2)i}}{dt} + \gamma^i_{jk} y^{(1)k} \left(\frac{dy^{(1)j}}{dt} + \gamma^j_{ms} y^{(1)m} \frac{dx^s}{dt} \right) +$$

$$+ \left[\frac{1}{2} \left(\frac{\partial \gamma^i_{jk}}{\partial x^r} - \gamma^i_{rh} \gamma^h_{kj} \right) y^{(1)k} y^{(1)r} + \gamma^i_{kj} y^{(2)k} \right] \frac{dx^j}{dt} = 0$$

- the v_1-paths are given by:

$$x^i(t) = x^i_0; \quad \frac{dx^i}{dt} = 0 \tag{1.6}$$

$$\frac{d^2 y^{(1)i}}{dt^2} + \underset{1}{\Lambda^i_{jk}} \frac{dy^{(1)j}}{dt} \frac{dy^{(1)k}}{dt} = 0$$

$$\frac{d^2y^{(2)i}}{dt^2}+\gamma^i_{jk}y^{(1)j}\frac{dy^{(1)k}}{dt}=0$$

- the v_2-paths are given by the following equations:

$$x^i(t)=x^i_0;\quad \frac{dx^i}{dt}=0 \tag{1.7}$$

$$y^{(1)i}(t)=y_0^{(1)i}\quad \frac{dy^{(1)i}}{dt}=0$$

$$\frac{d^2y^{(2)i}}{dt^2}+\underset{2}{\Lambda^i_{jk}}\frac{dy^{(2)j}}{dt}\frac{dy^{(2)k}}{dt}=0$$

2. Torsions and curvatures

Let $\mathcal{T}$ be the tensor of torsion of an N-linear connection D. For any vector fields $X,Y\in\chi(E)$ we have:

$$\mathcal{T}(X,Y)=D_XY-D_YX-[X,Y] \tag{2.1}$$

This tensor can be evaluated for the pairs of tensor fields (X^H,Y^H), (X^H,Y^{V_α}), $(X^{V_\alpha},Y^{V_\beta})$ and for the canonical metrical linear N-connection $C\Gamma(N)$. By direct calculations we obtain:

Theorem 2.1. *The d-tensor field of torsion for $C\Gamma(N)$ has the local components given by:*

$$T^i{}_{jk}=0,\qquad \underset{(0)}{\overset{1}{T}}{}^i{}_{jk}=r_m{}^i{}_{jk}\,y^{(1)m} \tag{2.2}$$

$$\underset{(0)}{\overset{2}{T}}{}^i{}_{jk}=\frac{1}{2}\left(\frac{\partial r_p{}^i{}_{jk}}{\partial x^q}-r_q{}^i{}_{mk}\gamma^m_{jp}+r_q{}^i{}_{mj}\gamma^m_{kp}\right)y^{(1)q}y^{(1)p}+r_m{}^i{}_{jk}y^{(1)m}$$

$$\underset{(\alpha)}{\overset{0}{P}}{}^i{}_{jk}=\underset{\alpha}{\Lambda^i_{jk}}\quad(\alpha=1,2)\quad \underset{(1)}{\overset{1}{P}}{}^i{}_{jk}=-\underset{0}{\Lambda^i_{jk}}\;;\quad \underset{(1)}{\overset{2}{P}}{}^m_{ij}=\gamma^m_{kr}\gamma^r_{ij}y^{(1)k}-\gamma^m_{ij};$$

$$\underset{(2)}{\overset{1}{P}}{}^i{}_{jk}=0;\quad \underset{(2)}{\overset{2}{P}}{}^i{}_{jk}=-\underset{0}{\Lambda^i_{jk}}\;;\quad \underset{(12)}{\overset{0}{P}}{}^i{}_{jk}=0;\quad \underset{(12)}{\overset{\alpha}{P}}{}^i{}_{jk}=\underset{(\alpha)}{C^i_{jk}}\;;$$

$$\underset{(\alpha)}{\overset{\beta}{S}}{}^i{}_{jk}=0\qquad(\alpha=1,2;\beta=0,1,2)$$

The curvature tensor $\mathcal{R}$ of an N-linear connection D is expressed by:

$$\mathcal{R}(X,Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z \quad \forall X,Y,Z \in \chi(E) \tag{2.3}$$

We know that the 2-tangent structure J has the property $D_X(JY) = J(D_X Y)$ for any vector fields $X, Y \in \chi(E)$. Hence, the essential components of the curvature tensor field are $\mathcal{R}(X,Y)Z^H \ \ \forall X,Y,Z \in \chi(E)$. By direct calculation we can prove:

Theorem 2.2. *The d-tensor field of curvature for the N-linear canonical metrical connection $C\Gamma(N)$ has the local components given by:*

$$\begin{aligned}(2.4)\quad R_b{}^a{}_{pq} &= r_b{}^a{}_{pq} + \delta^a_p \overset{0}{\sigma}_{bq} - \delta^a_q \overset{0}{\sigma}_{bp} - \gamma^{as}\gamma_{bp} \overset{0}{\sigma}_{sq} + \gamma^{as}\gamma_{bq} \overset{0}{\sigma}_{sp} + \\ &+ \delta^a_b (\overset{0}{\sigma}_{pq} - \overset{0}{\sigma}_{qp}) + \gamma_{bt}(r_s{}^a{}_{pq} \overset{1}{\sigma}{}^t - r_s{}^t{}_{pq} \overset{1}{\sigma}{}^a) y^{(1)s} + \\ &+ \gamma_{bt}(r_s{}^a{}_{pq} \overset{2}{\sigma}{}^t - r_s{}^t{}_{pq} \overset{2}{\sigma}{}^a) y^{(2)s}\end{aligned}$$

$$\begin{aligned}(2.5)\quad \underset{(1)}{P}{}_b{}^a{}_{pq} &= \delta^a_p \overset{01}{\sigma}_{bq} - \delta^a_q \overset{10}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp} \overset{01}{\sigma}_{sq} - \gamma_{bq} \overset{10}{\sigma}_{sp}) + \\ &+ \gamma^{as}(\gamma_{bp} \overset{0}{\sigma}_q \overset{1}{\sigma}_s - \gamma_{bq} \overset{0}{\sigma}_s \overset{1}{\sigma}_p) + \delta^a_b (\overset{01}{\sigma}_{pq} - \overset{10}{\sigma}_{qp}) - \\ &- 2\gamma^{as}\gamma_{pq} (\overset{0}{\sigma}_b \overset{1}{\sigma}_s - \overset{0}{\sigma}_s \overset{1}{\sigma}_b) - \delta^a_p \overset{0}{\sigma}_p \overset{1}{\sigma}_b - \delta^a_q \overset{0}{\sigma}_p \overset{1}{\sigma}_b\end{aligned}$$

$$\begin{aligned}(2.6)\quad \underset{(2)}{P}{}_b{}^a{}_{pq} &= \delta^a_p \overset{01}{\sigma}_{bq} - \delta^a_q \overset{20}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp} \overset{02}{\sigma}_{sq} - \gamma_{bq} \overset{20}{\sigma}_{sp}) + \\ &+ \gamma^{as}(\gamma_{bp} \overset{0}{\sigma}_q \overset{2}{\sigma}_s - \gamma_{bq} \overset{0}{\sigma}_s \overset{2}{\sigma}_p) + \delta^a_b (\overset{02}{\sigma}_{pq} - \overset{20}{\sigma}_{qp}) -\end{aligned}$$

$$-2\gamma^{as}\gamma_{pq}(\overset{0}{\sigma}_b\ \overset{2}{\sigma}_s - \overset{0}{\sigma}_s\ \overset{2}{\sigma}_b) - \delta^a_p\ \overset{0}{\sigma}_p\ \overset{2}{\sigma}_b - \delta^a_q\ \overset{0}{\sigma}_p\ \overset{2}{\sigma}_b$$

$$(2.7)\qquad \underset{(12)}{P}{}_b{}^a{}_{pq} = \delta^a_p\ \overset{12}{\sigma}_{bq} - \delta^a_q\ \overset{21}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\ \overset{12}{\sigma}_{sq} - \gamma_{bq}\ \overset{21}{\sigma}_{sp})+$$

$$+\gamma^{as}(\gamma_{qp}\ \overset{1}{\sigma}_s\ \overset{2}{\sigma}_b - \gamma_{qp}\ \overset{1}{\sigma}_b\ \overset{2}{\sigma}_s) + \delta^a_b(\overset{12}{\sigma}_{pq} - \overset{21}{\sigma}_{qp})$$

$$(2.8)\qquad \underset{(1)}{S}{}_b{}^a{}_{pq} = \delta^a_p\ \overset{1}{\sigma}_{bq} - \delta^a_q\ \overset{1}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\ \overset{1}{\sigma}_{sq} - \gamma_{bq}\ \overset{1}{\sigma}_{sp})+$$

$$+\delta^a_b(\overset{12}{\sigma}_{pt}\ \gamma^t_{lq} - \overset{12}{\sigma}_{qt}\ \gamma^t_{lp})y^{(1)l}$$

$$(2.9)\qquad \underset{(2)}{S}{}_b{}^a{}_{pq} = \delta^a_p\ \overset{2}{\sigma}_{bq} - \delta^a_q\ \overset{2}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\ \overset{2}{\sigma}_{sq} - \gamma_{bq}\ \overset{2}{\sigma}_{sp})$$

where

$$\overset{\alpha}{\sigma}_{cd} = \left(\frac{\delta\ \overset{\alpha}{\sigma}_c}{\delta y^{(\alpha)d}} - \overset{\alpha}{\sigma}_c\ \overset{\alpha}{\sigma}_d + \frac{1}{2}\gamma_{cd}\ \overset{\alpha}{\sigma}{}^t\ \overset{\alpha}{\sigma}_t\right) \quad (\alpha = 1, 2)$$

$$\overset{\alpha\beta}{\sigma}_{cd} = \left(\frac{\delta\ \overset{\alpha}{\sigma}_c}{\delta y^{(\beta)d}} - \overset{\beta}{\sigma}_c\ \overset{\alpha}{\sigma}_d + \frac{1}{2}\gamma_{cd}\ \overset{\beta}{\sigma}{}^t\ \overset{\alpha}{\sigma}_t\right) \quad (\alpha, \beta = 1, 2;\quad \alpha \neq \beta)$$

Considering this shape of Bianchi identities:

$$(2.10)\begin{cases} \underset{X,Y,Z}{\sigma} \quad \{D_X\mathcal{T}(Y,Z) - \Re(X,Y)Z + \mathcal{T}(\mathcal{T}(X,Y),Z)\} = 0 \\ \underset{X,Y,Z}{\sigma} \quad \{D_X\Re(U,Y,Z) + \Re(\mathcal{T}(X,Y),Z)U\} = 0 \end{cases}$$

we can rewrite them using the projectors h, v_1, v_2, taking the vector fields X, Y, Z as in the table:

X	A	A	A	A	A	A	B	B	B	C
Y	A	A	A	B	B	C	B	B	C	C
Z	A	B	C	B	C	C	B	C	C	C

where $A = \dfrac{\delta}{\delta x^i}$, $B = \dfrac{\delta}{\delta y^{(1)i}}$, $C = \dfrac{\delta}{\delta y^{(2)i}}$, and for U we take one by one $\dfrac{\delta}{\delta x^i}$, $\dfrac{\delta}{\delta y^{(1)i}}$, $\dfrac{\delta}{\delta y^{(2)i}}$. Hence, we can obtain all Bianchi identities. Those we use in this paper, we can group in:

Proposition 2.1. *For the space $GL^2(n)$ endowed with the considered metric tensor, for the N-linear canonical metrical connection, these Bianchi identities hold:*

$$\sum_{cicl\ p,q,r} \left(R_r{}^a{}_{qp} - \overset{1}{T}{}^b_{qp} \underset{(1)}{C}{}^a{}_{rb} - \overset{2}{T}{}^b_{qp} \underset{(2)}{C}{}^a{}_{rb} \right) = 0 \tag{2.11}$$

$$\sum_{cicl\ p,q,r} \underset{(\alpha)}{S}{}_r{}^a{}_{qp} = 0; \qquad \sum_{cicl\ p,q,r} \underset{(\alpha)}{S}{}_r{}^a{}_{qp} \overset{(\alpha)}{|}_r = 0$$

$$\sum_{cicl\ p,q,r} \left(R_r{}^a{}_{qp|b} - \overset{1}{T}{}^i_{pr} \underset{(1)}{P}{}_b{}^a{}_{qi} - \overset{2}{T}{}^i_{pr} \underset{(2)}{P}{}_b{}^a{}_{qi} \right) = 0$$

$$\underset{(1)}{P}{}_q{}^a{}_{pr} - \underset{(1)}{P}{}_r{}^a{}_{pq} + \underset{(1)}{\overset{1}{P}}{}^a_{pq} \overset{(1)}{|}_r - \underset{(1)}{\overset{1}{P}}{}^a_{pr} \overset{(1)}{|}_q + \underset{(1)}{\overset{2}{P}}{}^b_{pq} \underset{(2)}{C}{}^a{}_{rb} -$$

$$- \underset{(1)}{\overset{2}{P}}{}^b_{pr} \underset{(2)}{C}{}^a{}_{qb} + \underset{(1)}{\overset{1}{P}}{}^a_{lq} \underset{(1)}{C}{}^l{}_{pr} - \underset{(1)}{\overset{1}{P}}{}^a_{lr} \underset{(1)}{C}{}^l{}_{pq} = 0$$

$$\underset{(2)}{P}{}_q{}^a{}_{pr} - \underset{(2)}{P}{}_r{}^a{}_{pq} + \underset{(2)}{\overset{2}{P}}{}^a_{pq} \overset{(2)}{|}_r - \underset{(2)}{\overset{2}{P}}{}^a_{pr} \overset{(2)}{|}_q + \underset{(2)}{\overset{2}{P}}{}^b_{pq} \underset{(2)}{C}{}^a{}_{rb} -$$

$$- \underset{(2)}{\overset{2}{P}}{}^b_{pr} \underset{(2)}{C}{}^a{}_{qb} = 0$$

$$\underset{(12)}{P}{}_{q\ pr}^{\ a} - \underset{(12)}{P}{}_{r\ pq}^{\ a} + \underset{(12)}{\overset{2}{P}}{}^{a}_{pq} \overset{(2)}{|}_{r} - \underset{(12)}{\overset{2}{P}}{}^{a}_{pr} \overset{(2)}{|}_{q} + \underset{(12)}{\overset{2}{P}}{}^{b}_{pq} \underset{(2)}{C}{}^{a}_{rb} -$$

$$- \underset{(12)}{\overset{2}{P}}{}^{b}_{pr} \underset{(2)}{C}{}^{a}_{qb} = 0$$

Proposition 2.2. *The Ricci tensor fields of* $C\Gamma(N)$ *are:*

$$R_{bp} = r_{bp} + (1-n) \overset{0}{\sigma}_{bp} - \gamma_{bp}\gamma^{as} \overset{0}{\sigma}_{sa} + \overset{1}{\sigma}_{bp} + \tag{2.12}$$

$$+\gamma_{bt}\left(r_{0\ pa}^{\ a} \overset{1}{\sigma}{}^{t} - r_{0\ pa}^{\ t} \overset{1}{\sigma}{}^{a} \right) + \gamma_{bt}\left(r_{0\ pa}^{\ a} \overset{2}{\sigma}{}^{t} - r_{0\ pa}^{\ t} \overset{2}{\sigma}{}^{a} \right);$$

$$\underset{(\alpha)}{\overset{1}{P}}{}_{bp} = \overset{0\alpha}{\sigma}_{bp} + (1-n) \overset{\alpha 0}{\sigma}_{bp} + (1-n) \overset{0}{\sigma}_{p} \overset{\alpha}{\sigma}_{b} + \overset{0}{\sigma}_{b} \overset{\alpha}{\sigma}_{p} -$$

$$-\gamma_{bp}\gamma^{as}\left(\overset{0\alpha}{\sigma}_{sa} - \overset{0}{\sigma}_{a} \overset{\alpha}{\sigma}_{s} \right)$$

$$\underset{(\alpha)}{\overset{2}{P}}{}_{bp} = (1-n) \overset{0\alpha}{\sigma}_{bp} + \overset{\alpha 0}{\sigma}_{bp} + \overset{0}{\sigma}_{p} \overset{\alpha}{\sigma}_{b} + (1-n) \overset{0}{\sigma}_{b} \overset{\alpha}{\sigma}_{0}\ p -$$

$$-\gamma_{bp}\gamma^{as}\left(\overset{0\alpha}{\sigma}_{sa} - \overset{0}{\sigma}_{a} \overset{\alpha}{\sigma}_{s} \right)$$

$$\underset{(12)}{\overset{1}{P}}{}_{bp} = (1-n) \overset{21}{\sigma}_{bp} + \overset{12}{\sigma}_{bp} + \overset{12}{\sigma}_{pb} - \overset{21}{\sigma}_{bp} - \gamma_{bp}\gamma^{as} \overset{12}{\sigma}_{sa} +$$

$$+ \overset{1}{\sigma}_{p} \overset{2}{\sigma}_{b} - \overset{1}{\sigma}_{b} \overset{2}{\sigma}_{p}$$

$$\underset{(12)}{\overset{2}{P}}{}_{bp} = (1-n)\ \overset{12}{\sigma}_{bp} + \overset{21}{\sigma}_{bp} + \overset{21}{\sigma}_{pb} - \overset{12}{\sigma}_{bp} - \gamma_{bp}\gamma^{as}\ \overset{21}{\sigma}_{sa} +$$

$$+ \overset{2}{\sigma}_{p}\ \overset{1}{\sigma}_{b} - \overset{2}{\sigma}_{b}\ \overset{1}{\sigma}_{p}$$

$$\underset{(1)}{S}{}_{bp} = (1-n)\ \overset{1}{\sigma}_{bp} - \gamma^{as}\left(\gamma_{bp}\ \overset{1}{\sigma}_{sa} - \gamma_{ba}\ \overset{1}{\sigma}_{ps}\right) + 2\ \overset{12}{\sigma}_{bp}$$

$$\underset{(2)}{S}{}_{bp} = (1-n)\ \overset{2}{\sigma}_{bp} - \gamma^{as}\left(\gamma_{bp}\ \overset{2}{\sigma}_{sa} - \gamma_{ba}\ \overset{2}{\sigma}_{ps}\right)$$

and the curvature scalars are:

$$R = \Big[r + 2(1-n)\gamma^{bp}\ \overset{0}{\sigma}_{bp} + 2r_{st}y^{(1)s}\ \overset{1}{\sigma}{}^{t} + r_{ijt}y^{(1)i}y^{(1)j}\ \overset{2}{\sigma}{}^{t} +$$

$$+2r_{st}y^{(2)s}\ \overset{2}{\sigma}{}^{t} \Big] e^{-2\sigma} + \underset{(1)}{S} + \underset{(2)}{S} \tag{2.13}$$

$$\underset{\alpha}{\overset{1}{P}} = (1-n)\left(-\gamma^{bp}\ \overset{0\alpha}{\sigma}_{bp} + \gamma^{bp}\ \overset{\alpha 0}{\sigma}_{bp}\right) e^{-2\sigma}$$

$$\underset{\alpha}{\overset{2}{P}} = (1-n)\left(\gamma^{bp}\ \overset{0\alpha}{\sigma}_{bp} - \gamma^{bp}\ \overset{\alpha 0}{\sigma}_{bp}\right) e^{-2\sigma} \qquad (\alpha = 1, 2)$$

$$\underset{12}{\overset{1}{P}} = (1-n)\left(\gamma^{bp}\ \overset{21}{\sigma}_{bp} - \gamma^{bp}\ \overset{12}{\sigma}_{bp}\right) e^{-2\sigma};$$

$$\underset{12}{\overset{2}{P}} = (1-n)\left(\gamma^{bp}\ \overset{12}{\sigma}_{bp} - \gamma^{bp}\ \overset{21}{\sigma}_{bp}\right) e^{-2\sigma}$$

$$\underset{(1)}{S} = 2\left[(1-n)\gamma^{bp}\overset{1}{\sigma}_{bp} - \gamma^{bp}\overset{2}{\sigma}_{bp}\right]e^{-2\sigma};$$

$$\underset{(2)}{S} = 2(1-n)\gamma^{bp}\overset{2}{\sigma}_{bp}\; e^{-2\sigma}$$

Now we can formulate the Einstein equations for this generalized Lagrange space:

Theorem 2.3. *With respect to the N-linear canonical metrical connection the Einstein equations for the space $GL^2(n)$ endowed with the metric tensor*

$$g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)$$

are given by:

$$R_{bp} - \frac{1}{2}\gamma_{bp}R = \chi \underset{()}{\overset{0}{T}}{}_{bp} \tag{2.14}$$

$$\underset{(1)}{S}{}_{bp} - \frac{1}{2}\gamma_{bp}R = \chi \underset{()}{\overset{1}{T}}{}_{bp} \quad ; \quad \underset{(2)}{S}{}_{bp} - \frac{1}{2}\gamma_{bp}R = \chi \underset{()}{\overset{2}{T}}{}_{bp}$$

$$\underset{(\alpha)}{\overset{1}{P}}{}_{bp} = \chi \underset{(\alpha)}{\overset{01}{T}}{}_{bp} \quad ; \quad \underset{(\alpha)}{\overset{2}{P}}{}_{bp} = \chi \underset{(\alpha)}{\overset{02}{T}}{}_{bp} \qquad (\alpha = 1, 2) \tag{2.15}$$

$$\underset{(12)}{\overset{1}{P}}{}_{bp} = \chi \underset{(1)}{\overset{12}{T}}{}_{bp} \quad ; \quad \underset{(12)}{\overset{2}{P}}{}_{bp} = \chi \underset{(2)}{\overset{12}{T}}{}_{bp}$$

Theorem 2.4. *We have the folowing conservation law*

$$\left(R^b_p - \frac{1}{2}R\delta^b_p\right)_{|b} + \underset{(\alpha)}{\overset{1}{P}}{}^b_p \overset{(\alpha)}{|}_b + \underset{(\alpha)}{\overset{2}{P}}{}^b_p \overset{(\alpha)}{|}_b = 0 \tag{2.16}$$

summation on α, $\alpha = 1, 2$

$$\left(\underset{(1)}{S}{}^b_p - \frac{1}{2}R\delta^b_p\right)\overset{(1)}{|}_b - \underset{(1)}{\overset{2}{P}}{}^b_{p|b} + \underset{(12)}{\overset{2}{P}}{}^b_p \overset{(2)}{|}_b = 0$$

$$\left(\underset{(2)}{S^b}_p - \frac{1}{2}R\delta^b_p \right) \overset{(2)}{|_b} - \overset{2}{\underset{(2)}{P^b}}_{p|b} + \overset{2}{\underset{(12)}{P^b}}_p \overset{(1)}{|_b} = 0$$

Corollary 2.1. *If the nonlinear connection N is integrable then the conservation law is:*

$$(2.17)\quad R^b_{p|b} - \frac{1}{2}R_{|p} = 0 \quad ; \quad \underset{(\alpha)}{S^b}_p \overset{(\alpha)}{|_b} - \frac{1}{2}R \overset{(\alpha)}{|_p} = 0 \quad (\alpha = 1,2)$$

3 . Deviation of the connection $C\Gamma(N)$ from Berwald connection

We know that for a nonlinear connection N the corresponding Berwald connection is defined by $B\Gamma(N) = \left(\hat{L}^i_{jk},\ 0,\ 0\right)$, where $\hat{L}^i_{jk} = \dfrac{\delta \underset{(1)}{N^i}_j}{\delta y^{(1)k}}$.
For the space we consider, we have $B\Gamma(N) = \left(\gamma^i_{jk}, 0, 0\right)$

We denote by $_{\|}$ and $\overset{(\alpha)}{\|_i}$ the $h-$ and $v_\alpha-$ covariant derivatives with respect to $B\Gamma(N)$ and we have for any tensor field $T^{i_1\ldots i_r}_{j_1\ldots j_s}$:

$$(3.1)\qquad T^{i_1\ldots i_r}_{j_1\ldots j_s\|m} = \frac{\delta T^{i_1\ldots i_r}_{j_1\ldots j_s}}{\delta x^m} + B^{i_1}_{hm}T^{h\ldots i_r}_{j_1\ldots j_s} + \ldots - B^h_{j_s m}T^{i_1\ldots i_r}_{j_1\ldots h}$$

$$T^{i_1\ldots i_r}_{j_1\ldots j_s} \overset{(\alpha)}{\|_m} = \frac{\delta T^{i_1\ldots i_r}_{j_1\ldots j_s}}{\delta y^{(\alpha)m}} \qquad (\alpha = 1,2)$$

.

Proposition 3.1. *The local coefficients of torsion $\hat{T}$ for $B\Gamma(N)$ are given by:*

$$(3.2)\qquad \underset{(0\alpha)}{\hat{R}^i}_{jk} = \underset{(0\alpha)}{R^i}_{jk} \quad , \quad \overset{2}{\underset{(1)}{\hat{P}^i}}_{jk} = \gamma^i_{js}\gamma^s_{kl}y^{(1)l}, \quad (\alpha = 1,2)$$

all the other components are 0.

Proposition 3.2. *The local coefficients of curvature $\Re$ for $B\Gamma(N)$ are given by:*

$$(3.3)\qquad \hat{R}_j{}^i{}_{kl} = r_j{}^i{}_{kl}$$

all the other components are 0.

Let us consider now that $C\Gamma(N)$ is a deformation of $B\Gamma(N)$. The deformations tensors are given by Proposition 1.1 and the formulas (1.4). So we obtain a transformation of N-linear connections.

Theorem 3.1. *The deformation tensors* $\left(\underset{0}{\Lambda^i_{jk}}, \underset{1}{\Lambda^i_{jk}}, \underset{2}{\Lambda^i_{jk}} \right)$ *can be expressed by:*

$$\underset{0}{\Lambda^i_{jk}} = \frac{1}{2} g^{is} \left(g_{js\|k} + g_{sk\|j} - g_{jk\|s} \right) \tag{3.4}$$

$$\underset{\alpha}{\Lambda^i_{jk}} = \frac{1}{2} g^{is} \left(g_{js} \overset{(\alpha)}{\|_k} + g_{sk} \overset{(\alpha)}{\|_j} - g_{jk} \overset{(\alpha)}{\|_s} \right) \quad (\alpha = 1, 2)$$

Proposition 3.3. *The d-tensor field of torsion for* $C\Gamma(N)$ *is:*

$$T^i{}_{jk} = 0; \qquad \underset{(0\alpha)}{R^i{}_{jk}} = \underset{(0\alpha)}{\hat{R}^i{}_{jk}} \tag{3.5}$$

$$\underset{(\alpha)}{\overset{0}{P^i{}_{jk}}} = \underset{\alpha}{\Lambda^i_{jk}}; \quad \underset{(1)}{\overset{1}{P^i{}_{jk}}} = -\underset{0}{\Lambda^i_{jk}}; \quad \underset{(1)}{\overset{2}{P^i{}_{jk}}} = \underset{(1)}{\overset{2}{\hat{P}^i{}_{jk}}} - \gamma^i_{jk}$$

$$\underset{(2)}{\overset{1}{P^i{}_{jk}}} = 0; \quad \underset{(2)}{\overset{2}{P^i{}_{jk}}} = -\underset{0}{\Lambda^i_{jk}}; \quad \underset{(12)}{\overset{0}{P^i{}_{jk}}} = 0: \quad \underset{(12)}{\overset{\alpha}{P^i{}_{jk}}} = \underset{\alpha}{\Lambda^i_{jk}}$$

$$\underset{(\alpha)}{\overset{\beta}{S^i{}_{jk}}} = 0 \qquad (\alpha = 1, 2; \beta = 0, 1, 2)$$

and the d-tensor field of curvature can be written in the form:

$$\underset{}{R_j{}^i{}_{kp}} = \underset{(0)}{\hat{R}_j{}^i{}_{kp}} + \underset{(0)}{\bar{R}_j{}^i{}_{kp}}; \quad \underset{(\alpha)}{P_j{}^i{}_{kp}} = \underset{(\alpha)}{\bar{P}_j{}^i{}_{kp}} \tag{3.6}$$

$$\underset{(12)}{P_j{}^i{}_{kp}} = \underset{(12)}{\bar{P}_j{}^i{}_{kp}}; \quad \underset{(\alpha)}{S_j{}^i{}_{kp}} = \underset{(\alpha)}{\bar{S}_j{}^i{}_{kp}} \quad (\alpha = 1, 2)$$

where

$$\underset{(0)}{\bar{R}_j{}^i{}_{kp}} = \underset{0}{\Lambda^i_{jk\|p}} - \underset{0}{\Lambda^i_{jp\|k}} + \underset{0}{\Lambda^h_{jk}} \underset{0}{\Lambda^i_{hp}} -$$

$$- \underset{0}{\Lambda^h_{jp}} \underset{0}{\Lambda^i_{hk}} + \underset{1}{\Lambda^i_{jh}} \underset{(01)}{R^h{}_{kp}} + \underset{2}{\Lambda^i_{jh}} \underset{(02)}{R^h{}_{kp}}$$

$$\underset{(\alpha)}{\bar{P}_j{}^i{}_{kp}} = \underset{0}{\Lambda^i_{jk}} \overset{(\alpha)}{\|}_p - \underset{\alpha}{\Lambda^i_{jp\|k}} + \underset{\alpha}{\Lambda^h_{jk}} \underset{0}{\Lambda^i_{hp}} - \underset{\alpha}{\Lambda^h_{jp}} \underset{0}{\Lambda^i_{hk}}$$

$$\underset{(12)}{\bar{P}_j{}^i{}_{kp}} = \underset{1}{\Lambda^i_{jk}} \overset{(2)}{\|}_p - \underset{2}{\Lambda^i_{jp}} \overset{(1)}{\|}_k + \underset{2}{\Lambda^h_{jk}} \underset{1}{\Lambda^i_{hp}} - \underset{2}{\Lambda^h_{jp}} \underset{1}{\Lambda^i_{hk}}$$

$$\underset{(\alpha)}{\bar{S}_j{}^i{}_{kp}} = \underset{\alpha}{\Lambda^i_{jk}} \overset{(\alpha)}{\|}_p - \underset{\alpha}{\Lambda^i_{jp}} \overset{(\alpha)}{\|}_k + \underset{\alpha}{\Lambda^h_{jk}} \underset{\alpha}{\Lambda^i_{hp}} - \underset{\alpha}{\Lambda^h_{jp}} \underset{\alpha}{\Lambda^i_{hk}} \quad (\alpha = 1, 2)$$

Hence, we obtain this form for the Einstein equations:

Theorem 3.2. *With respect to the N-linear canonical metrical connection $C\Gamma(N)$ and the linear transformation of N-linear connections, Einstein equations for the space $GL^2(n)$ endowed with the metric tensor*

$$g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)$$

are:

$$r_{ij} - \frac{1}{2}g_{ij}r + \bar{R}_{ij} - \frac{1}{2}g_{ij}\bar{R} = \chi\hat{T}_{ij} \tag{3.7}$$

$$\underset{(\alpha)}{\bar{S}}{}_{ij} - \frac{1}{2}g_{ij} \underset{(\alpha)}{\bar{S}} = \chi \overset{\alpha}{T}{}_{ij}$$

$$\underset{(\alpha)}{\overset{\beta}{\bar{P}}}{}_{ij} = (-1)^{\beta+1}\chi \underset{\alpha}{\overset{0\beta}{T}}{}_{ij} \quad ; \quad \underset{(12)}{\overset{\alpha}{\bar{P}}}{}_{ij} = (-1)^{\alpha+1}\chi \underset{\alpha}{\overset{12}{T}}{}_{ij} \quad (\alpha, \beta = 1, 2)$$

4. Special example

Let us consider now that the function σ is given by:

$$\sigma(x^i, y^{(1)i}, y^{(2)i}) = \frac{1}{2}\|y^{(1)i}\|^2 = \frac{1}{2}\gamma_{ij} z_i^{(1)} z_j^{(1)} \tag{4.1}$$

We obtain the following results:

Proposition 4.1.

$$\overset{0}{\bar{\sigma}_k} = 0; \quad \overset{1}{\bar{\sigma}_k} = z_k^{(1)}; \quad \overset{2}{\bar{\sigma}_k} = 0 \tag{4.2}$$

$$\bar{L}^i_{jk} = \gamma^i_{jk}; \quad \underset{(1)}{\bar{C}^i_{jk}} = \delta^i_k z_j^{(1)} + \delta^i_j z_k^{(1)} - \gamma_{jk} z^{(1)i}; \quad \underset{(2)}{\bar{C}^i_{jk}} = 0$$

$$\overset{0}{\bar{\sigma}_{ij}} = 0; \quad \overset{01}{\bar{\sigma}_{ij}} = 0; \quad \overset{02}{\bar{\sigma}_{ij}} = 0; \quad \overset{0}{\bar{\sigma}^t} \overset{\alpha}{\bar{\sigma}_t} = \overset{0}{\bar{\sigma}_t} \overset{\alpha}{\bar{\sigma}^t} = 0 \quad (\alpha = 0, 1, 2)$$

$$\overset{1}{\bar{\sigma}_{ij}} = \delta^i_j + \frac{1}{2}\gamma_{ij}\|y^{(1)}\|^2 - z_i^{(1)} z_j^{(1)}; \quad \overset{1}{\bar{\sigma}^t} \overset{1}{\bar{\sigma}_t} = \|y^{(1)}\|^2;$$

$$\overset{1}{\bar{\sigma}^t} \overset{\alpha}{\bar{\sigma}_t} = 0 \ (\alpha = 0, 2); \quad \overset{1\alpha}{\bar{\sigma}_{ij}} = 0; \ \alpha = 1, 2; \quad \overset{2}{\bar{\sigma}_{ij}} = 0$$

Proposition 4.2. *The local components of the d-tensor field of torsion are:*

$$\underset{(0)}{\overset{\alpha}{\bar{T}^i}}{}_{jk} = \underset{(0)}{\overset{\alpha}{T^i}}{}_{jk}; \quad \underset{(1)}{\overset{0}{\bar{P}^i}}{}_{jk} = \underset{(1)}{C^i_{jk}}; \quad \underset{(2)}{\overset{0}{\bar{P}^i}}{}_{jk} = 0 \tag{4.3}$$

$$\underset{(1)}{\overset{1}{\bar{P}^i}}{}_{jk} = 0; \quad \underset{(1)}{\overset{2}{\bar{P}^i}}{}_{jk} = \underset{(1)}{\overset{2}{P^i}}{}_{jk}; \quad \underset{(21)}{\overset{1}{\bar{P}^i}}{}_{jk} = \underset{(1)}{C^i_{jk}} \tag{4.4}$$

$$\underset{(2)}{\overset{1}{\bar{P}^i}}{}_{jk} = \underset{(2)}{\overset{2}{\bar{P}^i}}{}_{jk} = \underset{(21)}{\overset{0}{\bar{P}^i}}{}_{jk} = \underset{(21)}{\overset{2}{\bar{P}^i}}{}_{jk} = 0$$

$$\underset{(\alpha)}{\overset{\beta}{\bar{S}^i}}{}_{jk} = 0; \quad (\alpha, \beta = 0, 1, 2) \tag{4.5}$$

Proposition 4.3. *The local components of the d-tensor field of curvature are:*

$$\bar{R}_{b\ pq}^{\ a} = r_{b\ pq}^{\ a} + g_{bt}\left(r_{s\ pq}^{\ a} y^{(1)t} y^{(1)s} - r_{s\ pq}^{\ t} y^{(1)a} y^{(1)s}\right) \tag{4.6}$$

$$\underset{(\alpha)}{\bar{P}}{}_{b\ pq}^{\ a} = 0 \qquad (\alpha = 1, 2); \qquad \underset{(12)}{\bar{P}}{}_{b\ pq}^{\ a} = 0 \tag{4.7}$$

$$\underset{(1)}{\bar{S}}{}_{b\ pq}^{\ a} = \delta_p^a \overset{1}{\bar{\sigma}}_{bq} - \delta_q^a \overset{1}{\bar{\sigma}}_{bp} + g^{as}\left(g_{bq} \overset{1}{\bar{\sigma}}_{sp} - g_{bp} \overset{1}{\bar{\sigma}}_{sq}\right) \tag{4.8}$$

$$\underset{(2)}{\bar{S}}{}_{b\ pq}^{\ a} = 0$$

Proposition 4.4. *Ricci tensor fields and the scalars of curvature are expressed like:*

$$\bar{R}_{bp} = r_{bp} + \gamma_{bt}\left(r_{0\ pa}^{\ a} y^{(1)t} - r_{0\ pa}^{\ t} y^{(1)a}\right) \tag{4.9}$$

$$\underset{(\alpha)}{\overset{1}{\bar{P}}}{}_{bp} = \underset{(\alpha)}{\overset{2}{\bar{P}}}{}_{bp} = 0 \quad (\alpha = 1, 2); \qquad \underset{(12)}{\overset{1}{\bar{P}}}{}_{bp} = \underset{(12)}{\overset{2}{\bar{P}}}{}_{bp} = 0; \tag{4.10}$$

$$\underset{(1)}{\bar{S}}{}_{bp} = (1-n) \overset{1}{\bar{\sigma}}_{bp} + \gamma^{as}\left(\gamma_{bp} \overset{1}{\bar{\sigma}}_{sa} - \gamma_{ba} \overset{1}{\bar{\sigma}}_{sp}\right); \qquad \underset{(2)}{\bar{S}}{}_{bp} = 0 \tag{4.11}$$

$$\bar{R} = \left(r + 2r_{st} y^{(1)s} y^{(1)t} + 2(1-n)\gamma^{bp} \overset{1}{\sigma}_{bp}\right) e^{-2\sigma} \tag{4.12}$$

$$\underset{(\alpha)}{\overset{1}{\bar{P}}} = \underset{(\alpha)}{\overset{2}{\bar{P}}} = 0; \qquad \underset{(12)}{\overset{1}{\bar{P}}} = \underset{(12)}{\overset{1}{\bar{P}}} = 0; \quad (\alpha = 1, 2) \tag{4.13}$$

$$\underset{(1)}{\bar{S}} = 2(1-n)\gamma^{bp} \overset{1}{\bar{\sigma}}_{bp}\, e^{-2\sigma}; \qquad \underset{(2)}{\bar{S}} = 0 \tag{4.14}$$

Proposition 4.5. *Einstein equations for the space $GL^{2(n)}$ endowed with the metric tensor $g_{ij} = e^{\frac{1}{2}\|y^{(1)i}\|^2}\gamma_{ij}(x)$ are:*

$$\frac{1}{2}r\gamma_{bp} + \gamma_{bp}\left[(n-1)\gamma^{ij} \overset{1}{\sigma}_{ij} - r_{st}y^{(1)s}y^{(1)t}\right] = \tag{4.15}$$

$$= \chi \underset{()}{\overset{0}{T}}_{bp} - \gamma_{bt}\left(r_0{}^a{}_{pa}y^{(1)t} - r_0{}^t{}_{pa}y^{(1)a}\right)$$

$$\underset{(1)}{\bar{S}}_{bp} - \frac{1}{2}g_{bp}R = \chi \underset{()}{\overset{1}{T}}_{bp} \ ; \quad \frac{1}{2}g_{bp}R = -\chi \underset{()}{\overset{1}{T}}_{bp}$$

First equation was presented in the form in order to emphasize the relation between Einstein equations of the considered space and Einstein equations for the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

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